THE GENERALIZED LIE ALGEBROIDS AND THEIR APPLICATIONS

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In memory of my uncles
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Dedicated to Acad. Prof. Dr. Doc. Radu MIRON at his 83^{th} anniversary

Abstract

In this paper we introduce the notion of generalized Lie algebroid and we develop a new formalism necessary to obtain a new solution for the Weistein's Problem [53]. Many applications emphasize the importance and the utility of this new framework determined by the introduction of generalized Lie algebroids.

We introduce and develop the exterior differential calculus for generalized Lie algebroids and, in this general framework, we establish the structure equations of Maurer-Cartan type. In particular, we obtain a new point of view over the exterior differential calculus for Lie algebroids.

Using the (generalized) Lie algebroids theory, we build the Lie algebroid generalized tangent bundle and, using that, we obtain a new method by determining the (linear) connections for fiber bundles, in general, and for vector bundles, in particular.

Using the linear connections theory we develop the study of the geometry of vector bundles. Moreover, using the connections theory, we develop the geometry of total space of the generalized tangent bundle for a vector bundle.

We present a geometric description of metrizability for the total space of the Lie algebroid generalized tangent bundle, where we extend the notions of generalized Lagrange space, Lagrange space and Finsler space. Using the Lie algebroid generalized tangent bundle of a generalized Lie algebroid, we introduce and develop a mechanical systems theory and we present a Lagrangian formalism for these mechanical systems. In particular, using the Lie algebroid generalized tangent bundle of a Lie algebroid, we obtain a new solution for the Weinstein's Problem.

A geometric description of metrizability for the total space of the Lie algebroid generalized tangent bundle for dual vector bundle is presented. We extend the notions of generalized Hamilton space, Hamilton space and Cartan space. Using the Lie algebroid generalized tangent bundle of dual of a generalized Lie algebroid, we introduce and develop the dual mechanical systems theory and we present a Hamiltonian formalism for dual mechanical systems.

Finally, we introduce and develop the concept of (horizontal) Legendre equivalence between a vector bundle and its dual vector bundle.

We remark that, if the morphisms used are identities morphisms, then we obtain similar results to the classical results, but which are not classical results though.

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Contents

1 Introduction

The motivation for our researches was the

> **Weinstein's Problem:**
>
> *Develop a Lagrangian formalism directly on the given Lie algebroid similar to Klein's formalism for ordinary Lagrangian Mechanics* [25].

This problem was formulated by A. Weinstein in [53], where the author gave the theory of Lagrangians on Lie algebroids and obtained the Euler-Lagrange equations using the dual of a Lie algebroid and the Legendre transformation defined by a regular Lagrangian.

In [28], P. Liberman showed that such a formalism is not possible if one consider the tangent bundle of a Lie algebroid as space for developing the theory. Using the prolongation of a Lie algebroid over a smooth map introduced by P.J. Higgins and K. Mackenzie in [15], E. Martinez solved the *Weinstein's Problem* in [53] (see also [13], [27]).

Finding an other space for developing the theory, we discovered the generalized Lie algebroids which are presented in Subsection 3.3.

Since any Lie algebroid can be regarded as a generalized Lie algebroid, we proposed to obtain a new solution for the *Weinstein's Problem* using the new notion of generalized Lie algebroid.

To solve this problem it was necessary to introduce and develop a new formalism. In order to develop our researches, new and interesting notions and results appeared, which determined the apparition of new theories which are naturally integrated in our paper.

So, in Subsection 3.2 we introduce and develop the exterior differential calculus for generalized Lie algebroids and, using that, we establish the structure equations of Maurer-Cartan type for generalized Lie algebroids. In particular, we obtain a new point of view over the exterior differential calculus for Lie algebroids.

Inspired by the general framework of Yang-Mills theory, presented synthetically in the following diagram:

$$\begin{array}{ccc} (E,\langle,\rangle_E) & & (TM,[,]_{TM},(Id_{TM},Id_M),g) \\ \downarrow\pi & & \downarrow\tau_M \\ M & \xrightarrow{Id_M} & M \end{array}$$

where:

1. (E,π,M) is a vector bundle,

2. $\langle,\rangle_E$ is an inner product for the module of sections $\Gamma(E,\pi,M)$,

3. $((Id_{TM},Id_M),[,]_{TM})$ is the usual Lie algebroid structure for the tangent vector bundle (TM,τ_M,M) and

4. $g\in\Gamma((T^*M,\tau_M^*,M)\otimes(T^*M,\tau_M^*,M))$ such that (M,g) is a Riemannian manifold,

we build the *Lie algebroid generalized tangent bundle* in Subsection 3.3.

Using this in Subsection 3.4, we introduce and develop a (linear) connections theory for fiber bundles, in general, and for vector bundles, in particular.

We can define the covariant derivatives with respect to sections of the generalized Lie algebroid

$$\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right).$$

In particular, if we use the generalized Lie algebroid structure

$$\left([,]_{TM,Id_M},(Id_{TM},Id_M)\right)$$

for the tangent bundle (TM,τ_M,M) in our theory, then the linear connections obtained are similar with the classical linear connections for the vector bundle (E,π,M), but not classical linear connections.

It is known that in Yang-Mills theory the set

$$Cov^0_{(E,\pi,M)}$$

of covariant derivatives for the vector bundle (E,π,M) such that

$$X\left(\langle u,v\rangle_E\right)=\langle D_X(u),v\rangle_E+\langle u,D_X(v)\rangle_E,$$

for any $X\in\mathcal{X}(M)$ and $u,v\in\Gamma(E,\pi,M)$, is very important, because the Yang-Mills theory is a variational theory which use (cf. [6]) the Yang-Mills functional

$$\begin{array}{rcl} Cov^0_{(E,\pi,M)} & \stackrel{\mathcal{YM}}{\longrightarrow} & \mathbb{R} \\ D_X & \longmapsto & \dfrac{1}{2}\int\limits_M \left\|\mathbb{R}^{D_X}\right\|^2 v_g \end{array}$$

where $\mathbb{R}^{D_X}$ is the curvature.

Using the linear connections theory, we succeed to extend at maximum the set $Cov^0_{(E,\pi,M)}$ of Yang-Mills theory, because using all generalized Lie algebroid structures for the tangent bundle (TM,τ_M,M), we obtain all possible linear connections for the vector bundle (E,π,M).

We emphasize the importance and the utility of linear connections theory for vector bundles in Chapter IV of our paper, where we present many applications. In particular, we obtain similar results to the classical results, but which are not classical results though.

After that we study the geometry of total space of the Lie algebroid generalized tangent bundle for a vector bundle in Section 5 of our paper, where we emphasize the importance and the utility of connections theory presented in Subsection 3.4.

The geometry of Lagrange spaces, introduced and studied in [24] and [35], was extensively examined in the last two decades by geometers and physicists from Romania, Japan, Hungary, Canada, Germany, Italy, Russia and USA. Many international conferences devoted to debate this subject, proceedings and monographs where published [3], [4], [41], [42]. A large area of applicability of this geometry is suggested by the connections to Biology, Mechanics and Physics and also by its general setting as a generalization of Finsler and Riemann geometries.

As the (generalized) Lagrange space has been certified as an excellent model for some important problems in Relativity, Gauge Theory and Electromagnetism, in Subsections

5.8 and 5.9 we continue and we present a geometric description of metrizability for the total space of the Lie algebroid generalized tangent bundle for a vector bundle. We extend the notions of generalized Lagrange space, Lagrange space and Finsler space and we define the Einstein equations in this general framework.

Subsection 5.11 is devoted to introduce and study of a new class of mechanical systems called by us *mechanical (ρ,η)-systems, generalized Lagrange mechanical (ρ,η)-systems*, *Lagrange mechanical (ρ,η)-systems and Finsler mechanical (ρ,η)-systems.*

For these mechanical systems we develop a theory of semisprays and sprays. We develop a Lagrangian formalism for Lagrange mechanical systems.

We determine and we study the (ρ,η)-semispray associated to a regular Lagrangian L and external force F_e which are applied on the total space of a generalized Lie algebroid and we derive the equations of Euler-Lagrange type.

In particular, using the Lie algebroid generalized tangent bundle of a Lie algebroid, we obtain a new solution for the *Weinstein's Problem* different by the Martinez's solution [53].

Moreover, if the Lie algebroid used is

$$\left(\left(TM,\tau_M,M\right),\left[,\right]_{TM},\left(Id_{TM},Id_M\right)\right),$$

then we obtain similar results to those presented by I. Bucataru and R. Miron in [7].

It is known that in 1918, immediately after the birth of general relativity, Weyl proposed the first unified theory of gravitation and electromagnetism, by generalizing the Riemannian space.

We are interested in finding the answer to the following question:

- *Could we to extend the study of the Riemannian geometry from the usual Lie algebroid*

  $$\left(\left(TM,\tau_M,M\right),\left[,\right]_{TM},\left(Id_{TM},Id_M\right)\right),$$

  to an arbitrary (generalized) Lie algebroid and can we obtain a general framework necessary to unify the theory of gravitation with the theory of electromagnetism?

The future will show how far our theory can be used in this direction.

Our researches continue in Section 6, where we study the geometry of total space of the Lie algebroid generalized tangent bundle of a dual vector bundle and so, we emphasize the importance and the utility of the generalized connections theory presented in paragraph 3.4.1.

We present the adapted (ρ,η)-basis and adapted dual (ρ,η)-basis and remarkable endomorphisms of $(\Gamma((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}),+,\cdot)$ module (projectors, almost product structure, almost tangent structure, almost complex structure, (ρ,η)-tension endomorphism) and we present the (ρ,η)-torsion and the (ρ,η)-curvature of a (ρ,η)-connection $(\rho,\eta)\,\Gamma$. We introduce and studied distinguished linear (ρ,η)-connections and we build the (g,h)-lift of accelerations for a differentiable curve. Using the distinguished linear (ρ,η)-connections theory, we introduced and study the (ρ,η)-torsion, the (ρ,η)-curvature and we present the formulas of Ricci type and the identities of Cartan and Bianchi type.

The concept of Hamilton space, introduced in [36], [40], was intensively studied in [19], [20], [21], and it has been successful, as a geometric theory of the Hamiltonian function. The modern formulation of the geometry of Cartan spaces was given by R. Miron

([36], [38]) although some results where obtained by É. Cartan [9] and A. Kawaguchi [23] . Since the fundamental entity in Mechanics and Physics is the (generalized) Hamilton space, in Subsections 4.8 and 4.9 we continue to present a geometric description of metrizability for the total space of the Lie algebroid generalized tangent bundle of dual vector bundle. We extend the notions of generalized Hamilton space, Hamilton space and Cartan space and we define the Einstein equations in this general framework.

Subsection 4.11 is devoted to the introduction and the study of a new class of mechanical systems, called by us *dual mechanical* (ρ, η)*-systems, generalized Hamilton mechanical* (ρ, η)*-systems, Hamilton mechanical* (ρ, η)*-systems and Cartan mechanical* (ρ, η)*-systems*. For dual mechanical systems we develop a theory of semisprays and sprays. For Hamilton mechanical systems we develop a Hamiltonian formalism. We determine and study the (ρ, η)-semispray associated to a regular Hamiltonian H and external force $\overset{*}{F}_e$, which are applied on the total space of the dual of a generalized Lie algebroid and we derive the equations of Hamilton-Jacobi type. One remarks that, if the morphisms used are identities, then similar results can be obtained by classical results, but not classical ones.

The classical Legendre's duality makes possible a natural connection between Lagrange and Hamilton spaces. It reveals new concepts and geometrical objects of Hamilton spaces that are dual to those which are similar in Lagrange spaces. The geometrical theory of Hamilton (Cartan) spaces was investigated from the Legendre duality point of view in the papers [36], [38], [20], [21].

In our paper, we propose a new point of view over the Legendre duality. We introduce and develop the notion of *(horizontal) Legendre* (ρ, η, h)*-equivalence* between an arbitrary vector bundle and its dual. For this new theory it was necessary to build the (ρ, η)*-tangent application* of the Legendre bundle morphism associated to a Lagrangian or a Hamiltonian.

We consider that this new theory can be used in the develop of the Poisson Geometry and Symplectic Geometry.

2 Preliminaries

In general, if $\mathcal{C}$ is a category, then we denoted by $|\mathcal{C}|$ the class of objects and we denoted by $\overrightarrow{\mathcal{C}}$ the class of arrows (morphisms). For any $A, B \in |\mathcal{C}|$, we denote by $\mathcal{C}(A, B)$ the morphisms set of A source and B target.

Let **Vect**, **Liealg**, **Mod**, **Man**, **B** and $\mathbf{B^v}$ be the category of real vector spaces, Lie algebras, modules, manifolds, fiber bundles and vector bundles respectively.

2.1 The category of Lie algebroids

Let $N \in |\mathbf{Man}|$ and $[,]_{TN}$ be the usual Lie bracket such that

$$\left(\Gamma\left(TN, \tau_N, N\right), +, \cdot, [,]_{TN}\right) \in |\mathbf{LieAlg}|.$$

Definition 2.1.1 If $(F, \nu, N) \in |\mathbf{B^v}|$ such that there exists

$$(\rho, Id_N) \in \mathbf{B^v}\left((F, \nu, N), (TN, \tau_N, N)\right)$$

and an operation

$$\begin{array}{ccc}\Gamma(F,\nu,N)\times\Gamma(F,\nu,N) & \xrightarrow{[,]_F} & \Gamma(F,\nu,N)\\ (u,v) & \longmapsto & [u,v]_F\end{array}$$

with the following properties:

LA_1. the equality holds good

$$[u, f\cdot v]_F = f\,[u,v]_F + \Gamma(\rho, Id_N)(u)\,f\cdot v,$$

for all $u,v\in\Gamma(F,\nu,N)$ and $f\in\mathcal{F}(N)$,

LA_2. the 4-tuple

$$(\Gamma(F,\nu,N),+,\cdot,[,]_F)$$

is a Lie $\mathcal{F}(N)$-algebra,

LA_3. the **Mod**-morphism $\Gamma(\rho, Id_N)$ is a **LieAlg**-morphism of

$$(\Gamma(F,\nu,N),+,\cdot,[,]_F)$$

source and

$$(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$$

target,

then we will say that *the triple*

$$((F,\nu,N),[,]_F,(\rho,Id_N)) \tag{2.1.1}$$

is a Lie algebroid.

The couple

$$([,]_F,(\rho,Id_N))$$

is called *Lie algebroid structure.*

Definition 2.1.2 We define the morphisms set of

$$((F,\nu,N),[,]_F,(\rho,Id_N))$$

source and

$$\left(\left(F',\nu',N'\right),[,]_{F'},\left(\rho',Id_{N'}\right)\right)$$

target as being the set

$$\{(\varphi,\varphi_0)\in\mathbf{B^v}\left((F,\nu,N),(F',\nu',N')\right)\}$$

such that the **Mod**-morphism $\Gamma(\varphi,\varphi_0)$ is a **LieAlg**-morphism of

$$(\Gamma(F,\nu,N),+,\cdot,[,]_F)$$

source and

$$\left(\Gamma\left(F',\nu',N'\right),+,\cdot,[,]_{F'}\right)$$

target.

Remark 2.1.1 Note that we can discuss about *the category of Lie algebroids.* This category is denoted by **LA**.

If

$$\left(\left(F,\nu,N\right),\left[,\right]_F,\left(\rho,Id_N\right)\right)$$

is a Lie algebroid, then we assume that (F,ν,N) is a vector bundle with type fibre the real vector space $(\mathbb{R}^p,+,\cdot)$ and structure group a Lie subgroup of $(\mathbf{GL}(p,\mathbb{R}),\cdot)$.

We take $(\varkappa^{\tilde{\imath}},z^{\alpha})$ as canonical local coordinates on (F,ν,N), where $\tilde{\imath}\in\overline{1,n}$, $\alpha\in\overline{1,p}$.

Consider

$$\left(\varkappa^{\tilde{\imath}},z^{\alpha}\right)\longrightarrow\left(\varkappa^{\tilde{\imath}},z^{\alpha'}\right)$$

a change of coordinates on (F,ν,N). Then the coordinates z^{α} change to $z^{\alpha'}$ by the rule:

$$z^{\alpha'}=\Lambda_{\alpha}^{\alpha'}z^{\alpha}. \tag{2.1.2}$$

The coefficients $\rho_{\alpha}^{\tilde{\imath}}$ change to $\rho_{\alpha'}^{\tilde{\imath}}$ by the rule:

$$\rho_{\alpha'}^{\tilde{\imath}}=\Lambda_{\alpha'}^{\alpha}\rho_{\alpha}^{\tilde{\imath}}\frac{\partial\varkappa^{\tilde{\imath}}}{\partial\varkappa^{\tilde{\imath}}}, \tag{2.1.3}$$

where

$$\left\|\Lambda_{\alpha'}^{\alpha}\right\|=\left\|\Lambda_{\alpha}^{\alpha'}\right\|^{-1}.$$

Locally, we obtain

$$\left[t_{\alpha},t_{\beta}\right]_F\overset{put}{=}L_{\alpha\beta}^{\gamma}t_{\gamma}. \tag{2.1.4}$$

The real local functions

$$L_{\alpha\beta}^{\gamma},\ \alpha,\beta,\gamma\in\overline{1,p}$$

will be called *structure functions of the Lie algebroid*

$$\left(\left(F,\nu,N\right),\left[,\right]_F,\left(\rho,Id_N\right)\right).$$

It is easy to prove that

$$L_{\alpha\beta}^{\gamma}=-L_{\beta\alpha}^{\gamma},\ \forall\alpha,\beta,\gamma\in\overline{1,p}.$$

2.2 The pull-back Lie algebroid of a Lie algebroid

We consider the following diagram:

$$\begin{array}{ccc} & & \left(F,\left[,\right]_F,\left(\rho,Id_N\right)\right)\\ & & \downarrow\nu\\ E & \xrightarrow{\quad\pi\quad} & N\end{array} \tag{2.2.1}$$

where (E,π,M) is a fiber bundle and $\left(\left(F,\nu,N\right),\left[,\right]_F,\left(\rho,Id_N\right)\right)$ is a Lie algebroid.

We assume that (E,π,M) has the type fibre a manifold of dimension r and structure group a Lie group $(\mathbf{G},\cdot)$.

Proposition 2.2.1 *Using the tangent* $\mathbf{B^v}$*-morphism* $(T\pi,\pi)$ *of* (TE,τ_E,E) *source and* (TN,τ_N,N) *target, we obtain that*

$$\frac{\partial f\circ\pi}{\partial x^{\tilde{\imath}}}=\frac{\partial f}{\partial x^{\tilde{\imath}}}\circ\pi,\ \forall f\in\mathcal{F}(N) \tag{2.2.2}$$

and

$$\frac{\partial f\circ\pi}{\partial y^{a}}=0,\ \forall f\in\mathcal{F}\left(N\right). \tag{2.2.3}$$

Let $\mathcal{AF}_F$ be a representative of vector fibred $(n+p)$-structure for the vector bundle (F,ν,N) and let $\mathcal{AF}_E$ be a representative of fibred $(n+r)$-structure for the fiber bundle (E,π,N). Let $(\pi^*F,\pi^*\nu,E)$ be the pull-back vector bundle through π.

If $(U,\xi_U)\in\mathcal{AF}_E$ and $(V,s_V)\in\mathcal{AF}_F$ such that $U\cap V\neq\phi$, then we define the application

$$\begin{array}{ccc}\pi^*\nu^{-1}(\pi^{-1}\left(U\cap V\right)) & \xrightarrow{\tilde{s}_{\pi^{-1}(U\cap V)}} & \pi^{-1}\left(U\cap V\right)\times\mathbb{R}^p\\ \left(u,\tilde{Z}\left(u\right)\right) & \longmapsto & \left(\varkappa,t_{V,\pi(u)}^{-1}\tilde{Z}\left(u\right)\right).\end{array}$$

Proposition 2.2.2 *The set*

$$\widetilde{\mathcal{AF}}_{\pi^*F}\overset{put}{=}\bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_E,\ (V,s_V)\in\mathcal{AF}_F\\ U\cap V\neq\phi}}\left\{\left(\pi^{-1}\left(U\cap V\right),\tilde{s}_{\pi^{-1}(U\cap V)}\right)\right\}$$

is a vector fibred $(m+r)+p$-atlas for the vector bundle $(\pi^*F,\pi^*\nu,E)$.

If

$$z=z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right),$$

then, using the vector fibred $(m+r)+p$-structure $\left[\widetilde{\mathcal{AF}}_{\pi^*F}\right]$, we obtain the section

$$\tilde{Z}=\left(z^{\alpha}\circ h\right)\tilde{T}_{\alpha}\in\Gamma\left(\pi^*F,\pi^*\nu,E\right)$$

such that

$$\tilde{Z}\left(u_x\right)=z\left(x\right),$$

for any $u_x\in\pi^{-1}\left(U\cap V\right)$.

The set $\left\{\tilde{T}_{\alpha},\ \alpha\in\overline{1,p}\right\}$ is a base for the module of sections

$$\left(\Gamma\left(\pi^*F,\pi^*\nu,E\right),+,\cdot\right).$$

Let $\left(\overset{\pi^*F}{\rho},Id_E\right)$ be the $\mathbf{B^v}$-morphism of

$$\left(\pi^*F,\pi^*\nu,E\right)$$

source and

$$\left(TE,\tau_E,E\right)$$

target, where

$$\begin{array}{ccc}\pi^*F & \xrightarrow{\overset{\pi^*F}{\rho}} & TE\\ \tilde{Z}^{\alpha}\tilde{T}_{\alpha}\left(u_x\right) & \longmapsto & \left(\tilde{Z}^{\alpha}\cdot\rho_{\alpha}^{\tilde{\imath}}\circ\pi\dfrac{\partial}{\partial x^{\tilde{\imath}}}\right)\left(u_x\right)\end{array} \tag{2.2.4}$$

We consider the operation

$$\Gamma\left(\pi^*F,\pi^*\nu,E\right)\times\Gamma\left(\pi^*F,\pi^*\nu,E\right)\xrightarrow{[,]_{\pi^*F}}\Gamma\left(\pi^*F,\pi^*\nu,E\right)$$

defined by

$$(2.2.5)\qquad \begin{aligned}\left[\tilde{T}_\alpha,\tilde{T}_\beta\right]_{\pi^*F} &= \left(L_{\alpha\beta}^\gamma\circ\pi\right)\tilde{T}_\gamma,\\ \left[\tilde{T}_\alpha,f\tilde{T}_\beta\right]_{\pi^*F} &= f\left(L_{\alpha\beta}^\gamma\circ\pi\right)\tilde{T}_\gamma+\left(\rho_\alpha^{\tilde{\imath}}\circ\pi\right)\frac{\partial f}{\partial x^{\tilde{\imath}}}\tilde{T}_\beta,\\ \left[f\tilde{T}_\alpha,\tilde{T}_\beta\right]_{\pi^*F} &= -\left[\tilde{T}_\beta,f\tilde{T}_\alpha\right]_{\pi^*F},\end{aligned}$$

for any $f\in\mathcal{F}(E)$.

Lemma 2.2.1 *The following equality holds good*

$$\left[\tilde{U},f\tilde{V}\right]_{\pi^*F}=f\left[\tilde{U},\tilde{V}\right]_{\pi^*F}+\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\left(\tilde{U}\right)f\cdot\tilde{V},$$

for any $\tilde{U},\tilde{V}\in\Gamma(\pi^*F,\pi^*\nu,E)$ *and for any* $f\in\mathcal{F}(E)$.

Proof. We observe that for any $\alpha,\beta\in\overline{1,p}$, we obtain

$$\left[\tilde{T}_\alpha,f\tilde{T}_\beta\right]_{\pi^*F}=f\left[\tilde{T}_\alpha,\tilde{T}_\beta\right]_{\pi^*F}+\Gamma\left(\overset{h^*F}{\rho},Id_E\right)\tilde{T}_\alpha(f),\ \forall f\in\mathcal{F}(E).$$

Using this equality and the definition of the operation $[,]_{\pi^*F}$ it results the conclusion of the lemma. *q.e.d.*

Lemma 2.2.2 *The* $\mathcal{F}(E)$*-algebra*

$$\left(\Gamma(\pi^*F,\pi^*\nu,E),+,\cdot,[,]_{\pi^*F}\right)$$

is a Lie $\mathcal{F}(E)$*-algebra.*

Proof. Using the definition of the operation $[,]_{\pi^*F}$ it results that

$$\left[\tilde{U},\tilde{V}\right]_{\pi^*F}=-\left[\tilde{V},\tilde{U}\right]_{\pi^*F},$$

for any $\tilde{U},\tilde{V}\in\Gamma(\pi^*F,\pi^*\nu,E)$. Therefore, we obtain

$$(1)\qquad \left[\tilde{U},\tilde{U}\right]_{\pi^*F}=0,\ \forall\tilde{U}\in\Gamma(\pi^*F,\pi^*\nu,E).$$

Since

$$\left(\Gamma(F,\nu,N),+,\cdot,[,]_F\right)$$

is a Lie $\mathcal{F}(N)$-algebra, we obtain the equality:

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(L_{\beta\gamma}^\varepsilon L_{\alpha\varepsilon}^\delta+\rho_\alpha^{\tilde{\imath}}\frac{\partial L_{\beta\gamma}^\delta}{\partial\varkappa^{\tilde{\imath}}}\right)=0.$$

Using (2.2.2), we obtain the equality:

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(\left(L_{\beta\gamma}^\varepsilon\circ\pi\right)\left(L_{\alpha\varepsilon}^\delta\circ\pi\right)+\rho_\alpha^{\tilde{\imath}}\circ\pi\frac{\partial\left(L_{\beta\gamma}^\delta\circ\pi\right)}{\partial\varkappa^{\tilde{\imath}}}\right)=0.$$

and multiplying with $\tilde{T}_\delta$, we obtain

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(\left(L_{\beta\gamma}^{\varepsilon}\circ\pi\right)\left(L_{\alpha\varepsilon}^{\delta}\circ\pi\right)\tilde{T}_\delta+\rho_\alpha^{\tilde{\imath}}\circ\pi\frac{\partial\left(L_{\beta\gamma}^{\delta}\circ\pi\right)}{\partial\varkappa^{\tilde{\imath}}}\tilde{T}_\delta\right)=0.$$

which is equivalent with

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(\left(L_{\beta\gamma}^{\varepsilon}\circ\pi\right)\left[\tilde{T}_\alpha,\tilde{T}_\varepsilon\right]_{\pi^*F}+\rho_\alpha^{\tilde{\imath}}\circ\pi\frac{\partial L_{\beta\gamma}^{\delta}\circ\pi}{\partial\varkappa^{\tilde{\imath}}}\tilde{T}_\delta\right)=0.$$

Since this equality implies

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left[\tilde{T}_\alpha,\left(L_{\beta\gamma}^{\varepsilon}\circ\pi\right)\tilde{T}_\varepsilon\right]_{\pi^*F}=0,$$

it results that the following Jacobi identity is satisfied

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left[\tilde{T}_\alpha,\left[\tilde{T}_\beta,\tilde{T}_\gamma\right]_{\pi^*F}\right]_{\pi^*F}=0.$$

In general, for any $\tilde{U},\tilde{V},\tilde{Z}\in\Gamma\left(\pi^*F,\pi^*\nu,E\right)$, we obtain the Jacobi identity:

$$\left[\tilde{U},\left[\tilde{V},\tilde{Z}\right]_{\pi^*F}\right]_{\pi^*F}+\left[\tilde{Z},\left[\tilde{U},\tilde{V}\right]_{\pi^*F}\right]_{\pi^*F}+\left[\tilde{V},\left[\tilde{Z},\tilde{U}\right]_{\pi^*F}\right]_{\pi^*F}=0. \tag{2}$$

Using the affirmations (1) and (2) it results the conclusion of the lemma. *q.e.d.*

Lemma 2.2.3 *The* **Mod**-*morphism*

$$\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)$$

is a **Liealg**-*morphism of*

$$\left(\Gamma\left(\pi^*F,\pi^*\nu,E\right),+,\cdot,[,]_{\pi^*F}\right)$$

source and

$$(\Gamma(TE,\tau_E,E),+,\cdot,[,]_{TE})$$

target.

Proof. As the **Mod**-morphism $\Gamma(\rho Id_N)$ is a **Liealg**-morphism of

$$\left(\Gamma\left(F,\nu,N\right),+,\cdot,[,]_F\right)$$

source and

$$(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$$

target, then we obtain

$$L_{\alpha\beta}^{\gamma}\rho_\gamma^{\tilde{k}}=\rho_\alpha^{\tilde{\imath}}\frac{\partial\left(\rho_\beta^{\tilde{k}}\right)}{\partial\varkappa^{\tilde{\imath}}}-\rho_\beta^{\tilde{\jmath}}\frac{\partial\left(\rho_\alpha^{\tilde{k}}\right)}{\partial\varkappa^{\tilde{\jmath}}}$$

Using relations (2.2.2), we obtain:

$$\left(L_{\alpha\beta}^{\gamma}\circ\pi\right)\left(\rho_\gamma^{\tilde{k}}\circ\pi\right)=\rho_\alpha^{\tilde{\imath}}\circ\pi\frac{\partial\left(\rho_\beta^{\tilde{k}}\circ\pi\right)}{\partial\varkappa^{\tilde{\imath}}}-\rho_\beta^{\tilde{\jmath}}\circ\pi\frac{\partial\left(\rho_\alpha^{\tilde{k}}\circ\pi\right)}{\partial\varkappa^{\tilde{\jmath}}}$$

Multiplying with $\frac{\partial}{\partial \varkappa^{\tilde{k}}}$, we obtain the equality

$$\left(L_{\alpha\beta}^{\gamma}\circ\pi\right)\left(\rho_{\gamma}^{\tilde{k}}\circ\pi\right)\frac{\partial}{\partial\varkappa^{\tilde{k}}}=\rho_{\alpha}^{\tilde{\imath}}\circ\pi\frac{\partial\left(\rho_{\beta}^{\tilde{k}}\circ\pi\right)}{\partial\varkappa^{\tilde{\imath}}}\frac{\partial}{\partial\varkappa^{\tilde{k}}}-\rho_{\beta}^{\tilde{\jmath}}\circ\pi\frac{\partial\left(\rho_{\alpha}^{\tilde{k}}\circ\pi\right)}{\partial\varkappa^{\tilde{\jmath}}}\frac{\partial}{\partial\varkappa^{\tilde{k}}}$$

which is equivalent with the equality

$$\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\left[\tilde{T}_\alpha,\tilde{T}_\beta\right]_{\pi^*F}=\left[\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\tilde{T}_\alpha,\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\tilde{T}_\beta\right]_{TN}$$

for any base sections $\tilde{T}_\alpha,\tilde{T}_\beta$.

In general, we obtain the equality

$$\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\left[\tilde{U},\tilde{V}\right]_{\pi^*F}=\left[\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\tilde{U},\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\tilde{V}\right]_{TN},$$

for any $\tilde{U},\tilde{V}\in\Gamma\left(h^*F,h^*\nu,M\right).$ *q.e.d.*

Using *Lemmas 2.2.1, 2.2.2* and *2.2.3*, we obtain the following

Theorem 2.2.1 *The couple*

$$\left(\left[,\right]_{\pi^*F},\left(\overset{\pi^*F}{\rho},Id_E\right)\right)$$

is a Lie algebroid structure for the vector bundle $\left(\pi^*F,\pi^*\nu,E\right).$

This Lie algebroid will be called *the pull-back Lie algebroid of the Lie algebroid*

$$\left(\left(F,\nu,N\right),\left[,\right]_F,\left(\rho,Id_N\right)\right).$$

3 Generalized Lie algebroids, exterior differential calculus and (linear) connections

3.1 The category of generalized Lie algebroids

We assume that $N\in|\mathbf{Man}|$ and let $\left[,\right]_{TN}$ be the usual Lie bracket such that

$$\left(\Gamma\left(TN,\tau_N,N\right),+,\cdot,\left[,\right]_{TN}\right)\in|\mathbf{LieAlg}|.$$

Let $h\in\mathbf{Man}\left(M,N\right)$ be a surjective application.

Definition 3.1.1 If $\left(F,\nu,N\right)\in|\mathbf{B^v}|$ such that there exists

$$\left(\rho,\eta\right)\in\mathbf{B^v}\left(\left(F,\nu,N\right),\left(TM,\tau_M,M\right)\right)$$

and an operation

$$\begin{array}{ccc}\Gamma\left(F,\nu,N\right)\times\Gamma\left(F,\nu,N\right) & \overset{\left[,\right]_{F,h}}{\longrightarrow} & \Gamma\left(F,\nu,N\right)\\ \left(u,v\right) & \longmapsto & \left[u,v\right]_{F,h}\end{array}$$

with the following properties:

GLA_1. the equality holds good

$$[u, f\cdot v]_{F,h} = f\,[u,v]_{F,h} + \Gamma\left(Th\circ\rho, h\circ\eta\right)(u)\,f\cdot v,$$

for all $u, v \in \Gamma\left(F,\nu,N\right)$ and $f \in \mathcal{F}\left(N\right)$.

GLA_2. the 4-tuple

$$\left(\Gamma\left(F,\nu,N\right), +, \cdot, [,]_{F,h}\right)$$

is a Lie $\mathcal{F}\left(N\right)$-algebra,

GLA_3. the **Mod**-morphism $\Gamma\left(Th\circ\rho, h\circ\eta\right)$ is a **LieAlg**-morphism of

$$\left(\Gamma\left(F,\nu,N\right), +, \cdot, [,]_{F,h}\right)$$

source and

$$\left(\Gamma\left(TN,\tau_N,N\right), +, \cdot, [,]_{TN}\right)$$

target,

then we will say that *the triple*

$$\left(\left(F,\nu,N\right), [,]_{F,h}, \left(\rho,\eta\right)\right) \tag{3.1.1}$$

is a generalized Lie algebroid.

The couple

$$\left([,]_{F,h}, \left(\rho,\eta\right)\right)$$

will be called *generalized Lie algebroid structure.*

Definition 3.1.2 We define the morphisms set of

$$\left(\left(F,\nu,N\right), [,]_{F,h}, \left(\rho,\eta\right)\right)$$

source and

$$\left(\left(F',\nu',N'\right), [,]_{F',h'}, \left(\rho',\eta'\right)\right)$$

target as being the set

$$\left\{\left(\varphi,\varphi_0\right) \in \mathbf{B^v}\left(\left(F,\nu,N\right), \left(F',\nu',N'\right)\right)\right\}$$

such that the **Mod**-morphism $\Gamma\left(\varphi,\varphi_0\right)$ is a **LieAlg**-morphism of

$$\left(\Gamma\left(F,\nu,N\right), +, \cdot, [,]_{F,h}\right)$$

source and

$$\left(\Gamma\left(F',\nu',N'\right), +, \cdot, [,]_{F',h'}\right)$$

target.

Remark 3.1.1 Note that we discuss about *the category of generalized Lie algebroids.* This category will be denoted by **GLA**.

In the following we will build some examples of generalized Lie algebroids.

We assume that $((F,\nu,N), [,]_F, (\rho, Id_N))$ is a Lie algebroid and let $h \in \mathbf{Man}(N,N)$ be a surjective application.

Let $\mathcal{AF}_F$ be a representative of vector fibred $(n+p)$-structure for the vector bundle (F,ν,N) and let $\mathcal{AF}_{TN}$ be a representative of vector fibred $(n+n)$-structure for the vector bundle (TN,τ_N,N).

If $(U,\xi_U) \in \mathcal{AF}_{TN}$ and $(V,s_V) \in \mathcal{AF}_F$ such that $U \cap h^{-1}(V) \neq \phi$, then we define the application

$$\begin{array}{ccc} \tau_N^{-1}(U\cap h^{-1}(V))) & \xrightarrow{\bar{\xi}_{U\cap h^{-1}(V)}} & \left(U\cap h^{-1}(V)\right) \times \mathbb{R}^n \\ (\varkappa, u(\varkappa)) & \longmapsto & \left(\varkappa, \xi_{U,\varkappa}^{-1} u(\varkappa)\right). \end{array}$$

Proposition 3.1.1 *The set*

$$\overline{\mathcal{AF}}_{TN} \overset{put}{=} \bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_{TN},\ (V,s_V)\in\mathcal{AF}_F \\ U\cap h^{-1}(V)\neq\phi}} \left\{\left(U\cap h^{-1}(V), \bar{\xi}_{U\cap h^{-1}(V)}\right)\right\}$$

is a vector fibred $n+n$-atlas for the vector bundle (TN,τ_N,N).

If

$$X = X^{\tilde{\imath}} \frac{\partial}{\partial \varkappa^{\tilde{\imath}}} \in \Gamma(TN,\tau_N,N)$$

then, using the vector fibred $n+n$-structure $\left[\overline{\mathcal{AF}}_{TN}\right]$, we obtain the section

$$\bar{X} = \bar{X}^{\tilde{\imath}} \circ h \frac{\partial}{\partial \bar{\varkappa}^{\tilde{\imath}}} \in \Gamma(TN,\tau_N,N),$$

such that

$$\bar{X}(\bar{\varkappa}) = X(h(\bar{\varkappa})),$$

for any $\bar{\varkappa} \in U \cap h^{-1}(V)$.

The set $\left\{\frac{\partial}{\partial \bar{\varkappa}^{\tilde{\imath}}},\ \tilde{\imath} \in \overline{1,n}\right\}$ is a base for the $\mathcal{F}(N)$-module $(\Gamma(TN,\tau_N,N), +, \cdot)$.

We consider the operation

$$\Gamma(F,\nu,N) \times \Gamma(F,\nu,N) \xrightarrow{[,]_{F,h}} \Gamma(F,\nu,N)$$

defined by

$$\begin{aligned} [t_\alpha, t_\beta]_{F,h} &= \left(L_{\alpha\beta}^\gamma \circ h\right) t_\gamma, \\ [t_\alpha, f t_\beta]_{F,h} &= f\left(L_{\alpha\beta}^\gamma \circ h\right) t_\gamma + \rho_\alpha^{\tilde{\imath}} \circ h \frac{\partial f}{\partial \bar{\varkappa}^{\tilde{\imath}}} t_\beta, \\ [f t_\alpha, t_\beta]_{F,h} &= -[t_\beta, f t_\alpha]_{F,h}, \end{aligned}$$

for any $f \in \mathcal{F}(N)$.

Lemma 3.1.1 *The following equality holds good*

$$[z, fv]_{F,h} = f[z,v]_{F,h} + \Gamma(Th\circ\rho, h)(z) f \cdot v,$$

for any $z, v \in \Gamma(F,\nu,N)$ and for any $f \in \mathcal{F}(N)$.

Proof. We obtain easily that

$$[t_\alpha, f t_\beta]_{F,h} = [t_\alpha, t_\beta]_{F,h} + \Gamma\left(Th \circ \rho, h\right)(t_\alpha) f \cdot t_\beta$$

for any $\forall f \in \mathcal{F}(N)$.

Using this equality and the definition of the operation $[,]_{F,h}$ it results the conclusion of the lemma. *q.e.d.*

Lemma 3.1.2 *The $\mathcal{F}(N)$-algebra*

$$\left(\Gamma\left(F,\nu,N\right), +, \cdot, [,]_{F,h}\right)$$

is a Lie $\mathcal{F}(N)$-algebra.

Proof. Using the definition of the operation $[,]_{F,h}$ it results that

$$[u,v]_{F,h} = -[v,u]_{F,h},$$

for any $u, v \in \Gamma(F,\nu,N)$. Therefore, we obtain

$$(1) \qquad [u,u]_{F,h} = 0,\ \forall u \in \Gamma(F,\nu,N).$$

Since $(\Gamma(F,\nu,N), +, \cdot, [,]_F)$ is a Lie $\mathcal{F}(N)$-algebra, it results that

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(L_{\beta\gamma}^{\varepsilon} L_{\alpha\varepsilon}^{\delta} + \rho_\alpha^{\tilde{\imath}} \frac{\partial L_{\beta\gamma}^{\delta}}{\partial \varkappa^{\tilde{\imath}}}\right) = 0.$$

Therefore,

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(\left(L_{\beta\gamma}^{\varepsilon} \circ h\right)\left(L_{\alpha\varepsilon}^{\delta} \circ h\right) + \rho_\alpha^{\tilde{\imath}} \circ h \frac{\partial\left(L_{\beta\gamma}^{\delta} \circ h\right)}{\partial \bar{\varkappa}^{\tilde{\imath}}}\right) = 0.$$

Multiplying with t_δ, we obtain the equality

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(\left(L_{\beta\gamma}^{\varepsilon} \circ h\right)\left(L_{\alpha\varepsilon}^{\delta} \circ h\right) t_\delta + \rho_\alpha^{\tilde{\imath}} \circ h \frac{\partial\left(L_{\beta\gamma}^{\delta} \circ h\right)}{\partial \bar{\varkappa}^{\tilde{\imath}}} t_\delta\right) = 0$$

which is equivalent with the following equality:

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(\left(L_{\beta\gamma}^{\varepsilon} \circ h\right)[t_\alpha, t_\varepsilon]_{F,h} + \rho_\alpha^{\tilde{\imath}} \circ h \frac{\partial\left(L_{\beta\gamma}^{\delta} \circ h\right)}{\partial \bar{\varkappa}^{\tilde{\imath}}} t_\delta\right) = 0.$$

Therefore, we obtain the Jacobi identity

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left[t_\alpha, [t_\beta, t_\gamma]_{F,h}\right]_{F,h} = 0.$$

For any $u, v, w \in \Gamma(F,\nu,N)$, we obtain the Jacobi identity

$$(2) \qquad \left[u, [v,w]_{F,h}\right]_{F,h} + \left[v, [w,u]_{F,h}\right]_{F,h} + \left[w, [u,v]_{F,h}\right]_{F,h} = 0,$$

Using (1) and (2) it results the conclusion of lemma. *q.e.d.*

Lemma 3.1.3 *The* **Mod**-*morphism* $\Gamma(Th\circ\rho,h)$ *is a* **Liealg**-*morphism of*

$$\left(\Gamma(F,\nu,N),+,\cdot,[,]_{F,h}\right)$$

source and

$$(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$$

target.

Proof. As the **Mod**-morphism $\Gamma(\rho,Id_N)$ is a **LieAlg**-morphisms of

$$(\Gamma(F,\nu,N),+,\cdot,[,]_F)$$

source and

$$(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$$

target, then we obtain

$$L_{\alpha\beta}^{\gamma}\rho_{\gamma}^{\tilde{k}}=\rho_{\alpha}^{\tilde{\imath}}\frac{\partial\rho_{\beta}^{\tilde{k}}}{\partial\varkappa^{\tilde{\imath}}}-\rho_{\beta}^{\tilde{\jmath}}\frac{\partial\rho_{\alpha}^{\tilde{k}}}{\partial\varkappa^{\tilde{\jmath}}}.$$

Therefore, we obtain

$$\left(L_{\alpha\beta}^{\gamma}\circ h\right)\left(\rho_{\gamma}^{\tilde{k}}\circ h\right)=\rho_{\alpha}^{\tilde{\imath}}\circ h\frac{\partial\rho_{\beta}^{\tilde{k}}\circ h}{\partial\bar{\varkappa}^{\tilde{\imath}}}-\rho_{\beta}^{\tilde{\jmath}}\circ h\frac{\partial\rho_{\alpha}^{\tilde{k}}\circ h}{\partial\bar{\varkappa}^{\tilde{\jmath}}}.$$

Moreover, we obtain

$$\left(L_{\alpha\beta}^{\gamma}\circ h\right)\left(\rho_{\gamma}^{\tilde{k}}\circ h\right)\frac{\partial}{\partial\bar{\varkappa}^{\tilde{k}}}=\rho_{\alpha}^{\tilde{\imath}}\circ h\frac{\partial\rho_{\beta}^{\tilde{k}}\circ h}{\partial\bar{\varkappa}^{\tilde{\imath}}}\frac{\partial}{\partial\bar{\varkappa}^{\tilde{k}}}-\rho_{\beta}^{\tilde{\jmath}}\circ h\frac{\partial\rho_{\alpha}^{\tilde{k}}\circ h}{\partial\bar{\varkappa}^{\tilde{\jmath}}}\frac{\partial}{\partial\bar{\varkappa}^{\tilde{k}}}.$$

After some calculations, we obtain that

$$\Gamma(Th\circ\rho,h)[t_\alpha,t_\beta]_{F,h}=[\Gamma(Th\circ\rho,h)t_\alpha,\Gamma(Th\circ\rho,h)t_\beta]_{TN}.$$

We obtain easily that

$$\Gamma(Th\circ\rho,h)[u,v]_{F,h}=[\Gamma(Th\circ\rho,h)u,\Gamma(Th\circ\rho,h)v]_{TN},$$

for any $u,v\in\Gamma(F,\nu,N)$. *q.e.d.*

Using *Lemmas 3.1.1, 3.1.2* and *3.1.3* we obtain the following

Theorem 3.1.1 (example of generalized Lie algebroid) *The couple*

$$\left([,]_{F,h},(\rho,Id_N)\right)$$

is a generalized Lie algebroid structure for the vector bundle (F,ν,N).

Definition 3.1.3 The generalized Lie algebroid

$$\left((F,\nu,N),[,]_{F,h},(\rho,Id_N)\right)$$

given by the previous theorem, will be called *the generalized Lie algebroid associated to the Lie algebroid*

$$((F,\nu,N),[,]_F,(\rho,Id_N))$$

and to the surjective application

$$h \in \mathbf{Man}(N, N).$$

In particular, if $h = Id_N$, then the generalized Lie algebroid

$$\left((F, \nu, N), [,]_{F, Id_N}, (\rho, Id_N)\right)$$

will be called *the generalized Lie algebroid associated to the Lie algebroid*

$$((F, \nu, N), [,]_F, (\rho, Id_N)).$$

Note that any Lie algebroid can be regarded as a generalized Lie algebroid.

Theorem 3.1.2 (example of generalized Lie algebroid) *Let* $M \in |\mathbf{Man}_m|$ *and* $g, h \in Iso_{\mathbf{Man}}(M)$.

Let $[,]_{TM}$ *be the usual Lie bracket such that*

$$(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM}) \in |\mathbf{LieAlg}|.$$

Using the tangent $\mathbf{B^v}$*-morphism* (Tg, g) *and the operation*

$$\begin{array}{ccc} \Gamma(TM, \tau_M, M) \times \Gamma(TM, \tau_M, M) & \xrightarrow{[,]_{TM,h}} & \Gamma(TM, \tau_M, M) \\ (u, v) & \longmapsto & [u, v]_{TM,h} \end{array}$$

where

$$[u, v]_{TM,h} = \Gamma\left(T(h \circ g)^{-1}, (h \circ g)^{-1}\right)([\Gamma(T(h \circ g), h \circ g) u, \Gamma(T(h \circ g), h \circ g) v]_{TM}),$$

for any $u, v \in \Gamma(TM, \tau_M, M)$*, then we obtain that*

$$\left((TM, \tau_M, M), [u, v]_{TM,h}, (Tg, g)\right)$$

is a generalized Lie algebroid.

Proof: As the operation $[,]_{TM,h}$ is biadditive, then we obtain that

$$\left(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM,h}\right) \in |\mathbf{Alg}|.$$

Using the definition of the operation $[,]_{TM,h}$ we obtain that

$$\Gamma(T(h \circ g), h \circ g)\left([u, v]_{TM,h}\right) = [\Gamma(T(h \circ g), h \circ g) u, \Gamma(T(h \circ g), h \circ g) v]_{TM}$$

for any $u, v \in \Gamma(TM, \tau_M, M)$.

1) Therefore, $\Gamma(T(h \circ g), h \circ g)$ is a **Alg**-morphism of

$$\left(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM,h}\right)$$

source and

$$(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM})$$

target.

For any $u, v \in \Gamma(TM, \tau_M, M)$ and $f \in \mathcal{F}(M)$ we obtain that

$$\begin{aligned}
[u, fv]_{TM,h} &= \Gamma\left(T(h\circ g)^{-1}, (h\circ g)^{-1}\right)\left([\Gamma(T(h\circ g), h\circ g)u, \Gamma(T(h\circ g), h\circ g) fv]_{TM}\right)\\
&= \Gamma\left(T(h\circ g)^{-1}, (h\circ g)^{-1}\right)\left(f\cdot[\Gamma(T(h\circ g), h\circ g)u, \Gamma(T(h\circ g), h\circ g) v]_{TM}\right)\\
&+\Gamma\left(T(h\circ g)^{-1}, (h\circ g)^{-1}\right)\left(\Gamma(T(h\circ g), h\circ g)u\right)(f)\cdot\Gamma(T(h\circ g), h\circ g)v\\
&= f\cdot\Gamma\left(T(h\circ g)^{-1}, (h\circ g)^{-1}\right)[\Gamma(T(h\circ g), h\circ g)u, \Gamma(T(h\circ g), h\circ g)v]_{TM}\\
&+(\Gamma(T(h\circ g), h\circ g)u)(f)\cdot v
\end{aligned}$$

2) Therefore, we obtain that

$$[u, fv]_{TM,h} = f\cdot[u,v]_{TM,h} + (\Gamma(T(h\circ g), h\circ g)u)(f)\cdot v$$

for any $u, v \in \Gamma(TM, \tau_M, M)$ and $f \in \mathcal{F}(M)$.

We remark that

$$\Gamma\left(T(h\circ g)^{-1}, (h\circ g)^{-1}\right)(0) = 0.$$

As

$$(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM}) \in |\mathbf{LieAlg}|$$

and

$$\begin{aligned}
\left[u,[v,z]_{TM,h}\right]_{TM,h} &= \Gamma\left(T(h\circ g)^{-1},(h\circ g)^{-1}\right)\left[\Gamma(T(h\circ g),h\circ g)u,\Gamma(T(h\circ g),h\circ g)[v,z]_{TM,h}\right]_{TM}\\
&= \Gamma\left(T(h\circ g)^{-1},(h\circ g)^{-1}\right)\left[\Gamma(T(h\circ g),h\circ g)u,[\Gamma(T(h\circ g),h\circ g)v,\Gamma(T(h\circ g),h\circ g)z]_{TM}\right]_{TM}
\end{aligned}$$

for any $u, v, z \in \Gamma(TM, \tau_M, M)$, it results that

$$[u,u]_{TM,h} = 0$$

for any $u \in \Gamma(TM, \tau_M, M)$ and

$$\left[u,[v,z]_{TM,h}\right]_{TM,h} + \left[z,[u,v]_{TM,h}\right]_{TM,h} + \left[v,[z,u]_{TM,h}\right]_{TM,h} = 0,$$

for any $u, v, z \in \Gamma(TM, \tau_M, M)$.

3) Therefore, we have that

$$\left(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM,h}\right) \in |\mathbf{LieAlg}|.$$

Using the affirmations 1), 2) and 3) it results the conclusion of the theorem.

Remark 3.1.2 For any **Man**-isomorphisms g and h we obtain new and interesting generalized Lie algebroid structures for the tangent vector bundle (TM, τ_M, M).

For any base $\{t_\alpha,\ \alpha \in \overline{1,m}\}$ of the module of sections $(\Gamma(TM, \tau_M, M), +, \cdot)$ we obtain the structure functions

$$L_{\alpha\beta}^{\gamma} = \left(\theta_\alpha^i \frac{\partial\theta_\beta^j}{\partial x^i} - \theta_\beta^i \frac{\partial\theta_\alpha^j}{\partial x^i}\right)\tilde{\theta}_j^\gamma,\ \alpha,\beta,\gamma \in \overline{1,m}$$

where

$$\theta_\alpha^i,\ i,\alpha \in \overline{1,m}$$

are real local functions such that

$$\Gamma\left(T\left(h\circ g\right),h\circ g\right)\left(t_{\alpha}\right)=\theta_{\alpha}^{i}\frac{\partial}{\partial x^{i}}$$

and

$$\tilde{\theta}_{j}^{\gamma},\ i,\gamma\in\overline{1,m}$$

are real local functions such that

$$\Gamma\left(T\left(h\circ g\right)^{-1},\left(h\circ g\right)^{-1}\right)\left(\frac{\partial}{\partial x^{j}}\right)=\tilde{\theta}_{j}^{\gamma}t_{\gamma}.$$

In particular, using arbitrary basis for the module of sections and arbitrary isometries (symmetries, translations, rotations,...) we obtain a lot of generalized Lie algebroid structures for the tangent vector bundle $(T\Sigma,\tau_{\Sigma},\Sigma)$ and we can study its geometry using our theory which is develop in the next.

3.1.1 Structure functions for generalized Lie algebroids

Let

$$\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$$

be a generalized Lie algebroid given by the diagram:

$$\begin{array}{ccc} & & \left(F,\left[,\right]_{F,h},\left(\rho,\eta\right)\right)\\ & & \downarrow\nu\\ M & \xrightarrow{\quad h\quad} & N\end{array}\tag{3.1.1.1}$$

We assume that (F,ν,N) is a vector bundle with type fibre the real vector space $(\mathbb{R}^{p},+,\cdot)$ and structure group a Lie subgroup of $(\mathbf{GL}\left(p,\mathbb{R}\right),\cdot)$.

We take $\left(x^{i},y^{i}\right)$ as canonical local coordinates on (TM,τ_{M},M), where $i\in\overline{1,m}$.

Consider

$$\left(x^{i},y^{i}\right)\longrightarrow\left(x^{i\prime}\left(x^{i}\right),y^{i\prime}\left(x^{i},y^{i}\right)\right)$$

a change of coordinates on (TM,τ_{M},M). Then the coordinates y^{i} change to $y^{i\prime}$ by the rule:

$$y^{i\prime}=\frac{\partial x^{i\prime}}{\partial x^{i}}y^{i}.\tag{3.1.1.2}$$

We take $(\varkappa^{\tilde{i}},z^{\alpha})$ as canonical local coordinates on (F,ν,N), where $\tilde{\imath}\in\overline{1,n}$, $\alpha\in\overline{1,p}$.

Consider

$$\left(\varkappa^{\tilde{i}},z^{\alpha}\right)\longrightarrow\left(\varkappa^{\tilde{i}},z^{\alpha\prime}\right)$$

a change of coordinates on (F,ν,N). Then the coordinates z^{α} change to $z^{\alpha\prime}$ by the rule:

$$z^{\alpha\prime}=\Lambda_{\alpha}^{\alpha\prime}z^{\alpha}.\tag{3.1.3}$$

We assume that $(\theta,\mu)\overset{put}{=}(Th\circ\rho,h\circ\eta)$.

If $z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right)$ is arbitrary, then

$$\begin{array}{c}\Gamma\left(Th\circ\rho,h\circ\eta\right)\left(z^{\alpha}t_{\alpha}\right)f\left(h\circ\eta\left(\varkappa\right)\right)=\\ =\left(\theta_{\alpha}^{\tilde{i}}z^{\alpha}\dfrac{\partial f}{\partial\varkappa^{\tilde{i}}}\right)\left(h\circ\eta\left(\varkappa\right)\right)=\left(\left(\rho_{\alpha}^{i}\circ h\right)\left(z^{\alpha}\circ h\right)\dfrac{\partial f\circ h}{\partial x^{i}}\right)\left(\eta\left(\varkappa\right)\right),\end{array}\tag{3.1.1.4}$$

for any $f \in \mathcal{F}(N)$ and $\varkappa \in N$.

The coefficients ρ_{α}^{i} respectively $\theta_{\alpha}^{\tilde{i}}$ change to $\rho_{\alpha'}^{i'}$ respectively $\theta_{\alpha'}^{\tilde{i}}$ by the rule:

$$\rho_{\alpha'}^{i'} = \Lambda_{\alpha'}^{\alpha}\rho_{\alpha}^{i}\frac{\partial x^{i'}}{\partial x^{i}}, \tag{3.1.1.5}$$

respectively

$$\theta_{\alpha'}^{\tilde{i}'} = \Lambda_{\alpha'}^{\alpha}\theta_{\alpha}^{\tilde{i}}\frac{\partial \varkappa^{\tilde{i}'}}{\partial \varkappa^{\tilde{i}}}, \tag{3.1.1.6}$$

where

$$\left\|\Lambda_{\alpha'}^{\alpha}\right\| = \left\|\Lambda_{\alpha}^{\alpha'}\right\|^{-1}.$$

Locally, we set

$$[t_{\alpha}, t_{\beta}]_{F} \overset{put}{=} L_{\alpha\beta}^{\gamma}t_{\gamma}. \tag{3.1.1.7}$$

We easily obtain that

$$L_{\alpha\beta}^{\gamma} = -L_{\beta\alpha}^{\gamma},\ \forall \alpha, \beta, \gamma \in \overline{1, p}.$$

The real local functions

$$\left\{L_{\alpha\beta}^{\gamma},\ \alpha, \beta, \gamma \in \overline{1, p}\right\}$$

will be called the *structure functions of the generalized Lie algebroid*

$$\left(\left(F, \nu, N\right), [,]_{F,h}, (\rho, \eta)\right).$$

Theorem 3.1.1.1 *The following equalities hold good:*

$$\rho_{\alpha}^{i} \circ h\frac{\partial f \circ h}{\partial x^{i}} = \left(\theta_{\alpha}^{\tilde{i}}\frac{\partial f}{\partial \varkappa^{\tilde{i}}}\right) \circ h, \forall f \in \mathcal{F}(N). \tag{3.1.1.8}$$

and

$$\left(L_{\alpha\beta}^{\gamma} \circ h\right)\left(\rho_{\gamma}^{k} \circ h\right) = \left(\rho_{\alpha}^{i} \circ h\right)\frac{\partial\left(\rho_{\beta}^{k} \circ h\right)}{\partial x^{i}} - \left(\rho_{\beta}^{j} \circ h\right)\frac{\partial\left(\rho_{\alpha}^{k} \circ h\right)}{\partial x^{j}}. \tag{3.1.1.9}$$

Proof. Using the relation (3.1.1.4), we obtain the equality (3.1.1.8). Since

$$\begin{aligned}
&\Gamma(Th \circ \rho, h \circ \eta)[t_{\alpha}, t_{\beta}]_{F}(f) \\
&= [\Gamma((Th, h) \circ (\rho, \eta))t_{\alpha}, \Gamma((Th, h) \circ (\rho, \eta))t_{\beta}]_{F}(f) \\
&= \Gamma(Th, h)\left([\Gamma(\rho, \eta)t_{\alpha}, \Gamma(\rho, \eta)t_{\beta}]_{TM}\right)(f),\ \forall f \in \mathcal{F}(N),
\end{aligned}$$

it results that

$$\left(L_{\alpha\beta}^{\gamma} \circ h\right)\left(\rho_{\gamma}^{k} \circ h\right)\tfrac{\partial f \circ h}{\partial x^{k}} = \left(\left(\rho_{\alpha}^{i} \circ h\right)\tfrac{\partial\left(\rho_{\beta}^{k} \circ h\right)}{\partial x^{i}} - \left(\rho_{\beta}^{j} \circ h\right)\tfrac{\partial\left(\rho_{\alpha}^{k} \circ h\right)}{\partial x^{j}}\right)\tfrac{\partial f \circ h}{\partial x^{k}},$$

for any $f \in \mathcal{F}(N)$. *q.e.d.*

3.1.2 The pull-back Lie algebroid of a generalized Lie algebroid

Let

$$\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$$

be a generalized Lie algebroid given by the diagram (3.1.1.1).

Let $\mathcal{AF}_F$ be a representative of vector fibred $(n+p)$-structure for the vector bundle (F,ν,N) and let $\mathcal{AF}_{TM}$ be a representative of vector fibred $(m+m)$-structure for the vector bundle (TM,τ_M,M).

Let $(h^*F,h^*\nu,M)$ be the pull-back vector bundle through h.

If $(U,\xi_U)\in\mathcal{AF}_{TM}$ and $(V,s_V)\in\mathcal{AF}_F$ such that $U\cap h^{-1}\left(V\right)\neq\phi$, then we define the application

$$\begin{array}{ccc} h^*\nu^{-1}(U\cap h^{-1}(V))) & \xrightarrow{\bar{s}_{U\cap h^{-1}(V)}} & \left(U\cap h^{-1}(V)\right)\times\mathbb{R}^p \\ \left(\varkappa,z\left(h\left(\varkappa\right)\right)\right) & \longmapsto & \left(\varkappa,t_{V,h(\varkappa)}^{-1}z\left(h\left(\varkappa\right)\right)\right). \end{array}$$

Proposition 3.1.2.1 *The set*

$$\overline{\mathcal{AF}}_F\overset{put}{=}\bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_{TM},\ (V,s_V)\in\mathcal{AF}_F\\ U\cap h^{-1}(V)\neq\phi}}\left\{\left(U\cap h^{-1}\left(V\right),\bar{s}_{U\cap h^{-1}(V)}\right)\right\}$$

is a vector fibred $m+p$-atlas for the vector bundle $\left(h^*F,h^*\nu,M\right)$.

If

$$z=z^\alpha t_\alpha\in\Gamma\left(F,\nu,N\right),$$

then, using the vector fibred $m+p$-structure $\left[\overline{\mathcal{AF}}_F\right]$, we obtain the section

$$Z=\left(z^\alpha\circ h\right)T_\alpha\in\Gamma\left(h^*F,h^*\nu,M\right)$$

such that

$$Z\left(x\right)=z\left(h\left(x\right)\right),$$

for any $x\in U\cap h^{-1}\left(V\right)$.

The set $\left\{T_\alpha,\ \alpha\in\overline{1,p}\right\}$ is a base for the module of sections

$$\left(\Gamma\left(h^*F,h^*\nu,M\right),+,\cdot\right).$$

Let $\left(\overset{h^*F}{\rho},Id_M\right)$ be the $\mathbf{B^v}$-morphism of

$$(h^*F,h^*\nu,M)$$

source and

$$(TM,\tau_M,M)$$

target, where

$$\begin{array}{ccc} h^*F & \overset{\overset{h^*F}{\rho}}{\longrightarrow} & TM \\ Z^\alpha T_\alpha\left(x\right) & \longmapsto & \left(Z^\alpha\cdot\rho_\alpha^i\circ h\right)\dfrac{\partial}{\partial x^i}\left(x\right) \end{array}\qquad(3.1.2.1)$$

We consider the operation

$$\Gamma\left(h^*F, h^*\nu, M\right) \times \Gamma\left(h^*F, h^*\nu, M\right) \quad \xrightarrow{[,]_{h^*F}} \quad \Gamma\left(h^*F, h^*\nu, M\right)$$

defined by

$$\begin{aligned} [T_\alpha, T_\beta]_{h^*F} &= \left(L_{\alpha\beta}^\gamma \circ h\right) T_\gamma, \\ [T_\alpha, fT_\beta]_{h^*F} &= f\left(L_{\alpha\beta}^\gamma \circ h\right) T_\gamma + \left(\rho_\alpha^i \circ h\right) \frac{\partial f}{\partial x^i} T_\beta, \\ [fT_\alpha, T_\beta]_{h^*F} &= -\left[T_\beta, fT_\alpha\right]_{h^*F}, \end{aligned} \tag{3.1.2.2}$$

for any $f \in \mathcal{F}(M)$.

Lemma 3.1.2.1 *The following equality holds good*

$$[U, fV]_{h^*F} = f\,[U, V]_{h^*F} + \Gamma\left(\overset{h^*F}{\rho}, Id_M\right)(U)\, f \cdot V,$$

for any $U, V \in \Gamma\left(h^*F, h^*\nu, M\right)$ *and for any* $f \in \mathcal{F}(M)$.

Proof. We observe that for any $\alpha, \beta \in \overline{1, p}$, we obtain

$$[T_\alpha, fT_\beta]_{h^*F} = f\,[T_\alpha, T_\beta]_{h^*F} + \Gamma\left(\overset{h^*F}{\rho}, Id_M\right)(T_\alpha)\, f \cdot T_\beta,\ \forall f \in \mathcal{F}(N).$$

Using this equality and the definition of the operation $[,]_{h^*F}$ it results the conclusion of the lemma. *q.e.d.*

Lemma 3.1.2.1 *The* $\mathcal{F}(M)$*-algebra*

$$\left(\Gamma\left(h^*F, h^*\nu, M\right), +, \cdot, [,]_{h^*F}\right)$$

is a Lie $\mathcal{F}(M)$*-algebra.*

Proof. Using the definition of the operation $[,]_{h^*F}$ it results that

$$[U, V]_{h^*F} = -\left[V, U\right]_{h^*F},$$

for any $U, V \in \Gamma\left(h^*F, h^*\nu, M\right)$. Therefore, we obtain

$$[U, U]_{h^*F} = 0,\ \forall U \in \Gamma\left(h^*F, h^*\nu, M\right). \tag{1}$$

Since $\left(\Gamma\left(F, \nu, M\right), +, \cdot, [,]_F\right)$ is a Lie $\mathcal{F}(M)$-algebra, we obtain the equality:

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(L_{\beta\gamma}^\varepsilon L_{\alpha\varepsilon}^\delta + \theta_\alpha^{\tilde{i}} \frac{\partial L_{\beta\gamma}^\delta}{\partial \varkappa^{\tilde{i}}}\right) = 0.$$

Using (3.1.1.9), we obtain the equality:

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(\left(L_{\beta\gamma}^\varepsilon \circ h\right)\left(L_{\alpha\varepsilon}^\delta \circ h\right) + \rho_\alpha^i \circ h \frac{\partial\left(L_{\beta\gamma}^\delta \circ h\right)}{\partial x^i}\right) = 0.$$

and multiplying with T_δ, we obtain

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(\left(L_{\beta\gamma}^{\varepsilon}\circ h\right)\left(L_{\alpha\varepsilon}^{\delta}\circ h\right)T_\delta\ +\rho_\alpha^i\circ h\frac{\partial\left(L_{\beta\gamma}^{\delta}\circ h\right)}{\partial x^i}T_\delta\right)\ =0.$$

which is equivalent with

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(\left(L_{\beta\gamma}^{\varepsilon}\circ h\right)[T_\alpha,T_\varepsilon]_{h^*F}+\rho_\alpha^i\circ h\frac{\partial L_{\beta\gamma}^{\delta}\circ h}{\partial x^i}T_\delta\right)=0. \tag{2}$$

Since this equality implies

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left[T_\alpha,\left(L_{\beta\gamma}^{\varepsilon}\circ h\right)T_\varepsilon\right]_{h^*F}=0,$$

it results that it is satisfied the Jacobi identity

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left[T_\alpha,[T_\beta,T_\gamma]_{h^*F}\right]_{h^*F}=0.$$

In general, for any $U,V,Z\in\Gamma\left(h^*F,h^*\nu,M\right)$, we obtain the Jacobi identity:

$$[U,[V,Z]_{h^*F}]_{h^*F}+[Z,[U,V]_{h^*F}]_{h^*F}+[V,[Z,U]_{h^*F}]_{h^*F}=0. \tag{2}$$

Using affirmations (1) and (2), we get the conclusion of the lemma. *q.e.d.*

Lemma 3.1.2.3 *The* **Mod**-*morphism*

$$\Gamma\left(\overset{h^*F}{\rho},Id_M\right)$$

is a **Liealg**-*morphism of*

$$\left(\Gamma\left(h^*F,h^*\nu,M\right),+,\cdot,[,]_{h^*F}\right)$$

source and

$$(\Gamma(TM,\tau_M,M),+,\cdot,[,]_{TM})$$

target.

Proof. Using relations (3.1.1.9), we obtain:

$$\left(L_{\alpha\beta}^{\gamma}\circ h\right)\left(\rho_\gamma^k\circ h\right)\frac{\partial}{\partial x^k}=\left(\rho_\alpha^i\circ h\right)\frac{\partial\left(\rho_\beta^k\circ h\right)}{\partial x^i}\frac{\partial}{\partial x^k}-\left(\rho_\beta^j\circ h\right)\frac{\partial\left(\rho_\alpha^k\circ h\right)}{\partial x^j}\frac{\partial}{\partial x^k},$$

Therefore,

$$\Gamma\left(\overset{h^*F}{\rho},Id_M\right)[T_\alpha,T_\beta]_{h^*F}=\left[\Gamma\left(\overset{h^*F}{\rho},Id_M\right)T_\alpha,\Gamma\left(\overset{h^*F}{\rho},Id_M\right)T_\beta\right]_{TM},$$

for any base sections T_α,T_β.

In general, we obtain the equality

$$\Gamma\left(\overset{h^*F}{\rho},Id_M\right)[U,V]_{h^*F}=\left[\Gamma\left(\overset{h^*F}{\rho},Id_M\right)U,\Gamma\left(\overset{h^*F}{\rho},Id_M\right)U\right]_{TM},$$

for any $U, V \in \Gamma\left(h^{*}F, h^{*}\nu, M\right)$. *q.e.d.*

Using Lemmas 3.1.2.1, 3.1.2.2 and 3.1.2.3, we obtain the following

Theorem 3.1.2.1 *The couple*

$$\left([,]_{h^{*}F}, \left(\overset{h^{*}F}{\rho}, Id_{M}\right)\right)$$

is a Lie algebroid structure for the vector bundle $\left(h^{*}F, h^{*}\nu, M\right)$.

This Lie algebroid will be called *the pull-back Lie algebroid of the generalized Lie algebroid*

$$\left(\left(F, \nu, N\right), [,]_{F,h}, \left(\rho, \eta\right)\right).$$

3.1.3 Interior Differential Systems

We consider a generalized Lie algebroid $\left(\left(F, \nu, N\right), [,]_{F,h}, \left(\rho, \eta\right)\right)$ given by the diagrams:

$$\begin{array}{ccccccc} & & F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\ & & \downarrow \nu & & \downarrow \tau_{M} & & \downarrow \tau_{N} \\ M & \xrightarrow{h} & N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \end{array}$$

Let $\left(\left(h^{*}F, h^{*}\nu, M\right), [,]_{h^{*}F}, \left(\overset{h^{*}F}{\rho}, Id_{M}\right)\right)$ be the pull-back Lie algebroid.

Definition 3.1.3.1 Any vector subbundle $\left(E, \pi, M\right)$ of the vector bundle $\left(h^{*}F, h^{*}\nu, M\right)$ will be called *interior differential system (IDS) of the generalized Lie algebroid* $\left(\left(F, \nu, N\right), [,]_{F,h}, \left(\rho, \eta\right)\right)$.

In particular, if $h = Id_{N} = \eta$, then any vector subbundle $\left(E, \pi, N\right)$ of the vector bundle $\left(F, \nu, N\right)$ will be called *interior differential system of the Lie algebroid* $\left(\left(F, \nu, N\right), [,]_{F}, \left(\rho, Id_{N}\right)\right)$.

Remark 3.1.3.1 If $\left(E, \pi, M\right)$ is an IDS of the generalized Lie algebroid $\left(\left(F, \nu, N\right), [,]_{F,h}, \left(\rho, \eta\right)\right)$, then $\left(\Gamma\left(E, \pi, M\right), +, \cdot\right)$ is a $\mathcal{F}\left(M\right)$-submodule of the $\mathcal{F}\left(M\right)$-module $\left(\Gamma\left(h^{*}F, h^{*}\nu, M\right), +, \cdot\right)$.

In addition, if

$$\Gamma\left(E^{\perp}, \pi^{\perp}, M\right) \overset{put}{=} \left\{\Omega \in \Gamma\left(\overset{*}{h^{*}F}, \overset{*}{h^{*}\nu}, M\right) : \Omega\left(S\right) = 0,\ \forall S \in \Gamma\left(E, \pi, M\right)\right\},$$

then $\left(\Gamma\left(E^{\perp}, \pi^{\perp}, M\right), +, \cdot\right)$ is $\mathcal{F}\left(M\right)$-submodule of the $\mathcal{F}\left(M\right)$-module $\left(\Gamma\left(\overset{*}{h^{*}F}, \overset{*}{h^{*}\nu}, M\right), +, \cdot\right)$.

We obtain a vector subbundle $\left(E^{\perp}, \pi^{\perp}, M\right)$ of the vector bundle $\left(\overset{*}{h^{*}F}, \overset{*}{h^{*}\nu}, M\right)$ which will be called *the annihilator vector subbundle for the IDS* $\left(E, \pi, M\right)$.

Proposition 3.1.3.1 *Let* $\left(E, \pi, M\right)$ *be an IDS of the generalized Lie algebroid* $\left(\left(F, \nu, N\right), [,]_{F,h}, \left(\rho, \eta\right)\right)$.

If $dim_{\mathcal{F}(M)}\Gamma\left(E, \pi, M\right) = r \leq p = dim_{\mathcal{F}(M)}\Gamma\left(h^{*}F, h^{*}\nu, M\right)$, *then* $dim_{\mathcal{F}(M)}\Gamma\left(E^{\perp}, \pi^{\perp}, M\right) = p - r$.

Therefore, if $\Gamma\left(E, \pi, M\right) = \langle S_{1}, ..., S_{r}\rangle$, *then it exists* $\Theta^{r+1}, ..., \Theta^{p} \in \Gamma\left(\overset{*}{h^{*}F}, \overset{*}{h^{*}\nu}, M\right)$ *linearly independent such that* $\Gamma\left(E^{\perp}, \pi^{\perp}, M\right) = \left\langle \Theta^{r+1}, ..., \Theta^{p}\right\rangle$.

Conversely, if $\Gamma\left(E^{\perp},\pi^{\perp},M\right)=\left\langle \Theta^{r+1},...,\Theta^{p}\right\rangle$, *then it exists* $S_1,...,S_r\in\Gamma\left(E,\pi,M\right)$ *linearly independent such that* $\Gamma\left(E,\pi,M\right)=\left\langle S_1,...,S_r\right\rangle$.

Definition 3.1.3.2 The IDS (E,π,M) of the generalized Lie algebroid $\left(\left(F,\nu,N\right),[,]_{F,h},(\rho,\eta)\right)$ will be called *involutive* if

$$[S,T]_{h^*F}\in\Gamma\left(E,\pi,M\right),\ \forall S,T\in\Gamma\left(E,\pi,M\right).$$

Proposition 3.1.3.2 *Let* (E,π,M) *be an IDS of the generalized Lie algebroid* $\left(\left(F,\nu,N\right),[,]_{F,h},(\rho,\eta)\right)$.

If $\{S_1,...,S_r\}$ *is a base for the* $\mathcal{F}(M)$*-submodule* $\left(\Gamma\left(E,\pi,M\right),+,\cdot\right)$ *then* (E,π,M) *is involutive if and only if*

$$[S_a,S_b]_{h^*F}\in\Gamma\left(E,\pi,M\right),\ \forall a,b\in\overline{1,r}.$$

3.2 Exterior differential calculus for generalized Lie algebroids

We propose an exterior differential calculus in the general framework of generalized Lie algebroids. As any Lie algebroid can be regarded as a generalized Lie algebroid, in particular, we obtain a new point of view over the exterior differential calculus for Lie algebroids.

Let

$$\left(\left(F,\nu,N\right),[,]_{F,h},(\rho,\eta)\right)$$

be a generalized Lie algebroid given by the diagram $(3.1.1.1)$.

Definition 3.2.1 For any $q\in\mathbb{N}$ we denote by $(\Sigma_q,\circ)$ the permutations group of the set $\{1,2,...,q\}$.

Definition 3.2.2 We denoted by $\Lambda^q\left(F,\nu,N\right)$ the set of q-linear applications

$$\begin{array}{ccc}\Gamma\left(F,\nu,N\right)^q & \stackrel{\omega}{\longrightarrow} & \mathcal{F}(N)\\ (z_1,...,z_q) & \longmapsto & \omega\left(z_1,...,z_q\right)\end{array}$$

such that

$$\omega\left(z_{\sigma(1)},...,z_{\sigma(q)}\right)=sgn\left(\sigma\right)\cdot\omega\left(z_1,...,z_q\right)$$

for any $z_1,...,z_q\in\Gamma\left(F,\nu,N\right)$ and for any $\sigma\in\Sigma_q$.

The elements of $\Lambda^q\left(F,\nu,N\right)$ will be called *differential forms of degree q* or *differential q-forms*.

Remark 3.2.1 If $\omega\in\Lambda^q\left(F,\nu,N\right)$, then $\omega\left(z_1,...,z,...,z,...z_q\right)=0$. Therefore, if $\omega\in\Lambda^q\left(F,\nu,N\right)$, then

$$\omega\left(z_1,...,z_i,...,z_j,...z_q\right)=-\omega\left(z_1,...,z_j,...,z_i,...z_q\right).$$

Theorem 3.2.1 *If* $q\in N$, *then* $\left(\Lambda^q\left(F,\nu,N\right),+,\cdot\right)$ *is a* $\mathcal{F}(N)$*-module.*

Definition 3.2.3 If $\omega \in \Lambda^q(F,\nu,N)$ and $\theta \in \Lambda^r(F,\nu,N)$, then the $(q+r)$-form $\omega\wedge\theta$ defined by

$$\begin{aligned}\omega\wedge\theta(z_1,...,z_{q+r}) &= \sum_{\substack{\sigma(1)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma)\,\omega\left(z_{\sigma(1)},...,z_{\sigma(q)}\right)\theta\left(z_{\sigma(q+1)},...,z_{\sigma(q+r)}\right)\\ &= \frac{1}{q!r!}\sum_{\sigma\in\Sigma_{q+r}} sgn(\sigma)\,\omega\left(z_{\sigma(1)},...,z_{\sigma(q)}\right)\theta\left(z_{\sigma(q+1)},...,z_{\sigma(q+r)}\right),\end{aligned}$$

for any $z_1,...,z_{q+r}\in\Gamma(F,\nu,N)$, will be called *the exterior product of the forms* ω *and* θ.

Using the previous definition, we obtain

Theorem 3.2.2 *The following affirmations hold good:*

1. *If* $\omega\in\Lambda^q(F,\nu,N)$ *and* $\theta\in\Lambda^r(F,\nu,N)$, *then*

$$\omega\wedge\theta = (-1)^{q\cdot r}\,\theta\wedge\omega. \tag{3.2.1}$$

2. *For any* $\omega\in\Lambda^q(F,\nu,N)$, $\theta\in\Lambda^r(F,\nu,N)$ *and* $\eta\in\Lambda^s(F,\nu,N)$ *we obtain*

$$(\omega\wedge\theta)\wedge\eta = \omega\wedge(\theta\wedge\eta). \tag{3.2.2}$$

3. *For any* $\omega,\theta\in\Lambda^q(F,\nu,N)$ *and* $\eta\in\Lambda^s(F,\nu,N)$ *we obtain*

$$(\omega+\theta)\wedge\eta = \omega\wedge\eta+\theta\wedge\eta. \tag{3.2.3}$$

4. *For any* $\omega\in\Lambda^q(F,\nu,N)$ *and* $\theta,\eta\in\Lambda^s(F,\nu,N)$ *we obtain*

$$\omega\wedge(\theta+\eta) = \omega\wedge\theta+\omega\wedge\eta. \tag{3.2.4}$$

5. *For any* $f\in\mathcal{F}(N)$, $\omega\in\Lambda^q(F,\nu,N)$ *and* $\theta\in\Lambda^s(F,\nu,N)$ *we obtain*

$$(f\cdot\omega)\wedge\theta = f\cdot(\omega\wedge\theta) = \omega\wedge(f\cdot\theta). \tag{3.2.5}$$

Theorem 3.2.3 *If*

$$\Lambda(F,\nu,N) = \underset{q\geq 0}{\oplus}\Lambda^q(F,\nu,N),$$

then

$$(\Lambda(F,\nu,N),+,\cdot,\wedge)$$

is a $\mathcal{F}(N)$*-algebra.* This algebra will be called *the exterior differential algebra of the vector bundle* (F,ν,N).

Remark 3.2.2 If $\left\{t^\alpha,\ \alpha\in\overline{1,p}\right\}$ is the coframe associated to the frame $\left\{t_\alpha,\ \alpha\in\overline{1,p}\right\}$ of the vector bundle (F,ν,N) in the vector local $(n+p)$-chart U, then

$$t^{\alpha_1}\wedge...\wedge t^{\alpha_q}\left(z_1^\alpha t_\alpha,...,z_q^\alpha t_\alpha\right) = \tfrac{1}{q!}\det\left\|\begin{array}{ccc} z_1^{\alpha_1} & ... & z_1^{\alpha_q}\\ ... & ... & ...\\ z_q^{\alpha_1} & ... & z_q^{\alpha_q}\end{array}\right\|,$$

for any $q\in\overline{1,p}$.

Remark 3.2.3 If $\left\{t^{\alpha},\ \alpha\in\overline{1,p}\right\}$ is the coframe associated to the frame $\left\{t_{\alpha},\ \alpha\in\overline{1,p}\right\}$ of the vector bundle (F,ν,N) in the vector local $(n+p)$-chart U, then, for any $q\in\overline{1,p}$ we define C_p^q exterior differential forms of the type

$$t^{\alpha_1}\wedge...\wedge t^{\alpha_q}$$

such that $1\leq\alpha_1<...<\alpha_q\leq p$.

The set

$$\{t^{\alpha_1}\wedge...\wedge t^{\alpha_q},1\leq\alpha_1<...<\alpha_q\leq p\}$$

is a base for the $\mathcal{F}(N)$-module

$$\left(\Lambda^q\left(F,\nu,N\right),+,\cdot\right).$$

Therefore, if $\omega\in\Lambda^q\left(F,\nu,N\right)$, then

$$\omega=\omega_{\alpha_1...\alpha_q}t^{\alpha_1}\wedge...\wedge t^{\alpha_q}.$$

In particular, if ω is an exterior differential p-form ω, then we can written

$$\omega=a\cdot t^1\wedge...\wedge t^p,$$

where $a\in\mathcal{F}\left(N\right)$.

Definition 3.2.4 If

$$\omega=\omega_{\alpha_1...\alpha_q}t^{\alpha_1}\wedge...\wedge t^{\alpha_q}\in\Lambda^q\left(F,\nu,N\right)$$

such that

$$\omega_{\alpha_1...\alpha_q}\in C^r\left(N\right),$$

for any $1\leq\alpha_1<...<\alpha_q\leq p$, then we will say that *the q-form ω is differentiable of C^r-class.*

Definition 3.2.5 For any $z\in\Gamma\left(F,\nu,N\right)$, the $\mathcal{F}(N)$-multilinear application

$$\Lambda\left(F,\nu,N\right)\ \xrightarrow{L_z}\ \Lambda\left(F,\nu,N\right)\ ,$$

defined by

$$L_z\left(f\right)=\Gamma\left(Th\circ\rho,h\circ\eta\right)z\left(f\right),\ \forall f\in\mathcal{F}\left(N\right)$$

and

$$\begin{array}{ll}L_z\omega\left(z_1,...,z_q\right) & =\Gamma\left(Th\circ\rho,h\circ\eta\right)z\left(\omega\left(\left(z_1,...,z_q\right)\right)\right)\\ & -\sum\limits_{i=1}^{q}\omega\left(\left(z_1,...,\left[z,z_i\right]_{F,h},...,z_q\right)\right),\end{array}$$

for any $\omega\in\Lambda^q\ \left(F,\nu,N\right)$ and $z_1,...,z_q\in\Gamma\left(F,\nu,N\right)$, will be called *the covariant Lie derivative with respect to the section z.*

Theorem 3.2.4 *If $z\in\Gamma\left(F,\nu,N\right)$, $\omega\in\Lambda^q\left(F,\nu,N\right)$ and $\theta\in\Lambda^r\left(F,\nu,N\right)$, then*

$$L_z\left(\omega\wedge\theta\right)=L_z\omega\wedge\theta+\omega\wedge L_z\theta. \tag{3.2.6}$$

Proof. Let $z_1, ..., z_{q+r} \in \Gamma(F, \nu, N)$ be arbitrary. Since

$$
\begin{aligned}
& L_z(\omega \wedge \theta)(z_1, ..., z_{q+r}) = \Gamma(Th \circ \rho, h \circ \eta)\, z\left((\omega \wedge \theta)(z_1, ..., z_{q+r})\right) \\
& -\sum_{i=1}^{q+r} (\omega \wedge \theta)\left(\left(z_1, ..., [z, z_i]_{F,h}, ..., z_{q+r}\right)\right) \\
& = \Gamma(Th \circ \rho, h \circ \eta)\, z\left(\sum_{\substack{\sigma(1)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \cdot \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right)\right. \\
& \qquad \left.\cdot \theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right)\right) - \sum_{i=1}^{q+r} (\omega \wedge \theta)\left(\left(z_1, ..., [z, z_i]_{F,h}, ..., z_{q+r}\right)\right) \\
& = \sum_{\substack{\sigma(1)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \cdot \Gamma(Th \circ \rho, h \circ \eta)\, z\left(\omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right)\right) \\
& \qquad \cdot \theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right) + \sum_{\substack{\sigma(1)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \cdot \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right) \\
& \qquad \cdot \Gamma(Th \circ \rho, h \circ \eta)\, z\left(\theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right)\right) - \sum_{\substack{\sigma(1)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \\
& \qquad \cdot \sum_{i=1}^{q} \omega\left(z_{\sigma(1)}, ..., \left[z, z_{\sigma(i)}\right]_{F,h}, ..., z_{\sigma(q+r)}\right) \cdot \theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right) \\
& - \sum_{\substack{\sigma(1)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \sum_{i=q+1}^{q+r} \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right) \\
& \qquad \cdot \theta\left(z_{\sigma(q+1)}, ..., \left[z, z_{\sigma(i)}\right]_{F,h}, ..., z_{\sigma(q+r)}\right) \\
& = \sum_{\substack{\sigma(1)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma)\, L_z\omega\left(z_{\sigma(1)}, ..., \left[z, z_{\sigma(i)}\right]_{F,h}, ..., z_{\sigma(q+r)}\right) \\
& \qquad \cdot \theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right) + \sum_{\substack{\sigma(1)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \sum_{i=q+1}^{q+r} \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right) \\
& \qquad \cdot L_z\theta\left(z_{\sigma(q+1)}, ..., \left[z, z_{\sigma(i)}\right]_{F,h}, ..., z_{\sigma(q+r)}\right) \\
& = (L_z\omega \wedge \theta + \omega \wedge L_z\theta)(z_1, ..., z_{q+r})
\end{aligned}
$$

it results the conclusion of the theorem. *q.e.d.*

Definition 3.2.6 For any $z \in \Gamma(F, \nu, N)$, the $\mathcal{F}(N)$-multilinear application

$$
\begin{array}{ccc}
\Lambda(F, \nu, N) & \xrightarrow{i_z} & \Lambda(F, \nu, N) \\
\Lambda^q(F, \nu, N) \ni \omega & \longmapsto & i_z\omega \in \Lambda^{q-1}(F, \nu, N),
\end{array}
$$

where

$$i_z\omega(z_2, ..., z_q) = \omega(z, z_2, ..., z_q),$$

for any $z_2, ..., z_q \in \Gamma(F, \nu, N)$, will be called the *interior product associated to the section z.*

For any $f \in \mathcal{F}(N)$, we define $i_z f = 0$.

Remark 3.2.4 If $z \in \Gamma(F,\nu,N)$, $\omega \in \Lambda^p(F,\nu,N)$ and U is an open subset of N such that $z_{|U} = 0$ or $\omega_{|U} = 0$, then $(i_z\omega)_{|U} = 0$.

Theorem 3.2.5 *If* $z \in \Gamma(F,\nu,N)$, *then for any* $\omega \in \Lambda^q(F,\nu,N)$ *and* $\theta \in \Lambda^r(F,\nu,N)$ *we obtain*

$$i_z(\omega \wedge \theta) = i_z\omega \wedge \theta + (-1)^q \omega \wedge i_z\theta. \tag{3.2.7}$$

Proof. Let $z_1, ..., z_{q+r} \in \Gamma(F,\nu,N)$ be arbitrary. We observe that

$$\begin{aligned}
& i_{z_1}(\omega \wedge \theta)(z_2, ..., z_{q+r}) = (\omega \wedge \theta)(z_1, z_2, ..., z_{q+r}) \\
&= \sum_{\substack{\sigma(1)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \cdot \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right) \cdot \theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right) \\
&= \sum_{\substack{1=\sigma(1)<\sigma(2)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \cdot \omega\left(z_1, z_{\sigma(2)}, ..., z_{\sigma(q)}\right) \cdot \theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right) \\
&+ \sum_{\substack{\sigma(1)<...<\sigma(q) \\ 1=\sigma(q+1)<\sigma(q+2)<...<\sigma(q+r)}} sgn(\sigma) \cdot \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right) \cdot \theta\left(z_1, z_{\sigma(q+2)}, ..., z_{\sigma(q+r)}\right) \\
&= \sum_{\substack{\sigma(2)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \cdot i_{z_1}\omega\left(z_{\sigma(2)}, ..., z_{\sigma(q)}\right) \cdot \theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right) \\
&+ \sum_{\substack{\sigma(1)<...<\sigma(q) \\ \sigma(q+2)<...<\sigma(q+r)}} sgn(\sigma) \cdot \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right) \cdot i_{z_1}\theta\left(z_{\sigma(q+2)}, ..., z_{\sigma(q+r)}\right).
\end{aligned}$$

In the second sum, we have the permutation

$$\sigma = \begin{pmatrix} 1 & ... & q & q+1 & q+2 & ... & q+r \\ \sigma(1) & ... & \sigma(q) & 1 & \sigma(q+2) & ... & \sigma(q+r) \end{pmatrix}.$$

We observe that $\sigma = \tau \circ \tau'$, where

$$\tau = \begin{pmatrix} 1 & 2 & ... & q+1 & q+2 & ... & q+r \\ 1 & \sigma(1) & ... & \sigma(q) & \sigma(q+2) & ... & \sigma(q+r) \end{pmatrix}$$

and

$$\tau' = \begin{pmatrix} 1 & 2 & ... & q & q+1 & q+2 & ... & q+r \\ 2 & 3 & ... & q+1 & 1 & q+2 & ... & q+r \end{pmatrix}.$$

Since $\tau(2) < ... < \tau(q+1)$ and τ' has q inversions, it results that

$$sgn(\sigma) = (-1)^q \cdot sgn(\tau).$$

Therefore,

$$\begin{aligned}
& i_{z_1}(\omega \wedge \theta)(z_2, ..., z_{q+r}) = (i_{z_1}\omega \wedge \theta)(z_2, ..., z_{q+r}) \\
& + (-1)^q \sum_{\substack{\tau(2)<...<\tau(q) \\ \tau(q+2)<...<\tau(q+r)}} sgn(\tau) \cdot \omega\left(z_{\tau(2)}, ..., z_{\tau(q)}\right) \cdot i_{z_1}\theta\left(z_{\tau(q+2)}, ..., z_{\tau(q+r)}\right) \\
& = (i_{z_1}\omega \wedge \theta)(z_2, ..., z_{q+r}) + (-1)^q (\omega \wedge i_{z_1}\theta)(z_2, ..., z_{q+r}).
\end{aligned}$$

q.e.d.

Theorem 3.2.6 *For any* $z, v \in \Gamma(F, \nu, N)$ *we obtain*

$$L_v \circ i_z - i_z \circ L_v = i_{[z,v]_{F,h}}. \tag{3.2.8}$$

Proof. Let $\omega \in \Lambda^q(F, \nu, N)$ be arbitrary. Since

$$\begin{array}{l} i_z(L_v\omega)(z_2, ...z_q) = L_v\omega(z, z_2, ...z_q) \\ = \Gamma(Th \circ \rho, h \circ \eta)\, v(\omega(z, z_2, ..., z_q)) - \omega\left([v, z]_{F,h}, z_2, ..., z_q\right) \\ -\sum\limits_{i=2}^{q}\omega\left(\left(z, z_2, ..., [v, z_i]_{F,h}, ..., z_q\right)\right) \\ = \Gamma(Th \circ \rho, h \circ \eta)\, v(i_z\omega(z_2, ..., z_q)) - \sum\limits_{i=2}^{q} i_z\omega\left(z_2, ..., [v, z_i]_{F,h}, ..., z_q\right) \\ -i_{[v,z]_F}(z_2, ..., z_q) = \left(L_v(i_z\omega) - i_{[v,z]_{F,h}}\right)(z_2, ..., z_q), \end{array}$$

for any $z_2, ..., z_q \in \Gamma(F, \nu, N)$ it result the conclusion of the theorem. *q.e.d.*

Definition 3.2.7 If $f \in \mathcal{F}(N)$ and $z \in \Gamma(F, \nu, N)$, then we define

$$d^F f(z) = \Gamma(Th \circ \rho, h \circ \eta)(z) f.$$

Theorem 3.2.7 *The* $\mathcal{F}(N)$*-multilinear application*

$$\begin{array}{ccc} \Lambda^q\ (F, \nu, N) & \xrightarrow{d^F} & \Lambda^{q+1}\ (F, \nu, N) \\ \omega & \longmapsto & d\omega \end{array}$$

defined by

$$\begin{array}{rl} d^F\omega(z_0, z_1, ..., z_q) = & \sum\limits_{i=0}^{q}(-1)^i\,\Gamma(Th \circ \rho, h \circ \eta)\, z_i(\omega((z_0, z_1, ..., \hat{z}_i, ..., z_q))) \\ & + \sum\limits_{i<j}(-1)^{i+j}\,\omega\left(\left([z_i, z_j]_{F,h}, z_0, z_1, ..., \hat{z}_i, ..., \hat{z}_j, ..., z_q\right)\right), \end{array}$$

for any $z_0, z_1, ..., z_q \in \Gamma(F, \nu, N)$, *is unique with the following property:*

$$L_z = d^F \circ i_z + i_z \circ d^F,\ \forall z \in \Gamma(F, \nu, N). \tag{3.2.9}$$

This $\mathcal{F}(N)$-multilinear application d^F will be called *the exterior differentiation operator for the exterior differential algebra of the generalized Lie algebroid* $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$.

Proof. We verify the property (3.2.9) Since

$$
\begin{aligned}
&\left(i_{z_0}\circ d^F\right)\omega\left(z_1,...,z_q\right)=d\omega\left(z_0,z_1,...,z_q\right)\\
&=\sum_{i=0}^{q}\left(-1\right)^i\Gamma\left(Th\circ\rho,h\circ\eta\right)z_i\left(\omega\left(z_0,z_1,...,\hat{z}_i,...,z_q\right)\right)\\
&+\sum_{0\leq i<j}\left(-1\right)^{i+j}\omega\left(\left[z_i,z_j\right]_{F,h},z_0,z_1,...,\hat{z}_i,...,\hat{z}_j,...,z_q\right)\\
&=\Gamma\left(Th\circ\rho,h\circ\eta\right)z_0\left(\omega\left(z_1,...,z_q\right)\right)\\
&+\sum_{i=1}^{q}\left(-1\right)^i\Gamma\left(Th\circ\rho,h\circ\eta\right)z_i\left(\omega\left(z_0,z_1,...,\hat{z}_i,...,z_q\right)\right)\\
&+\sum_{i=1}^{q}\left(-1\right)^i\omega\left(\left[z_0,z_i\right]_{F,h},z_1,...,\hat{z}_i,...,z_q\right)\\
&+\sum_{1\leq i<j}\left(-1\right)^{i+j}\omega\left(\left[z_i,z_j\right]_{F,h},z_0,z_1,...,\hat{z}_i,...,\hat{z}_j,...,z_q\right)\\
&=\Gamma\left(Th\circ\rho,h\circ\eta\right)z_0\left(\omega\left(z_1,...,z_q\right)\right)\\
&-\sum_{i=1}^{q}\omega\left(z_1,...,\left[z_0,z_i\right]_{F,h},...,z_q\right)\\
&-\sum_{i=1}^{q}\left(-1\right)^{i-1}\Gamma\left(Th\circ\rho,h\circ\eta\right)z_i\left(i_{z_0}\omega\left(\left(z_1,...,\hat{z}_i,...,z_q\right)\right)\right)\\
&-\sum_{1\leq i<j}\left(-1\right)^{i+j-2}i_{z_0}\omega\left(\left(\left[z_i,z_j\right]_{F,h},z_1,...,\hat{z}_i,...,\hat{z}_j,...,z_q\right)\right)\\
&=\left(L_{z_0}-d^F\circ i_{z_0}\right)\omega\left(z_1,...,z_q\right),
\end{aligned}
$$

for any $z_0,z_1,...,z_q\in\Gamma\left(F,\nu,N\right)$ it results that the property (3.2.9) is satisfied.

In the following, we verify the uniqueness of the operator d^F.

Let d'^F be an another exterior differentiation operator satisfying the property (3.2.9).

Let $S=\left\{q\in\mathbb{N}:d^F\omega=d'^F\omega,\ \forall\omega\in\Lambda^q\left(F,\nu,N\right)\right\}$ be.

Let $z\in\Gamma\left(F,\nu,N\right)$ be arbitrary.

We observe that (3.2.9) is equivalent with

$$i_z\circ\left(d^F-d'^F\right)+\left(d^F-d'^F\right)\circ i_z=0. \tag{1}$$

Since $i_zf=0$, for any $f\in\mathcal{F}\left(N\right)$, it results that

$$\left(\left(d^F-d'^F\right)f\right)\left(z\right)=0,\ \forall f\in\mathcal{F}\left(N\right).$$

Therefore, we obtain that

$$0\in S. \tag{2}$$

In the following, we prove that

$$q\in S\Longrightarrow q+1\in S \tag{3}$$

Let $\omega\in\Lambda^{p+1}\left(F,\nu,N\right)$ be arbitrary. Since $i_z\omega\in\Lambda^q\left(F,\nu,N\right)$, using the equality (1), it results that

$$i_z\circ\left(d^F-d'^F\right)\omega=0.$$

We obtain that, $\left(\left(d^F-d'^F\right)\omega\right)\left(z_0,z_1,...,z_q\right)=0$, for any $z_1,...,z_q\in\Gamma\left(F,\nu,N\right)$. Therefore $d^F\omega=d'^F\omega$, namely $q+1\in S$.

Using the *Peano's Axiom* and the affirmations (2) and (3) it results that $S=\mathbb{N}$. Therefore, the uniqueness is verified. *q.e.d.*

Note that if $\omega = \omega_{\alpha_1 \ldots \alpha_q} t^{\alpha_1} \wedge \ldots \wedge t^{\alpha_q} \in \Lambda^q(F, \nu, N)$, then

$$\begin{aligned} d^F \omega\left(t_{\alpha_0}, t_{\alpha_1}, \ldots, t_{\alpha_q}\right) &= \sum_{i=0}^{q} (-1)^i \theta_{\alpha_i}^{\tilde{k}} \frac{\partial \omega_{\alpha_0, \ldots, \widehat{\alpha_i} \ldots \alpha_q}}{\partial \varkappa^{\tilde{k}}} \\ &+ \sum_{i<j} (-1)^{i+j} L_{\alpha_i \alpha_j}^{\alpha} \cdot \omega_{\alpha, \alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_q}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} d^F \omega = &\left(\sum_{i=0}^{q} (-1)^i \theta_{\alpha_i}^{\tilde{k}} \frac{\partial \omega_{\alpha_0, \ldots, \widehat{\alpha_i} \ldots \alpha_q}}{\partial \varkappa^{\tilde{k}}} \right. \\ &\left. + \sum_{i<j} (-1)^{i+j} L_{\alpha_i \alpha_j}^{\alpha} \cdot \omega_{\alpha, \alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_q} \right) t^{\alpha_0} \wedge t^{\alpha_1} \wedge \ldots \wedge t^{\alpha_q}. \end{aligned} \tag{3.2.10}$$

Remark 3.2.4 If d^F is the exterior differentiation operator for the generalized Lie algebroid

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right),$$

$\omega \in \Lambda^q(F, \nu, N)$ and U is an open subset of N such that $\omega_{|U} = 0$, then $\left(d^F \omega\right)_{|U} = 0$.

Theorem 3.2.8 *The exterior differentiation operator d^F given by the previous theorem has the following properties:*

1. *For any $\omega \in \Lambda^q(F, \nu, N)$ and $\theta \in \Lambda^r(F, \nu, N)$ we obtain*

$$d^F(\omega \wedge \theta) = d^F \omega \wedge \theta + (-1)^q \omega \wedge d^F \theta. \tag{3.2.11}$$

2. *For any $z \in \Gamma(F, \nu, N)$* we obtain

$$L_z \circ d^F = d^F \circ L_z. \tag{3.2.12}$$

3. $d^F \circ d^F = 0$.

Proof.
1. Let $S = \left\{ q \in \mathbb{N} : d^F(\omega \wedge \theta) = d^F \omega \wedge \theta + (-1)^q \omega \wedge d^F \theta,\ \forall \omega \in \Lambda^q(F, \nu, N) \right\}$ be. Since

$$\begin{aligned} & d^F(f \wedge \theta)(z, v) = d^F(f \cdot \theta)(z, v) \\ & = \Gamma(Th \circ \rho, h \circ \eta) z (f \omega(v)) - \Gamma(Th \circ \rho, h \circ \eta) v (f \omega(z)) - f \omega\left([z, v]_{F,h}\right) \\ & = \Gamma(Th \circ \rho, h \circ \eta) z(f) \cdot \omega(v) + f \cdot \Gamma(Th \circ \rho, h \circ \eta) z(\omega(v)) \\ & - \Gamma(Th \circ \rho, h \circ \eta) v(f) \cdot \omega(z) - f \cdot \Gamma(Th \circ \rho, h \circ \eta) v(\omega(z)) - f \omega\left([z, v]_{F,h}\right) \\ & = d^F f(z) \cdot \omega(v) - d^F f(v) \cdot \omega(z) + f \cdot d^F \omega(z, v) \\ & = \left(d^F f \wedge \omega\right)(z, v) + (-1)^0 f \cdot d^F \omega(z, v) \\ & = \left(d^F f \wedge \omega\right)(z, v) + (-1)^0 \left(f \wedge d^F \omega\right)(z, v),\ \forall z, v \in \Gamma(F, \nu, N), \end{aligned}$$

it results that

$$0 \in S. \tag{1.1}$$

In the following we prove that

$$q \in S \Longrightarrow q + 1 \in S. \tag{1.2}$$

Without restricting the generality, we consider that $\theta \in \Lambda^r(F,\nu,N)$. Since

$$
\begin{aligned}
&d^F(\omega \wedge \theta)(z_0, z_1, ..., z_{q+r}) = i_{z_0} \circ d^F(\omega \wedge \theta)(z_1, ..., z_{q+r}) \\
&= L_{z_0}(\omega \wedge \theta)(z_1, ..., z_{q+r}) - d^F \circ i_{z_0}(\omega \wedge \theta)(z_1, ..., z_{q+r}) \\
&= (L_{z_0}\omega \wedge \theta + \omega \wedge L_{z_0}\theta)(z_1, ..., z_{q+r}) \\
&- \left[d^F \circ (i_{z_0}\omega \wedge \theta + (-1)^q \omega \wedge i_{z_0}\theta)\right](z_1, ..., z_{q+r}) \\
&= \left(L_{z_0}\omega \wedge \theta + \omega \wedge L_{z_0}\theta - \left(d^F \circ i_{z_0}\omega\right) \wedge \theta\right)(z_1, ..., z_{q+r}) \\
&- \Big((-1)^{q-1} i_{z_0}\omega \wedge d^F\theta + (-1)^q d^F\omega \wedge i_{z_0}\theta\Big)(z_1, ..., z_{q+r}) \\
&- (-1)^{2q} \omega \wedge d^F \circ i_{z_0}\theta(z_1, ..., z_{q+r}) \\
&= \left(\left(L_{z_0}\omega - d^F \circ i_{z_0}\omega\right) \wedge \theta\right)(z_1, ..., z_{q+r}) \\
&+\omega \wedge \left(L_{z_0}\theta - d^F \circ i_{z_0}\theta\right)(z_1, ..., z_{q+r}) \\
&+ \left((-1)^q i_{z_0}\omega \wedge d^F\theta - (-1)^q d^F\omega \wedge i_{z_0}\theta\right)(z_1, ..., z_{q+r}) \\
&= \Big[\left(\left(i_{z_0} \circ d^F\right)\omega\right) \wedge \theta + (-1)^{q+1} d^F\omega \wedge i_{z_0}\theta\Big](z_1, ..., z_{q+r}) \\
&+ \left[\omega \wedge \left(\left(i_{z_0} \circ d^F\right)\theta\right) + (-1)^q i_{z_0}\omega \wedge d^F\theta\right](z_1, ..., z_{q+r}) \\
&= \left[i_{z_0}\left(d^F\omega \wedge \theta\right) + (-1)^q i_{z_0}\left(\omega \wedge d^F\theta\right)\right](z_1, ..., z_{q+r}) \\
&= \left[d^F\omega \wedge \theta + (-1)^q \omega \wedge d^F\theta\right](z_1, ..., z_{q+r}),
\end{aligned}
$$

for any $z_0, z_1, ..., z_{q+r} \in \Gamma(F,\nu,N)$, it results (1.2).

Using the *Peano's Axiom* and the affirmations (1.1) and (1.2) it results that $S = \mathbb{N}$.

Therefore, it results the conclusion of affirmation 1.

2. Let $z \in \Gamma(F,\nu,N)$ be arbitrary.

Let $S = \left\{q \in \mathbb{N} : \left(L_z \circ d^F\right)\omega = \left(d^F \circ L_z\right)\omega,\ \forall \omega \in \Lambda^q(F,\nu,N)\right\}$ be.

Let $f \in \mathcal{F}(N)$ be arbitrary. Since

$$
\begin{aligned}
&\left(d^F \circ L_z\right) f(v) = i_v \circ \left(d^F \circ L_z\right) f = \left(i_v \circ d^F\right) \circ L_z f \\
&= (L_v \circ L_z) f - \left(\left(d^F \circ i_v\right) \circ L_z\right) f \\
&= (L_v \circ L_z) f - L_{[z,v]_{F,h}} f + d^F \circ i_{[z,v]_{F,h}} f - d^F \circ L_z(i_v f) \\
&= (L_v \circ L_z) f - L_{[z,v]_{F,h}} f + d^F \circ i_{[z,v]_{F,h}} f - 0 \\
&= (L_v \circ L_z) f - L_{[z,v]_{F,h}} f + d^F \circ i_{[z,v]_{F,h}} f - L_z \circ d^F(i_v f) \\
&= (L_z \circ i_v)\left(d^F f\right) - L_{[z,v]_{F,h}} f + d^F \circ i_{[z,v]_{F,h}} f \\
&= (i_v \circ L_z)\left(d^F f\right) + L_{[z,v]_{F,h}} f - L_{[z,v]_{F,h}} f \\
&= i_v \circ \left(L_z \circ d^F\right) f = \left(L_z \circ d^F\right) f(v),\ \forall v \in \Gamma(F,\nu,N),
\end{aligned}
$$

it results that

$$0 \in S. \tag{2.1}$$

In the following we prove that

$$q \in S \Longrightarrow q + 1 \in S. \tag{2.2}$$

Let $\omega \in \Lambda^q(F,\nu,N)$ be arbitrary. Since

$$\begin{aligned}
&\left(d^F \circ L_z\right)\omega\left(z_0,z_1,...,z_q\right) = i_{z_0} \circ \left(d^F \circ L_z\right)\omega\left(z_1,...,z_q\right)\\
&= \left(i_{z_0} \circ d^F\right) \circ L_z\omega\left(z_1,...,z_q\right)\\
&= \left[\left(L_{z_0} \circ L_z\right)\omega - \left(\left(d^F \circ i_{z_0}\right) \circ L_z\right)\omega\right]\left(z_1,...,z_q\right)\\
&= \left[\left(L_{z_0} \circ L_z\right)\omega - L_{[z,z_0]_{F,h}}\omega\right]\left(z_1,...,z_q\right)\\
&+ \left[d^F \circ i_{[z,z_0]_{F,h}}\omega - d^F \circ L_z\left(i_{z_0}\omega\right)\right]\left(z_1,...,z_q\right)\\
&\overset{ip.}{=} \left[\left(L_{z_0} \circ L_z\right)\omega - L_{[z,z_0]_{F,h}}\omega\right]\left(z_1,...,z_q\right)\\
&+ \left[d^F \circ i_{[z,z_0]_{F,h}}\omega - L_z \circ d^F\left(i_{z_0}\omega\right)\right]\left(z_1,...,z_q\right)\\
&= \left[\left(L_z \circ i_{z_0}\right)\left(d^F\omega\right) - L_{[z,z_0]_{F,h}}\omega + d^F \circ i_{[z,z_0]_{F,h}}\omega\right]\left(z_1,...,z_q\right)\\
&= \left[\left(i_{z_0} \circ L_z\right)\left(d^F\omega\right) + L_{[z,z_0]_{F,h}}\omega - L_{[z,z_0]_{F,h}}\omega\right]\left(z_1,...,z_q\right)\\
&= i_{z_0} \circ \left(L_z \circ d^F\right)\omega\left(z_1,...,z_q\right)\\
&= \left(L_z \circ d^F\right)\omega\left(z_0,z_1,...,z_q\right),\ \forall z_0,z_1,...,z_q \in \Gamma(F,\nu,N),
\end{aligned}$$

it results (2.2).

Using the *Peano's Axiom* and the affirmations (2.1) and (2.2) it results that $S = \mathbb{N}$. Therefore, it results the conclusion of affirmation 2.

3. It is remarked that

$$\begin{aligned}
&i_z \circ \left(d^F \circ d^F\right) = \left(i_z \circ d^F\right) \circ d^F = L_z \circ d^F - \left(d^F \circ i_z\right) \circ d^F\\
&= L_z \circ d^F - d^F \circ L_z + d^F \circ \left(d^F \circ i_z\right) = \left(d^F \circ d^F\right) \circ i_z,
\end{aligned}$$

for any $z \in \Gamma(F,\nu,N)$.

Let $\omega \in \Lambda^q(F,\nu,N)$ be arbitrary. Since

$$\begin{aligned}
&\left(d^F \circ d^F\right)\omega\left(z_1,...,z_{q+2}\right) = i_{z_{q+2}} \circ ... \circ i_{z_1} \circ \left(d^F \circ d^F\right)\omega = ...\\
&= i_{z_{q+2}} \circ \left(d^F \circ d^F\right) \circ i_{z_{q+1}}\left(\omega\left(z_1,...,z_q\right)\right)\\
&= i_{z_{q+2}} \circ \left(d^F \circ d^F\right)(0) = 0,\ \forall z_1,...,z_{q+2} \in \Gamma(F,\nu,N),
\end{aligned}$$

it results the conclusion of affirmation 3. *q.e.d.*

Theorem 3.2.9 *If* $((F,\nu,N),[,]_{F,h},(\rho,\eta))$ *is a generalized Lie algebroid and* d^F *is the exterior differentiation operator for the exterior differential* $\mathcal{F}(N)$*-algebra* $(\Lambda(F,\nu,N),+,\cdot,\wedge)$, *then we obtain the structure equations of Maurer-Cartan type*

$$(\mathcal{C}_1)\qquad d^F t^\alpha = -\frac{1}{2}L^\alpha_{\beta\gamma}t^\beta \wedge t^\gamma,\ \alpha \in \overline{1,p}$$

and

$$(\mathcal{C}_2)\qquad d^F \varkappa^{\tilde{\imath}} = \theta^{\tilde{\imath}}_\alpha t^\alpha,\ \tilde{\imath} \in \overline{1,n},$$

where $\left\{t^\alpha, \alpha \in \overline{1,p}\right\}$ *is the coframe of the vector bundle* (F,ν,N).

This equations will be called *the structure equations of Maurer-Cartan type associated to the generalized Lie algebroid* $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.

Proof. Let $\alpha \in \overline{1,p}$ be arbitrary. Since

$$d^F t^\alpha (t_\beta, t_\gamma) = -L_{\beta\gamma}^\alpha,\ \forall \beta,\gamma \in \overline{1,p}$$

it results that

$$d^F t^\alpha = -\sum_{\beta<\gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma. \tag{1}$$

Since $L_{\beta\gamma}^\alpha = -L_{\gamma\beta}^\alpha$ and $t^\beta \wedge t^\gamma = -t^\gamma \wedge t^\beta$, for nay $\beta,\gamma \in \overline{1,p}$, it results that

$$\sum_{\beta<\gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma = \frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma \tag{2}$$

Using the equalities (1) and (2) it results the structure equation $(\mathcal{C}_1)$.

Let $\tilde{\imath} \in \overline{1,n}$ be arbitrarily. Since

$$d^F \varkappa^{\tilde{\imath}} (t_\alpha) = \theta_\alpha^{\tilde{\imath}},\ \forall \alpha \in \overline{1,p}$$

it results the structure equation $(\mathcal{C}_2)$. *q.e.d.*

Corollary 3.2.1 *If* $\left((h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M \right) \right)$ *is the pull-back Lie algebroid associated to the generalized Lie algebroid* $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta) \right)$ *and* d^{h^*F} *is the exterior differentiation operator for the exterior differential* $\mathcal{F}(M)$*-algebra* $(\Lambda(h^*F, h^*\nu, M), +, \cdot, \wedge)$*, then we obtain the following structure equations of Maurer-Cartan type*

$$d^{h^*F} T^\alpha = -\frac{1}{2} \left(L_{\beta\gamma}^\alpha \circ h \right) T^\beta \wedge T^\gamma,\ \alpha \in \overline{1,p} \tag{$\mathcal{C}_1'$}$$

and

$$d^{h^*F} x^i = \left(\rho_\alpha^i \circ h \right) T^\alpha,\ i \in \overline{1,m}. \tag{$\mathcal{C}_2'$}$$

This equations will be called *the structure equations of Maurer-Cartan type associated to the pull-back Lie algebroid*

$$\left((h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M \right) \right).$$

Theorem 3.2.10 (of Cartan type) *Let* (E,π,M) *be an IDS of the generalized Lie algebroid* $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta) \right)$. *If* $\left\{ \Theta^{r+1}, ..., \Theta^p \right\}$ *is a base for the* $\mathcal{F}(M)$*-submodule* $\left(\Gamma\left(E^\perp, \pi^\perp, M \right), +, \cdot \right)$*, then the IDS* (E,π,M) *is involutive if and only if it exists*

$$\Omega_\beta^\alpha \in \Lambda^1 (h^*F, h^*\nu, M),\ \alpha,\beta \in \overline{r+1,p}$$

such that

$$d^{h^*F} \Theta^\alpha = \Sigma_{\beta \in \overline{r+1,p}} \Omega_\beta^\alpha \wedge \Theta^\beta \in \mathcal{I}\left(\Gamma\left(E^\perp, \pi^\perp, M \right) \right).$$

Proof: Let $\{S_1, ..., S_r\}$ be a base for the $\mathcal{F}(M)$-submodule $(\Gamma(E,\pi,M), +, \cdot)$
Let $\{S_{r+1}, ..., S_p\} \in \Gamma(h^*F, h^*\nu, M)$ such that

$$\{S_1, ..., S_r, S_{r+1}, ..., S_p\}$$

is a base for the $\mathcal{F}(M)$-module

$$\left(\Gamma\left(h^{*}F, h^{*}\nu, M\right), +, \cdot\right).$$

Let $\Theta^{1}, ..., \Theta^{r} \in \Gamma\left(\overset{*}{h}F, \overset{*}{h}\nu, M\right)$ such that

$$\left\{\Theta^{1}, ..., \Theta^{r}, \Theta^{r+1}, ..., \Theta^{p}\right\}$$

is a base for the $\mathcal{F}(M)$-module

$$\left(\Gamma\left(\overset{*}{h}F, \overset{*}{h}\nu, M\right), +, \cdot\right).$$

For any $a, b \in \overline{1, r}$ and $\alpha, \beta \in \overline{r+1, p}$, we have the equalities:

$$\begin{array}{rcl}\Theta^{a}\left(S_{b}\right) & = & \delta_{b}^{a} \\ \Theta^{a}\left(S_{\beta}\right) & = & 0 \\ \Theta^{\alpha}\left(S_{b}\right) & = & 0 \\ \Theta^{\alpha}\left(S_{\beta}\right) & = & \delta_{\beta}^{\alpha}\end{array}$$

We remark that the set of the 2-forms

$$\left\{\Theta^{a} \wedge \Theta^{b}, \Theta^{a} \wedge \Theta^{\beta}, \Theta^{\alpha} \wedge \Theta^{\beta},\ a, b \in \overline{1, r} \wedge \alpha, \beta \in \overline{r+1, p}\right\}$$

is a base for the $\mathcal{F}(M)$-module

$$\left(\Lambda^{2}\left(h^{*}F, h^{*}\nu, M\right), +, \cdot\right).$$

Therefore, we have

$$d^{h^{*}F}\Theta^{\alpha} = \Sigma_{b<c}A_{bc}^{\alpha}\Theta^{b} \wedge \Theta^{c} + \Sigma_{b,\gamma}B_{b\gamma}^{\alpha}\Theta^{b} \wedge \Theta^{\gamma} + \Sigma_{\beta<\gamma}C_{\beta\gamma}^{\alpha}\Theta^{\beta} \wedge \Theta^{\gamma}, \tag{1}$$

where, $A_{bc}^{\alpha}, B_{b\gamma}^{\alpha}$ and $C_{\beta\gamma}^{\alpha},\ a, b, c \in \overline{1, r} \wedge \alpha, \beta, \gamma \in \overline{r+1, p}$ are real local functions such that $A_{bc}^{\alpha} = -A_{cb}^{\alpha}$ and $C_{\beta\gamma}^{\alpha} = -C_{\gamma\beta}^{\alpha}$.

Using the formula

(2)

$$d^{h^{*}F}\Theta^{\alpha}\left(S_{b}, S_{c}\right) = \Gamma\left(\overset{h^{*}F}{\rho}, Id_{M}\right)S_{b}\left(\Theta^{\alpha}\left(S_{c}\right)\right) - \Gamma\left(\overset{h^{*}F}{\rho}, Id_{M}\right)S_{c}\left(\Theta^{\alpha}\left(S_{b}\right)\right) - \Theta^{\alpha}\left(\left[S_{b}, S_{c}\right]_{h^{*}F}\right),$$

we obtain that

$$A_{bc}^{\alpha} = -\Theta^{\alpha}\left(\left[S_{b}, S_{c}\right]_{h^{*}F}\right),\ \forall\left(b, c \in \overline{1, r} \wedge \alpha \in \overline{r+1, p}\right). \tag{3}$$

We admit that (E, π, M) is an involutive IDS of the generalized Lie algebroid $\left(\left(F, \nu, N\right), [,]_{F,h}, (\rho, \eta)\right)$.

As

$$\left[S_{b}, S_{c}\right]_{h^{*}F} \in \Gamma\left(E, \pi, M\right),\ \forall b, c \in \overline{1, r}$$

it results that

$$\Theta^{\alpha}\left(\left[S_{b}, S_{c}\right]_{h^{*}F}\right) = 0,\ \forall\left(b, c \in \overline{1, r} \wedge \alpha \in \overline{r+1, p}\right).$$

Therefore,

$$A_{bc}^{\alpha} = 0,\ \forall\left(b, c \in \overline{1, r} \wedge \alpha \in \overline{r+1, p}\right)$$

and we obtain
$$\begin{array}{rcl} d^{h^{*}F}\Theta^{\alpha} & = & \Sigma_{b,\gamma}B_{b\gamma}^{\alpha}\Theta^{b}\wedge\Theta^{\gamma}+\frac{1}{2}C_{\beta\gamma}^{\alpha}\Theta^{\beta}\wedge\Theta^{\gamma}\\ & = & \left(B_{b\gamma}^{\alpha}\Theta^{b}+\frac{1}{2}C_{\beta\gamma}^{\alpha}\Theta^{\beta}\right)\wedge\Theta^{\gamma}. \end{array}$$

As
$$\Omega_{\gamma}^{\alpha}\overset{put}{=}B_{b\gamma}^{\alpha}\Theta^{b}+\frac{1}{2}C_{\beta\gamma}^{\alpha}\Theta^{\beta}\in\Lambda^{1}\left(h^{*}F,h^{*}\nu,M\right),\ \forall\alpha,\beta\in\overline{r+1,p}$$
it results the first implication.

Conversely, we admit that it exists
$$\Omega_{\beta}^{\alpha}\in\Lambda^{1}\left(h^{*}F,h^{*}\nu,M\right),\ \alpha,\beta\in\overline{r+1,p}$$
such that
$$d^{h^{*}F}\Theta^{\alpha}=\Sigma_{\beta\in\overline{r+1,p}}\Omega_{\beta}^{\alpha}\wedge\Theta^{\beta},\ \forall\alpha\in\overline{r+1,p}. \tag{4}$$

Using the affirmations $(1),(2)$ and (4) we obtain that
$$A_{bc}^{\alpha}=0,\ \forall\left(b,c\in\overline{1,r}\wedge\alpha\in\overline{r+1,p}\right).$$

Using the affirmation (3), we obtain
$$\Theta^{\alpha}\left(\left[S_{b},S_{c}\right]_{h^{*}F}\right)=0,\ \forall\left(b,c\in\overline{1,r}\wedge\alpha\in\overline{r+1,p}\right).$$

Therefore,
$$\left[S_{b},S_{c}\right]_{h^{*}F}\in\Gamma\left(E,\pi,M\right),\ \forall b,c\in\overline{1,r}.$$

Using the *Proposition 3.1.3.2*, we obtain the second implication. *q.e.d.*

Let $\left(\left(F',\nu',N'\right),[,]_{F',h'},\left(\rho',\eta'\right)\right)$ be an another generalized Lie algebroid.

Definition 3.2.8 For any morphism $\left(\varphi,\varphi_{0}\right)$ of $\left(\left(F,\nu,N\right),[,]_{F,h},\left(\rho,\eta\right)\right)$ source and $\left(\left(F',\nu',N'\right),[,]_{F',h'},\left(\rho',\eta'\right)\right)$ target we define the application
$$\begin{array}{ccc} \Lambda^{q}\left(F',\nu',N'\right) & \xrightarrow{\left(\varphi,\varphi_{0}\right)^{*}} & \Lambda^{q}\left(F,\nu,N\right)\\ \omega' & \longmapsto & \left(\varphi,\varphi_{0}\right)^{*}\omega' \end{array},$$
where
$$\left(\left(\varphi,\varphi_{0}\right)^{*}\omega'\right)\left(z_{1},...,z_{q}\right)=\omega'\left(\Gamma\left(\varphi,\varphi_{0}\right)\left(z_{1}\right),...,\Gamma\left(\varphi,\varphi_{0}\right)\left(z_{q}\right)\right),$$
for any $z_{1},...,z_{q}\in\Gamma\left(F,\nu,N\right)$.

Remark 3.2.5 It is remarked that the $\mathbf{B^v}$-morphism $(Th \circ \rho, h \circ \eta)$ is a **GLA**-morphism of

$$\left(\left(F, \nu, N\right), [,]_{F,h}, (\rho, \eta)\right)$$

source and

$$\left(\left(TN, \tau_N, N\right), [,]_{TN, Id_N}, (Id_{TN}, Id_N)\right)$$

target.

Moreover, for any $\tilde{\imath} \in \overline{1, n}$, we obtain

$$(Th \circ \rho, h \circ \eta)^* \left(d\varkappa^{\tilde{\imath}}\right) = d^F \varkappa^{\tilde{\imath}},$$

where d is the exterior differentiation operator associated to the exterior differential Lie $\mathcal{F}(N)$-algebra

$$\left(\Lambda\left(TN, \tau_N, N\right), +, \cdot, \wedge\right).$$

Theorem 3.2.11 *If* (φ, φ_0) *is a morphism of*

$$\left(\left(F, \nu, N\right), [,]_{F,h}, (\rho, \eta)\right)$$

source and

$$\left(\left(F', \nu', N'\right), [,]_{F',h'}, \left(\rho', \eta'\right)\right)$$

target, then the following affirmations are satisfied:

1. *For any* $\omega' \in \Lambda^q\left(F', \nu', N'\right)$ *and* $\theta' \in \Lambda^r\left(F', \nu', N'\right)$ *we obtain*

$$(\varphi, \varphi_0)^* \left(\omega' \wedge \theta'\right) = (\varphi, \varphi_0)^* \omega' \wedge (\varphi, \varphi_0)^* \theta'. \tag{3.2.13}$$

2. *For any* $z \in \Gamma\left(F, \nu, N\right)$ *and* $\omega' \in \Lambda^q\left(F', \nu', N'\right)$ *we obtain*

$$i_z \left((\varphi, \varphi_0)^* \omega'\right) = (\varphi, \varphi_0)^* \left(i_{\varphi(z)} \omega'\right). \tag{3.2.14}$$

3. *If* $N = N'$ *and*

$$(Th \circ \rho, h \circ \eta) = \left(Th' \circ \rho', h' \circ \eta'\right) \circ (\varphi, \varphi_0),$$

then we obtain

$$(\varphi, \varphi_0)^* \circ d^{F'} = d^F \circ (\varphi, \varphi_0)^*. \tag{3.2.15}$$

Proof.

1. Let $\omega' \in \Lambda^q\left(F', \nu', N'\right)$ and $\theta' \in \Lambda^r\left(F', \nu', N'\right)$ be arbitrary. Since

$$\begin{array}{l}
(\varphi, \varphi_0)^* \left(\omega' \wedge \theta'\right)(z_1, ..., z_{q+r}) = \left(\omega' \wedge \theta'\right)\left(\Gamma(\varphi, \varphi_0) z_1, ..., \Gamma(\varphi, \varphi_0) z_{q+r}\right) \\
\quad = \dfrac{1}{(q+r)!} \sum\limits_{\sigma \in \Sigma_{q+r}} sgn(\sigma) \cdot \omega'\left(\Gamma(\varphi, \varphi_0) z_1, ..., \Gamma(\varphi, \varphi_0) z_q\right) \\
\qquad\qquad \cdot \theta'\left(\Gamma(\varphi, \varphi_0) z_{q+1}, ..., \Gamma(\varphi, \varphi_0) z_{q+r}\right) \\
\quad = \dfrac{1}{(q+r)!} \sum\limits_{\sigma \in \Sigma_{q+r}} sgn(\sigma) \cdot (\varphi, \varphi_0)^* \omega'(z_1, ..., z_q)\, (\varphi, \varphi_0)^* \theta'(z_{q+1}, ..., z_{q+r}) \\
\quad = \left((\varphi, \varphi_0)^* \omega' \wedge (\varphi, \varphi_0)^* \theta'\right)(z_1, ..., z_{q+r}),
\end{array}$$

for any $z_1, ..., z_{q+r} \in \Gamma(F, \nu, N)$, it results the conclusion of affirmation 1.

2. Let $z \in \Gamma(F, \nu, N)$ and $\omega' \in \Lambda^q(F', \nu', N')$ be arbitrary. Since

$$\begin{aligned} i_z\left((\varphi, \varphi_0)^* \omega'\right)(z_2, ..., z_q) &= \omega'\left(\Gamma(\varphi, \varphi_0) z, \Gamma(\varphi, \varphi_0) z_2, ..., \Gamma(\varphi, \varphi_0) z_q\right) \\ &= i_{\Gamma(\varphi, \varphi_0) z}\omega'\left(\Gamma(\varphi, \varphi_0) z_2, ..., \Gamma(\varphi, \varphi_0) z_q\right) \\ &= (\varphi, \varphi_0)^* \left(i_{\Gamma(\varphi, \varphi_0) z}\omega'\right)(z_2, ..., z_q), \end{aligned}$$

for any $z_2, ..., z_q \in \Gamma(F, \nu, N)$, it results the conclusion of affirmation 2.

3. Let $\omega' \in \Lambda^q(F', \nu', N')$ and $z_0, ..., z_q \in \Gamma(F, \nu, N)$ be arbitrary. Since

$$\begin{aligned} &\left((\varphi, \varphi_0)^* d^{F'}\omega'\right)(z_0, ..., z_q) = \left(d^{F'}\omega'\right)\left(\Gamma(\varphi, \varphi_0) z_0, ..., \Gamma(\varphi, \varphi_0) z_q\right) \\ &= \sum_{i=0}^{q} (-1)^i \Gamma(Th' \circ \rho', h' \circ \eta')\left(\Gamma(\varphi, \varphi_0) z_i\right) \\ &\qquad \cdot\omega'\left(\left(\Gamma(\varphi, \varphi_0) z_0, \Gamma(\varphi, \varphi_0) z_1, ..., \Gamma\widehat{(\varphi, \varphi_0)} z_i, ..., \Gamma(\varphi, \varphi_0) z_q\right)\right) \\ &+ \sum_{0 \leq i < j} (-1)^{i+j} \cdot \omega'\Big(\Gamma(\varphi, \varphi_0)[z_i, z_j]_F, \Gamma(\varphi, \varphi_0) z_0, \Gamma(\varphi, \varphi_0) z_1, ..., \\ &\qquad\qquad \cdot\Gamma\widehat{(\varphi, \varphi_0)} z_i, ..., \Gamma\widehat{(\varphi, \varphi_0)} z_j, ..., \Gamma(\varphi, \varphi_0) z_q\Big) \end{aligned}$$

and

$$\begin{aligned} &d^F\left((\varphi, \varphi_0)^* \omega'\right)(z_0, ..., z_q) \\ &= \sum_{i=0}^{q} (-1)^i \Gamma(Th \circ \rho, h \circ \eta)(z_i) \cdot \left((\varphi, \varphi_0)^* \omega'\right)(z_0, ..., \widehat{z_i}, ..., z_q) \\ &+ \sum_{0 \leq i < j} (-1)^{i+j} \cdot \left((\varphi, \varphi_0)^* \omega'\right)\left([z_i, z_j]_{F,h}, z_0, ..., \widehat{z_i}, ..., \widehat{z_j}, ..., z_q\right) \\ &= \sum_{i=0}^{q} (-1)^i \Gamma(Th \circ \rho, h \circ \eta)(z_i) \cdot \omega'\left(\Gamma(\varphi, \varphi_0) z_0, ..., \Gamma\widehat{(\varphi, \varphi_0)} z_i, ..., \Gamma(\varphi, \varphi_0) z_q\right) \\ &+ \sum_{0 \leq i < j} (-1)^{i+j} \cdot \omega'\Big(\Gamma(\varphi, \varphi_0)[z_i, z_j]_{F,h}, \Gamma(\varphi, \varphi_0) z_0, \Gamma(\varphi, \varphi_0) z_1, ..., \\ &\Gamma\widehat{(\varphi, \varphi_0)} z_i, ..., \Gamma\widehat{(\varphi, \varphi_0)} z_j, ..., \Gamma(\varphi, \varphi_0) z_q\Big) \end{aligned}$$

it results the conclusion of affirmation 3. *q.e.d.*

Definition 3.2.9 For any $q \in \overline{1, n}$ we define

$$\mathcal{Z}^q(F, \nu, N) = \{\omega \in \Lambda^q(F, \nu, N) : d\omega = 0\},$$

the set of *closed differential exterior q-forms* and

$$\mathcal{B}^q(F, \nu, N) = \left\{\omega \in \Lambda^q(F, \nu, N) : \exists \eta \in \Lambda^{q-1}(F, \nu, N) \mid d\eta = \omega\right\},$$

the set of *exact differential exterior q-forms*.

3.2.1 Exterior Differential Systems

We consider a generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ given by the diagrams:

$$\begin{array}{ccccccc} & & F & \stackrel{\rho}{\longrightarrow} & TM & \stackrel{Th}{\longrightarrow} & TN \\ & & \downarrow\nu & & \downarrow\tau_M & & \downarrow\tau_N \\ M & \stackrel{h}{\longrightarrow} & N & \stackrel{\eta}{\longrightarrow} & M & \stackrel{h}{\longrightarrow} & N \end{array}$$

Let $\left((h^*F,h^*\nu,M),[,]_{h^*F},\left(\overset{h^*F}{\rho},Id_M\right)\right)$ be the pull-back Lie algebroid.

Definition 3.2.1.1 Any ideal $(\mathcal{I},+,\cdot)$ of the exterior differential algebra of the pull-back Lie algebroid $\left((h^*F,h^*\nu,M),[,]_{h^*F},\left(\overset{h^*F}{\rho},Id_M\right)\right)$ closed under differentiation operator d^{h^*F}, namely $d^{h^*F}\mathcal{I}\subseteq\mathcal{I}$, will be called *exterior differential system (EDS) of the generalized Lie algebroid* $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.

In particular, if $h=Id_N=\eta$, then any ideal $(\mathcal{I},+,\cdot)$ of the exterior differential algebra of the Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_M))$ closed under differentiation operator d^F , namely $d^F\mathcal{I}\subseteq\mathcal{I}$, will be called *exterior differential system (EDS) of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_M))$.

Theorem 3.2.1.1 *The IDS (E,π,M) of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is involutive, if and only if the ideal generated by the $\mathcal{F}(M)$-submodule $\left(\Gamma\left(E^{\perp},\pi^{\perp},M\right),+,\cdot\right)$ is an EDS of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.*

Proof: Let (E,π,M) be an involutive IDS of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.
Let $\left\{\Theta^{r+1},...,\Theta^p\right\}$ be a base for the $\mathcal{F}(M)$-submodule $\left(\Gamma\left(E^{\perp},\pi^{\perp},M\right),+,\cdot\right)$.
We know that

$$\mathcal{I}\left(\Gamma\left(E^{\perp},\pi^{\perp},M\right)\right)=\cup_{q\in\mathbb{N}}\left\{\Omega_\alpha\wedge\Theta^\alpha,\ \ \{\Omega_{r+1},...,\Omega_p\}\subset\Lambda^q(h^*F,h^*\nu,M)\right\}.$$

Let

$$S=\left\{q\in\mathbb{N}:d^{h^*F}(\Omega_\alpha\wedge\Theta^\alpha)\in\mathcal{I}\left(\Gamma\left(E^{\perp},\pi^{\perp},M\right)\right),\ \forall\{\Omega_{r+1},...,\Omega_p\}\subset\Lambda^q(h^*F,h^*\nu,M)\right\}.$$

Let $\{\Omega_{r+1},...,\Omega_p\}\subset\Lambda^0(h^*F,h^*\nu,M)$ be arbitrary.
Using the *Theorem 3.2.10*, we obtain

$$\begin{array}{rcl} d^{h^*F}(\Omega_\alpha\wedge\Theta^\alpha) & = & d^{h^*F}\Omega_\alpha\wedge\Theta^\alpha+(-1)^0\,\Omega_\alpha\wedge d^{h^*F}\Theta^\alpha \\ & = & \left(d^{h^*F}\Omega_\alpha+\Omega_\beta\cdot\Omega_\alpha^\beta\right)\wedge\Theta^\alpha. \end{array}$$

As

$$d^{h^*F}\Omega_\beta+\Omega_\alpha\cdot\Omega_\beta^\alpha\in\Lambda^1(h^*F,h^*\nu,M)$$

it results that

$$d^{h^*F}\left(\Omega_\beta\wedge\Theta^\beta\right)\in\mathcal{I}\left(\Gamma\left(E^{\perp},\pi^{\perp},M\right)\right)$$

Therefore,

$$0\in S. \tag{1}$$

In the following, we prove that

$$(2) \qquad q\in S\Longrightarrow q+1\in S.$$

Let $\{\Omega_{r+1},...,\Omega_p\}\subset\Lambda^{q+1}(h^*F,h^*\nu,M)$ be arbitrary.
Using the *Theorem 3.2.10*, we obtain

$$\begin{array}{rcl} d^{h^*F}(\Omega_\alpha\wedge\Theta^\alpha) & = & d^{h^*F}\Omega_\alpha\wedge\Theta^\alpha+(-1)^{q+1}\Omega_\beta\wedge d^{h^*F}\Theta^\beta \\ & = & \left(d^{h^*F}\Omega_\alpha+(-1)^{q+1}\Omega_\beta\wedge\Omega_\alpha^\beta\right)\wedge\Theta^\alpha. \end{array}$$

As

$$d^{h^*F}\Omega_\alpha+(-1)^{q+1}\Omega_\beta\wedge\Omega_\alpha^\beta\in\Lambda^{q+2}(h^*F,h^*\nu,M)$$

it results that

$$d^{h^*F}\left(\Omega_\beta\wedge\Theta^\beta\right)\in\mathcal{I}\left(\Gamma\left(E^\perp,\pi^\perp,M\right)\right)$$

Therefore,

$$q+1\in S.$$

Using the **Peano's Axiom** and the affirmations (1) and (2), it results that $S=\mathbb{N}$.
Therefore,

$$d^{h^*F}\mathcal{I}\left(\Gamma\left(E^\perp,\pi^\perp,M\right)\right)\subseteq\mathcal{I}\left(\Gamma\left(E^\perp,\pi^\perp,M\right)\right).$$

Conversely, let (E,π,M) be an IDS of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ such that the $\mathcal{F}(M)$-submodule $\left(\Gamma\left(E^\perp,\pi^\perp,M\right),+,\cdot\right)$ is an EDS of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$, namely

$$(3) \qquad d^{h^*F}\mathcal{I}\left(\Gamma\left(E^\perp,\pi^\perp,M\right)\right)\subseteq\mathcal{I}\left(\Gamma\left(E^\perp,\pi^\perp,M\right)\right).$$

Let $\left\{\Theta^{r+1},...,\Theta^p\right\}$ be a base for the $\mathcal{F}(M)$-submodule $\left(\Gamma\left(E^\perp,\pi^\perp,M\right),+,\cdot\right)$.
Using the affirmation (3), it results that it exists

$$\Omega_\beta^\alpha\in\Lambda^1(h^*F,h^*\nu,M),\ \alpha,\beta\in\overline{r+1,p}$$

such that

$$d^{h^*F}\Theta^\alpha=\Sigma_{\beta\in\overline{r+1,p}}\Omega_\beta^\alpha\wedge\Theta^\beta\in\mathcal{I}\left(\Gamma\left(E^\perp,\pi^\perp,M\right)\right).$$

Using the *Theorem 3.2.10*, it results that (E,π,M) is an involutive IDS of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$. *q.e.d.*

3.3 The generalized tangent bundle

We consider the following diagram:

$$(3.3.1) \qquad \begin{array}{ccc} E & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \pi\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array}$$

where (E, π, M) is a fiber bundle and $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right)$ is a generalized Lie algebroid.

We assume that the r-dimensional manifold $\mathbf{V}$ is the type fibre and the Lie group $(\mathbf{G}, \cdot)$ is the structure group for the fiber bundle (E, π, M).

We take $\left(x^i, y^a\right)$ as canonical local coordinates on (E, π, M), where $i \in \overline{1, m}$ and $a \in \overline{1, r}$.

Let

$$\left(x^i, y^a\right) \longrightarrow \left(x^{\acute{i}}\left(x^i\right), y^{\acute{a}}\left(x^i, y^a\right)\right)$$

be a change of coordinates on (E, π, M). Then the coordinates y^a change to $y^{\acute{a}}$ by the rule:

$$y^{\acute{a}} = \frac{\partial y^{\acute{a}}}{\partial y^a} y^a. \tag{3.3.2}$$

In particular, if (E, π, M) is vector bundle, then the coordinates y^a change to $y^{\acute{a}}$ by the rule:

$$y^{\acute{a}} = M_a^{\acute{a}} y^a. \tag{3.3.2'}$$

Let

$$\left(h^* F, h^* \nu, M\right), [,]_{h^* F}, \left(\overset{h^* F}{\rho}, Id_M\right)$$

be the pull-back Lie algebroid of the generalized Lie algebroid

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right).$$

Let

$$\left(\pi^*\left(h^* F\right), \pi^*\left(h^* \nu\right), E\right), [,]_{\pi^*\left(h^* F\right)}, \left(\overset{\pi^*\left(h^* F\right)}{\rho}, Id_E\right)$$

be the pull-back Lie algebroid of the Lie algebroid

$$\left(h^* F, h^* \nu, M\right), [,]_{h^* F}, \left(\overset{h^* F}{\rho}, Id_M\right).$$

If

$$z = z^\alpha t_\alpha \in \Gamma(F, \nu, N),$$

then, using the vector fibred $(m+r)+p$-structure $\left[\widetilde{\mathcal{AF}}_{\pi^*\left(h^* F\right)}\right]$, we obtain the section

$$\tilde{Z} = \left(z^\alpha \circ h \circ \pi\right) \tilde{T}_\alpha \in \Gamma\left(\pi^*\left(h^* F\right), \pi^*\left(h^* \nu\right), E\right)$$

such that

$$\tilde{Z}\left(u_x\right) = z(h(x)),$$

for any $u_x \in \pi^{-1}\left(U \cap h^{-1} V\right)$.

The set $\left\{\tilde{T}_\alpha,\ \alpha \in \overline{1, p}\right\}$ is a base for the module of sections

$$\left(\Gamma\left(\pi^*\left(h^* F\right), \pi^*\left(h^* \nu\right), E\right), +, \cdot\right).$$

Since $t_{\acute{\alpha}} = \Lambda_{\acute{\alpha}}^{\alpha} t_\alpha$, it results that

$$\tilde{T}_{\acute{\alpha}} = \Lambda_{\acute{\alpha}}^{\alpha} \circ (h \circ \pi) \tilde{T}_\alpha. \tag{3.3.3}$$

Therefore,

$$\left\| \Lambda_{\alpha}^{\acute{\alpha}} \circ (h \circ \pi) \right\| \tag{3.3.4}$$

is the matrix of coordinate transformation on $(\pi^* (h^* F), \pi^* (h^* \nu), E)$.

Let $\left(\overset{\pi^*(h^*F)}{\rho}, Id_E\right)$ be the $\mathbf{B^v}$-morphism of $(\pi^* (h^* F), \pi^* (h^* \nu), E)$ source and (TE, τ_E, E) target, where

$$\begin{array}{ccc} \pi^* (h^* F) & \xrightarrow{\overset{\pi^*(h^*F)}{\rho}} & TE \\ \tilde{Z}^{\alpha} \tilde{T}_{\alpha} (u_x) & \longmapsto & \tilde{Z}^{\alpha} \cdot \left(\rho_{\alpha}^{i} \circ h \circ \pi\right) \cdot \dfrac{\partial}{\partial x^i} (u_x). \end{array} \tag{3.3.5}$$

Using the operation

$$\Gamma (\pi^* (h^* F), \pi^* (h^* \nu), E)^2 \xrightarrow{[,]_{\pi^*(h^*F)}} \Gamma (\pi^* (h^* F), \pi^* (h^* \nu), E)$$

defined by

$$\begin{array}{ll} \left[\tilde{T}_{\alpha}, \tilde{T}_{\beta}\right]_{\pi^*(h^*F)} & = \left(L_{\alpha\beta}^{\gamma} \circ h \circ \pi\right) \tilde{T}_{\gamma}, \\ \left[\tilde{T}_{\alpha}, f\tilde{T}_{\beta}\right]_{\pi^*(h^*F)} & = f \left(L_{\alpha\beta}^{\gamma} \circ h \circ \pi\right) \tilde{T}_{\gamma} + \left(\rho_{\alpha}^{i} \circ h \circ \pi\right) \dfrac{\partial f}{\partial x^i} \tilde{T}_{\beta}, \\ \left[f\tilde{T}_{\alpha}, \tilde{T}_{\beta}\right]_{\pi^*(h^*F)} & = - \left[\tilde{T}_{\beta}, f\tilde{T}_{\alpha}\right]_{\pi^*(h^*F)}, \end{array} \tag{3.3.6}$$

for any $f \in \mathcal{F}(E)$, we obtain the following

Theorem 3.3.1 *The couple*

$$\left([,]_{\pi^*(h^*F)}, \left(\overset{\pi^*(h^*F)}{\rho}, Id_E\right)\right)$$

is a Lie algebroid structure for the vector bundle $(\pi^* (h^* F), \pi^* (h^* \nu), E)$.

It is known that the tangent bundle (TE, τ_E, E) is a vector bundle with type fibre the real space $(\mathbb{R}^{m+r}, +, \cdot)$ and structure group the Lie group $\mathbf{GL}(m + r, \mathbb{R})$.

Theorem 3.3.2 *The set*

$$\pi^* (h^* F) \oplus TE = \bigcup_{u \in E} \pi^* (h^* F)_u \oplus (TE)_u \tag{3.3.7}$$

is the total space of a vector bundle with the base E*, canonical projection denoted* $\overset{\oplus}{\pi}$*, type fibre the real space* $\left(\mathbb{R}^{p+(m+r)}, +, \cdot\right)$ *and structure group, a Lie subgroup of* $(\mathbf{GL}(p + (m + r), \mathbb{R}), \cdot)$.

Let

$$\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right)$$

be the natural base for the sections Lie algebra $(\Gamma (TE, \tau_E, E), +, \cdot, [,]_{TE})$.

Remark 3.3.1 The sections

$$(3.3.8)\qquad \left(\tilde{T}_\alpha, \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right)\right)$$

determined the bases for the module $\Gamma\left(\pi^*\left(h^*F\right)\oplus TE, \overset{\oplus}{\pi}, E\right)$.

The matrix of coordinate transformation on

$$\left(\pi^*\left(h^*F\right)\oplus TE, \overset{\oplus}{\pi}, E\right)$$

at a change of fibred charts is

$$(3.3.9)\qquad \left\|\begin{array}{ccc} \Lambda_\alpha^{\acute{\alpha}}\circ h\circ\pi & 0 & 0 \\ 0 & \dfrac{\partial x^{\acute{i}}}{\partial x_i}\circ\pi & 0 \\ 0 & \dfrac{\partial y^{\acute{a}}}{\partial x^i} & \dfrac{\partial y^{\acute{a}}}{\partial y^a} \end{array}\right\|.$$

In particular, if (E,π,M) is a vector bundle, then we consider that the local coordinates on (E,π,M) is changed by:

$$\left(x^i, y^a\right)\longrightarrow\left(x^{\acute{i}}\left(x^i\right), y^{\acute{a}} = M_a^{\acute{a}}\left(x^i\right)y^a\right).$$

Then the matrix of coordinate transformation on

$$\left(\pi^*\left(h^*F\right)^* F\oplus TE, \overset{\oplus}{\pi}, E\right)$$

at a change of fibred charts is

$$(3.3.10)\qquad \left\|\begin{array}{ccc} \Lambda_\alpha^{\acute{\alpha}}\circ h\circ\pi & 0 & 0 \\ 0 & \dfrac{\partial x^{\acute{i}}}{\partial x_i}\circ\pi & 0 \\ 0 & \dfrac{\partial M_a^{\acute{a}}\circ\pi}{\partial x^i}y^a & M_a^{\acute{a}}\circ\pi \end{array}\right\|.$$

For any sections

$$\tilde{Z}^\alpha\tilde{T}_\alpha\in\Gamma\left(\pi^*\left(h^*F\right), \pi^*\left(h^*F\right), E\right)$$

and

$$Y^a\frac{\partial}{\partial y^a}\in\Gamma\left(VTE, \tau_E, E\right)$$

we construct the section

$$\begin{aligned} \tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y^a\frac{\partial}{\partial\tilde{y}^a} &=: \tilde{Z}^\alpha\left(\tilde{T}_\alpha\oplus\left(\rho_\alpha^i\circ h\circ\pi\right)\frac{\partial}{\partial x^i}\right)+Y^a\left(0_{\pi^*(h^*F)}\oplus\frac{\partial}{\partial y^a}\right) \\ &= \tilde{Z}^\alpha\tilde{T}_\alpha\oplus\left(\tilde{Z}^\alpha\left(\rho_\alpha^i\circ h\circ\pi\right)\frac{\partial}{\partial x^i}+Y^a\frac{\partial}{\partial y^a}\right)\in\Gamma\left(\pi^*\left(h^*F\right)\oplus TE, \overset{\oplus}{\pi}, E\right). \end{aligned}$$

Since we have

$$\begin{gathered} \tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y^a\frac{\partial}{\partial\tilde{y}^a}=0 \\ \Updownarrow \\ \tilde{Z}^\alpha\tilde{T}_\alpha=0\ \wedge\tilde{Z}^\alpha\left(\rho_\alpha^i\circ h\circ\pi\right)\frac{\partial}{\partial x^i}+Y^a\frac{\partial}{\partial y^a}=0, \end{gathered}$$

it implies $\tilde{Z}^\alpha = 0,\ \alpha \in \overline{1,p}$ and $Y^a = 0,\ a \in \overline{1,r}$.

Therefore the sections $\frac{\partial}{\partial \tilde{z}^1}, ..., \frac{\partial}{\partial \tilde{z}^p}, \frac{\partial}{\partial \tilde{y}^1}, ..., \frac{\partial}{\partial \tilde{y}^r}$ are linearly independent.

We consider the vector subbundle $((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E)$ of the vector bundle $\left(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E\right)$, for which the $\mathcal{F}(E)$-module of sections is the $\mathcal{F}(E)$-submodule of $\left(\Gamma\left(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E\right), +, \cdot\right)$, generated by the family of sections $\left(\frac{\partial}{\partial \tilde{z}^\alpha}, \frac{\partial}{\partial \tilde{y}^a}\right)$.

The base sections

$$\left(\frac{\partial}{\partial \tilde{z}^\alpha}, \frac{\partial}{\partial \tilde{y}^a}\right) \overset{put}{=} \left(\tilde{\partial}_\alpha, \dot{\tilde{\partial}}_a\right) \tag{3.3.11}$$

will be called the *natural* (ρ,η)*-base.*

The matrix of coordinate transformation on $((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E)$ at a change of fibred charts is

$$\left\| \begin{array}{cc} \Lambda_\alpha^{\acute{\alpha}} \circ h \circ \pi & 0 \\ \left(\rho_\alpha^i \circ h \circ \pi\right) \frac{\partial y^{\acute{a}}}{\partial x^i} & \frac{\partial y^{\acute{a}}}{\partial y^a} \end{array} \right\|. \tag{3.3.12}$$

In particular, if (E,π,M) is a vector bundle, then the matrix of coordinate transformation on $((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E)$ at a change of fibred charts is

$$\left\| \begin{array}{cc} \Lambda_\alpha^{\acute{\alpha}} \circ h \circ \pi & 0 \\ \left(\rho_a^i \circ h \circ \pi\right) \frac{\partial M_b^{\acute{a}} \circ \pi}{\partial x_i} y^b & M_a^{\acute{a}} \circ \pi \end{array} \right\|. \tag{3.3.13}$$

Next, we consider the operation

$$\Gamma\left((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E\right)^2 \xrightarrow{[,]_{(\rho,\eta)TE}} \Gamma\left((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E\right)$$

defined by

$$\begin{array}{l} \left[\left(\tilde{Z}_1^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y_1^a \frac{\partial}{\partial \tilde{y}^a}\right), \left(\tilde{Z}_2^\beta \frac{\partial}{\partial \tilde{z}^\beta} + Y_2^b \frac{\partial}{\partial \tilde{y}^b}\right)\right]_{(\rho,\eta)TE} \\ = \left[\tilde{Z}_1^\alpha \tilde{T}_a, \tilde{Z}_2^\beta \tilde{T}_\beta\right]_{\pi^*(h^*F)} \oplus \left[\left(\rho_\alpha^i \circ h \circ \pi\right) \tilde{Z}_1^\alpha \frac{\partial}{\partial x^i} + Y_1^a \frac{\partial}{\partial y^a},\right. \\ \qquad\qquad \left.\left(\rho_\beta^i \circ h \circ \pi\right) \tilde{Z}_2^\beta \frac{\partial}{\partial x^i} + Y_2^b \frac{\partial}{\partial y^b}\right]_{TE}, \end{array} \tag{3.3.14}$$

for any $\left(\tilde{Z}_1^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y_1^a \frac{\partial}{\partial \tilde{y}^a}\right)$ and $\left(\tilde{Z}_2^\beta \frac{\partial}{\partial \tilde{z}^\beta} + Y_2^b \frac{\partial}{\partial \tilde{y}^b}\right)$.

Let $(\tilde{\rho}, Id_E)$ be the $\mathbf{B^v}$-morphism of $((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E)$ source and (TE, τ_E, E) target, where

$$\begin{array}{c} (\rho,\eta)\,TE \xrightarrow{\tilde{\rho}} TE \\ \left(\tilde{Z}^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y^a \frac{\partial}{\partial \tilde{y}^a}\right)(u_x) \longmapsto \left(\tilde{Z}^\alpha \left(\rho_\alpha^i \circ h \circ \pi\right) \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial y^a}\right)(u_x) \end{array} \tag{3.3.15}$$

Lemma 3.3.1 *The operation $[,]_{(\rho,\eta)TE}$ is a Lie bracket, namely it satisfies*

$$\left[\tilde{U}, f\tilde{Z}\right]_{(\rho,\eta)TE} = f\left[\tilde{U},\tilde{Z}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\tilde{U}\right)(f)\,\tilde{Z} \tag{3.3.16}$$

for any $\tilde{U},\tilde{Z}\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$ and $f\in F\left(E\right)$.

Proof. For any $f\in\mathcal{F}\left(E\right)$, we obtain:

$$\begin{array}{ll}
 & \left[\frac{\partial}{\partial\tilde{z}^{\alpha}}, f\frac{\partial}{\partial\tilde{z}^{\beta}}\right]_{(\rho,\eta)TE} = \left[T_{\alpha}, fT_{\beta}\right]_{\pi^*(h^*F)} \oplus \left[\rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial}{\partial x^{i}}, f\cdot\left(\rho_{\beta}^{j}\circ h\circ\pi\right)\frac{\partial}{\partial x^{j}}\right]_{TE} \\
 & = \left(f\left[\tilde{T}_{\alpha},\tilde{T}_{\beta}\right]_{\pi^*(h^*F)} + \Gamma\left(\overset{\pi^*(h^*F)}{\rho}, Id_E\right)\left(\tilde{T}_{\alpha}\right)f\cdot\tilde{T}_{\beta}\right) \\
 & \quad\oplus\left(f\left[\rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial}{\partial x^{i}},\rho_{\beta}^{j}\circ h\circ\pi\frac{\partial}{\partial x^{j}}\right]_{TE}\right. \\
(1) & \left.+\Gamma\left(Id_{TE}, Id_E\right)\left(\rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial}{\partial x^{i}}\right)f\cdot\rho_{\beta}^{j}\circ h\circ\pi\frac{\partial}{\partial x^{j}}\right) \\
 & = f\left(\left[\tilde{T}_{\alpha},\tilde{T}_{\beta}\right]_{\pi^*(h^*F)}\oplus\left[\rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial}{\partial x^{i}},\rho_{\beta}^{j}\circ h\circ\pi\frac{\partial}{\partial x^{j}}\right]_{TE}\right) \\
 & +\rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial f}{\partial x^{i}}\left(\tilde{T}_{\beta}\oplus\rho_{\beta}^{j}\circ h\circ\pi\frac{\partial}{\partial x^{j}}\right) \\
 & = f\left[\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{z}^{\beta}}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\frac{\partial}{\partial\tilde{z}^{\alpha}}\right)f\cdot\frac{\partial}{\partial\tilde{z}^{\beta}};
\end{array}$$

$$\begin{array}{ll}
 & \left[\frac{\partial}{\partial\tilde{z}^{\alpha}}, f\frac{\partial}{\partial\tilde{y}^{b}}\right]_{(\rho,\eta)TE} = \left[\tilde{T}_{\alpha},0\right]_{\pi^*(h^*F)}\oplus\left[\rho_{\alpha}^{i}\circ h\circ\pi\tfrac{\partial}{\partial x^{i}}, f\frac{\partial}{\partial y^{b}}\right]_{TE} \\
 & = 0_{\pi^*(h^*F)}\oplus\left(f\left[\rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial}{\partial x^{i}},\frac{\partial}{\partial y^{b}}\right]_{TE} + \rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial f}{\partial x^{i}}\frac{\partial}{\partial y^{b}}\right) \\
 & = 0_{\pi^*(h^*F)}\oplus\left(-f\frac{\partial\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)}{\partial y^{b}}\frac{\partial}{\partial x^{i}} + \rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial f}{\partial x^{i}}\frac{\partial}{\partial y^{b}}\right) \\
(2) & \overset{(2.2.3)}{=} 0_{\pi^*(h^*F)}\oplus\left(0_{TE} + \rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial f}{\partial x^{i}}\frac{\partial}{\partial y^{b}}\right) \\
 & = 0_{\pi^*(h^*F)}\oplus\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\frac{\partial f}{\partial x^{i}}\frac{\partial}{\partial y^{b}} \\
 & = \left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\frac{\partial f}{\partial x^{i}}\left(0_{\pi^*(h^*F)}\oplus\frac{\partial}{\partial y^{b}}\right) \\
 & = f\left[\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^{b}}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\frac{\partial}{\partial\tilde{z}^{\alpha}}\right)f\frac{\partial}{\partial\tilde{y}^{b}};
\end{array}$$

$$\left[\frac{\partial}{\partial \tilde{y}^a}, f\frac{\partial}{\partial \tilde{z}^\beta}\right]_{(\rho,\eta)TE} = \left[0, \tilde{T}_\beta\right]_{\pi^*(h^*F)} \oplus \left[\tfrac{\partial}{\partial y^a}, f\left(\rho_\beta^j \circ h \circ \pi\right)\tfrac{\partial}{\partial x^j}\right]_{TE}$$

$$= 0_{\pi^*(h^*F)} \oplus \left(f\left[\frac{\partial}{\partial y^a}, \left(\rho_\beta^j \circ h \circ \pi\right)\frac{\partial}{\partial x^j}\right]_{TE} + \frac{\partial f}{\partial y^a}\left(\rho_\beta^j \circ h \circ \pi\right)\frac{\partial}{\partial x^j}\right)$$

$$= 0_{\pi^*(h^*F)} \oplus \left(f\left(\rho_\beta^j \circ h \circ \pi\right)\left[\frac{\partial}{\partial y^a}, \frac{\partial}{\partial x^j}\right]_{TE}\right.$$

(3)

$$\left. + f\frac{\partial\left(\rho_\beta^j \circ h \circ \pi\right)}{\partial y^a}\frac{\partial}{\partial x^j} + \frac{\partial f}{\partial y^a}\left(\rho_\beta^j \circ h \circ \pi\right)\frac{\partial}{\partial x^j}\right)$$

$$\overset{(2.2.3)}{=} 0_{\pi^*(h^*F)} \oplus \left(0_{TE} + 0_{TE} + \frac{\partial f}{\partial y^a}\left(\rho_\beta^j \circ h \circ \pi\right)\frac{\partial}{\partial x^j}\right)$$

$$= f\left[\frac{\partial}{\partial \tilde{y}^a}, \frac{\partial}{\partial \tilde{z}^\beta}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\frac{\partial}{\partial \tilde{y}^a}\right) f\frac{\partial}{\partial \tilde{z}^\beta};$$

$$\left[\frac{\partial}{\partial \tilde{y}^a}, f\frac{\partial}{\partial \tilde{y}^b}\right]_{(\rho,\eta)TE} = [0,0]_{\pi^*(h^*F)} \oplus \left[\tfrac{\partial}{\partial y^a}, f\tfrac{\partial}{\partial y^b}\right]_{TE}$$

$$= 0_{\pi^*(h^*F)} \oplus \left(f\left[\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right]_{TE} + \frac{\partial f}{\partial y^a}\frac{\partial}{\partial y^b}\right)$$

(4)

$$= 0_{\pi^*(h^*F)} \oplus \left(0_{TE} + \frac{\partial f}{\partial y^a}\frac{\partial}{\partial y^b}\right) = \left(0_{\pi^*(h^*F)} \oplus \frac{\partial f}{\partial y^a}\frac{\partial}{\partial y^b}\right)$$

$$= \frac{\partial f}{\partial y^a}\frac{\partial}{\partial \tilde{y}^b} = f\left[\frac{\partial}{\partial \tilde{y}^a}, \frac{\partial}{\partial \tilde{y}^b}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\frac{\partial}{\partial \tilde{y}^a}\right) f \cdot \frac{\partial}{\partial \tilde{y}^b}.$$

In general, after some calculations, we obtain

$$\left[\tilde{U}, f\tilde{Z}\right]_{(\rho,\eta)TE} = f\left[\tilde{U}, \tilde{Z}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\tilde{U}\right)(f)\,\tilde{Z} \tag{5}$$

for any $\tilde{U}, \tilde{Z} \in \Gamma\left(\left(\rho,\eta\right)TE, \left(\rho,\eta\right)\tau_E, E\right)$ and $f \in \mathcal{F}(E)$. *q.e.d.*

Lemma 3.3.2 *For any* $\tilde{U}, \tilde{V} \in \Gamma\left(\left(\rho,\eta\right)TE, \left(\rho,\eta\right)\tau_E, E\right)$, *we have:*

$$\left[\tilde{U}, \tilde{V}\right]_{(\rho,\eta)TE} = -\left[\tilde{V}, \tilde{U}\right]_{(\rho,\eta)TE}. \tag{3.3.17}$$

In particular, we obtain:

$$\left[\tilde{U}, \tilde{U}\right]_{(\rho,\eta)TE} = 0_{(\rho,\eta)TE},\ \forall \tilde{U} \in \Gamma\left(\left(\rho,\eta\right)TE, \left(\rho,\eta\right)\tau_E, E\right). \tag{3.3.18}$$

Proof. Using relations (3.1.1.9) and (3.1.2.2), we obtain

$$\begin{array}{l}\left(L_{\alpha\beta}^\gamma \circ h \circ \pi\right)\left(\rho_\gamma^k \circ h \circ \pi\right)\\ \qquad = \left(\rho_\alpha^i \circ h \circ \pi\right)\dfrac{\partial\left(\rho_\beta^k \circ h \circ \pi\right)}{\partial x^i} - \left(\rho_\beta^j \circ h \circ \pi\right)\dfrac{\partial\left(\rho_\alpha^k \circ h \circ \pi\right)}{\partial x^j}.\end{array} \tag{1}$$

Using relation (1), we obtain

$$
\begin{array}{ll}
 & \left[\frac{\partial}{\partial \tilde{z}^{\alpha}},\frac{\partial}{\partial \tilde{z}^{\beta}}\right]_{(\rho,\eta)TE}= \\
 & =\left(\left[\tilde{T}_{\alpha},\tilde{T}_{\beta}\right]_{\pi^*(h^*F)}\oplus\left[\rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial}{\partial x^{i}},\rho_{\beta}^{j}\circ h\circ\pi\frac{\partial}{\partial x^{j}}\right]_{TE}\right) \\
(2) & =L_{\alpha\beta}^{\gamma}\circ(h\circ\pi)\,\tilde{T}_{\gamma}\oplus\left(\rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial\left(\rho_{\beta}^{k}\circ h\circ\pi\right)}{\partial x^{i}}-\rho_{\beta}^{j}\circ h\circ\pi\frac{\partial\left(\rho_{\alpha}^{k}\circ h\circ\pi\right)}{\partial x^{j}}\right)\frac{\partial}{\partial x^{k}} \\
 & =L_{\alpha\beta}^{\gamma}\circ(h\circ\pi)\left(\tilde{T}_{\gamma}\oplus\rho_{\gamma}^{k}\circ h\circ\pi\frac{\partial}{\partial x^{k}}\right)=L_{\alpha\beta}^{\gamma}\circ(h\circ\pi)\,\frac{\partial}{\partial\tilde{z}^{\gamma}}.
\end{array}
$$

Moreover, we obtain

$$
\begin{array}{ll}
(3) & \begin{aligned}\left[\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^{b}}\right]_{(\rho,\eta)TE}&=\left[\tilde{T}_{\alpha},0\right]_{\pi^*(h^*F)}\oplus\left[\rho_{\alpha}^{i}\circ h\circ\pi\tfrac{\partial}{\partial x^{i}},\tfrac{\partial}{\partial y^{b}}\right]_{TE}\\ &=0_{\pi^*(h^*F)}\oplus\frac{-\partial\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)}{\partial y^{b}}\frac{\partial}{\partial x^{i}}=0_{\pi^*(h^*F)}\oplus 0_{TE};\end{aligned} \\
(4) & \begin{aligned}\left[\frac{\partial}{\partial\tilde{y}^{a}},\frac{\partial}{\partial\tilde{z}^{\beta}}\right]_{(\rho,\eta)TE}&=\left[0,\tilde{T}_{\beta}\right]_{\pi^*(h^*F)}\oplus\left[\frac{\partial}{\partial y^{a}},\rho_{\beta}^{j}\circ h\circ\pi\frac{\partial}{\partial x^{j}}\right]_{TE}\\ &=0_{\pi^*(h^*F)}\oplus\frac{\partial\left(\rho_{\beta}^{j}\circ h\circ\pi\right)}{\partial y^{a}}\frac{\partial}{\partial x^{j}}=0_{\pi^*(h^*F)}\oplus 0_{TE};\end{aligned} \\
(5) & \left[\frac{\partial}{\partial\tilde{y}^{a}},\frac{\partial}{\partial\tilde{y}^{b}}\right]_{(\rho,\eta)TE}=[0,0]_{\pi^*(h^*F)}\oplus\left[\frac{\partial}{\partial y^{a}},\frac{\partial}{\partial y^{b}}\right]_{TE}=0_{\pi^*(h^*F)}\oplus 0_{TE}.
\end{array}
$$

Using relations (2) , (3) , (4) and (5), we obtain:

$$
(6)\qquad
\begin{aligned}
\left[\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{z}^{\beta}}\right]_{(\rho,\eta)TE}&=-\left[\frac{\partial}{\partial\tilde{z}^{\beta}},\frac{\partial}{\partial\tilde{z}^{\alpha}}\right]_{(\rho,\eta)TE},\\
\left[\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^{b}}\right]_{(\rho,\eta)TE}&=-\left[\frac{\partial}{\partial\tilde{y}^{b}},\frac{\partial}{\partial\tilde{z}^{\alpha}}\right]_{(\rho,\eta)TE},\\
\left[\frac{\partial}{\partial\tilde{y}^{a}},\frac{\partial}{\partial\tilde{z}^{\beta}}\right]_{(\rho,\eta)TE}&=-\left[\frac{\partial}{\partial\tilde{z}^{\beta}},\frac{\partial}{\partial\tilde{y}^{a}}\right]_{(\rho,\eta)TE},\\
\left[\frac{\partial}{\partial\tilde{y}^{a}},\frac{\partial}{\partial\tilde{y}^{b}}\right]_{(\rho,\eta)TE}&=-\left[\frac{\partial}{\partial\tilde{y}^{b}},\frac{\partial}{\partial\tilde{y}^{a}}\right]_{(\rho,\eta)TE}.
\end{aligned}
$$

In general, for any $\tilde{U},\tilde{V}\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$, we have:

$$
(7)\qquad \left[\tilde{U},\tilde{V}\right]_{(\rho,\eta)TE}=-\left[\tilde{V},\tilde{U}\right]_{(\rho,\eta)TE}.
$$

Since equality (7) implies

$$
2\left[\tilde{U},\tilde{U}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE},\ \forall\tilde{U}\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right),
$$

we obtain:

$$
\left[\tilde{U},\tilde{U}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE},\ \forall\tilde{U}\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right).
$$

q.e.d.

Lemma 3.3.3 *We have the Jacobi identity:*

$$\begin{aligned}(3.3.19)\qquad &\left[\tilde{U},\left[\tilde{V},\tilde{Z}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}+\left[\tilde{Z},\left[\tilde{U},\tilde{V}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}\\&+\left[\tilde{V},\left[\tilde{Z},\tilde{U}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE}.\end{aligned}$$

for any $\tilde{U},\tilde{V},\tilde{Z}\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right).$

Proof. After some calculations, using the sections of natural (ρ,η)-base, we obtain the following Jacobi identities:

$$\begin{aligned}(1)\quad &\left[\frac{\partial}{\partial\tilde{z}^{\alpha}},\left[\frac{\partial}{\partial\tilde{z}^{\beta}},\frac{\partial}{\partial\tilde{z}^{\gamma}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}+\left[\frac{\partial}{\partial\tilde{z}^{\beta}},\left[\frac{\partial}{\partial\tilde{z}^{\gamma}},\frac{\partial}{\partial\tilde{z}^{\alpha}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}\\&+\left[\frac{\partial}{\partial\tilde{z}^{\gamma}},\left[\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{z}^{\beta}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE},\end{aligned}$$

$$\begin{aligned}(2)\quad &\left[\frac{\partial}{\partial\tilde{y}^{a}},\left[\frac{\partial}{\partial\tilde{y}^{b}},\frac{\partial}{\partial\tilde{z}^{\gamma}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}+\left[\frac{\partial}{\partial\tilde{y}^{b}},\left[\frac{\partial}{\partial\tilde{z}^{\gamma}},\frac{\partial}{\partial\tilde{y}^{a}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}\\&+\left[\frac{\partial}{\partial\tilde{z}^{\gamma}},\left[\frac{\partial}{\partial\tilde{y}^{a}},\frac{\partial}{\partial\tilde{y}^{b}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE},\end{aligned}$$

$$\begin{aligned}(3)\quad &\left[\frac{\partial}{\partial\tilde{y}^{a}},\left[\frac{\partial}{\partial\tilde{y}^{b}},\frac{\partial}{\partial\tilde{y}^{c}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}+\left[\frac{\partial}{\partial\tilde{y}^{b}},\left[\frac{\partial}{\partial\tilde{y}^{c}},\frac{\partial}{\partial\tilde{y}^{a}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}\\&+\left[\frac{\partial}{\partial\tilde{y}^{c}},\left[\frac{\partial}{\partial\tilde{y}^{a}},\frac{\partial}{\partial\tilde{y}^{b}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE}.\end{aligned}$$

After some calculations, we obtain the Jacobi identity

$$\begin{aligned}&\left[\tilde{U},\left[\tilde{V},\tilde{Z}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}+\left[\tilde{Z},\left[\tilde{U},\tilde{V}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}\\&+\left[\tilde{V},\left[\tilde{Z},\tilde{U}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE},\end{aligned}$$

for any $\tilde{U},\tilde{V},\tilde{Z}\in\Gamma\left(\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)\right)$ *q.e.d.*

Lemma 3.3.4 *The* **Mod**-*morphism*

$$\Gamma\left(\tilde{\rho},Id_{E}\right)$$

is a **Liealg**-*morphism of*

$$\left(\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right),+,\cdot,[,]_{(\rho,\eta)TE}\right)$$

source and

$$\left(\Gamma\left(TE,\tau_E,E\right),+,\cdot,[,]_{TE}\right)$$

target.

Proof. Indeed, we have:

$$\begin{array}{ll} (1) & \begin{array}{l} \left[\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{z}^\alpha},\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{z}^\beta}\right]_{TE}\\ =\left[\left(\rho_\alpha^i\circ h\circ\pi\right)\frac{\partial}{\partial x^i},\left(\rho_\beta^j\circ h\circ\pi\right)\frac{\partial}{\partial x^j}\right]_{TE}\\ =\left(\left(\rho_\alpha^i\circ h\circ\pi\right)\frac{\partial\left(\rho_\beta^k\circ h\circ\pi\right)}{\partial x^i}-\left(\rho_\beta^j\circ h\circ\pi\right)\frac{\partial\left(\rho_\alpha^k\circ h\circ\pi\right)}{\partial x^j}\right)\frac{\partial}{\partial x^k}\\ \underset{(3.1.1.9)}{\overset{(2.2.2)}{=}}\left(L_{\alpha\beta}^\gamma\circ h\circ\pi\right)\left(\rho_\gamma^k\circ h\circ\pi\right)\frac{\partial}{\partial x^k}=\left(L_{\alpha\beta}^\gamma\circ h\circ\pi\right)\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{z}^\gamma}\\ =\Gamma\left(\tilde{\rho},Id_E\right)\left[\frac{\partial}{\partial\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{z}^\beta}\right]_{(\rho,\eta)TE}, \end{array} \end{array}$$

$$\begin{array}{ll} (2) & \begin{array}{l} \left[\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{z}^\alpha},\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{y}^b}\right]_{TE}=\left[\left(\rho_\alpha^i\circ h\circ\pi\right)\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^b}\right]_{TE}\\ =-\frac{\partial\left(\rho_\alpha^i\circ h\circ\pi\right)}{\partial y^b}\frac{\partial}{\partial x^i}=0_{TE}=\Gamma\left(\tilde{\rho},Id_E\right)\left(0_{(\rho,\eta)TE}\right)\\ =\Gamma\left(\tilde{\rho},Id_E\right)\left[\frac{\partial}{\partial\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{y}^b}\right]_{(\rho,\eta)TE}, \end{array} \end{array}$$

$$\begin{array}{ll} (3) & \begin{array}{l} \left[\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{y}^a},\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{z}^\beta}\right]_{TE}=\left[\frac{\partial}{\partial y^a},\left(\rho_\beta^j\circ h\circ\pi\right)\frac{\partial}{\partial x^j}\right]_{TE}\\ =\frac{\partial\left(\rho_\beta^j\circ h\circ\pi\right)}{\partial y^a}\frac{\partial}{\partial x^j}=0_{TE}=\Gamma\left(\tilde{\rho},Id_E\right)\left(0_{(\rho,\eta)TE}\right)\\ =\Gamma\left(\tilde{\rho},Id_E\right)\left[\frac{\partial}{\partial\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{y}^b}\right]_{(\rho,\eta)TE} \end{array} \end{array}$$

and

$$\begin{array}{ll} (4) & \begin{array}{l} \left[\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{y}^a},\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{y}^b}\right]_{TE}=\left[\frac{\partial}{\partial y^a},\frac{\partial}{\partial y^b}\right]_{TE}\\ =0_{TE}=\Gamma\left(\tilde{\rho},Id_E\right)\left(0_{(\rho,\eta)TE}\right)=\Gamma\left(\tilde{\rho},Id_E\right)\left[\frac{\partial}{\partial\tilde{y}^a},\frac{\partial}{\partial\tilde{y}^b}\right]_{(\rho,\eta)TE}. \end{array} \end{array}$$

In general, for any $\tilde{U},\tilde{Z}\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$, we obtain:

$$\left[\Gamma\left(\tilde{\rho},Id_E\right)(\tilde{U}),\Gamma\left(\tilde{\rho},Id_E\right)(\tilde{Z})\right]_{TE}=\Gamma\left(\tilde{\rho},Id_E\right)\left(\left[\tilde{U},\tilde{Z}\right]_{(\rho,\eta)TE}\right).$$

q.e.d.

Using *Lemmas 3.3.1, 3.3.2, 3.3.3* and *3.3.4,* we obtain the following

Theorem 3.3.4 *The couple*

$$\left([,]_{(\rho,\eta)TE},\left(\tilde{\rho},Id_E\right)\right)$$

is a Lie algebroid structure for the vector bundle

$$\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right).$$

Remark 3.3.2 In particular, if $h = Id_M$ and $[,]_{TM}$ is the usual Lie bracket, it results that the Lie algebroid

$$\left(\left((Id_{TM}, Id_M)\, TE, (Id_{TM}, Id_M)\, \tau_E, E\right), [,]_{(Id_{TM}, Id_M)TE}, \left(\widetilde{Id_{TM}}, Id_E\right)\right)$$

is isomorphic with the usual Lie algebroid

$$\left((TE, \tau_E, E), [,]_{TE}, (Id_{TE}, Id_E)\right).$$

This is a reason for which the Lie algebroid

$$\left(\left((\rho, \eta)\, TE, (\rho, \eta)\, \tau_E, E\right), [,]_{(\rho,\eta)TE}, (\tilde{\rho}, Id_E)\right),$$

will be called the *Lie algebroid generalized tangent bundle.*

The vector bundle $((\rho, \eta)\, TE, (\rho, \eta)\, \tau_E, E)$ will be called the *generalized tangent bundle.*

3.3.1 The generalized tangent bundle of dual vector bundle

Let (E, π, M) be a vector bundle. We build the generalized tangent bundle of dual vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ using the diagram:

(3.3.1.1)
$$\begin{array}{ccc} \overset{*}{E} & & \left(F, [,]_{F,h}, (\rho, \eta)\right) \\ \overset{*}{\pi} \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array},$$

where $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$ is a generalized Lie algebroid.

We take $\left(x^i, p_a\right)$ as canonical local coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, where $i \in \overline{1, m}$ and $a \in \overline{1, r}$.

Consider

$$\left(x^i, p_a\right) \longrightarrow \left(x^{\acute{\imath}}\left(x^i\right), p_{\acute{a}}\left(x^i, p_a\right)\right)$$

a change of coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$. Then the coordinates p_a change to $p_{\acute{a}}$ by the rule:

(3.3.1.2)
$$p_{\acute{a}} = M_{\acute{a}}^{a} p_a.$$

Let

$$\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_a}\right)$$

be the natural base for the sections Lie algebra $\left(\Gamma\left(T\overset{*}{E}, \tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot, [,]_{T\overset{*}{E}}\right)$.

The sections

(3.3.1.3)
$$\left(\tilde{T}_{\alpha}, \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_a}\right)\right)$$

determine a base for the module $\Gamma\left(\overset{**}{\pi}(h^*F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right)$.

The matrix of coordinate transformation on

$$\left(\overset{**}{\pi}(h^*F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right)$$

at a change of fibred charts is

$$\tag{3.3.1.4}\left\Vert\begin{array}{ccc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\overset{*}{\pi} & 0 & 0\\ 0 & \dfrac{\partial x^{\acute{i}}}{\partial x_i}\circ\overset{*}{\pi} & 0\\ 0 & \dfrac{\partial M_{\acute{a}}^{a}\circ\overset{*}{\pi}}{\partial x^i}p_a & M_{\acute{a}}^{a}\circ\overset{*}{\pi}\end{array}\right\Vert.$$

For any sections

$$\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\in\Gamma\left(\overset{**}{\pi}(h^*F),\overset{**}{\pi}(h^*\nu),\overset{*}{E}\right)$$

and

$$Y_a\frac{\partial}{\partial p_a}\in\Gamma\left(VT\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

we construct the section

$$\begin{array}{c}\tilde{Z}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_a\dfrac{\partial}{\partial\tilde{p}_a}=:\tilde{Z}^{\alpha}\left(T_{\alpha}\oplus\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial}{\partial x^i}\right)+Y_a\left(0_{\overset{**}{\pi}(h^*F)}\oplus\dfrac{\partial}{\partial p_a}\right)\\ =\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\oplus\left(\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial}{\partial x^i}+Y_a\dfrac{\partial}{\partial p_a}\right)\in\Gamma\left(\overset{**}{\pi}(h^*F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right).\end{array}$$

Since we have

$$\begin{array}{c}\tilde{Z}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_a\dfrac{\partial}{\partial\tilde{p}_a}=0_{\overset{**}{\pi}(h^*F)\oplus T\overset{*}{E}}\\ \Updownarrow\\ \tilde{Z}^{\alpha}\tilde{T}_{\alpha}=0_{\overset{**}{\pi}(h^*F)}\wedge\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial}{\partial x^i}+Y_a\dfrac{\partial}{\partial p_a}=0_{T\overset{*}{E}},\end{array}$$

it implies $\tilde{Z}^{\alpha}=0,\ \alpha\in\overline{1,p}$ and $Y_a=0,\ a\in\overline{1,r}$.

Therefore, the sections

$$\frac{\partial}{\partial\tilde{z}^1},...,\frac{\partial}{\partial\tilde{z}^p},\frac{\partial}{\partial\tilde{p}_1},...,\frac{\partial}{\partial\tilde{p}_r}$$

are linearly independent.

We consider the vector subbundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

of vector bundle

$$\left(\overset{**}{\pi}(h^*F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right),$$

for which the $\mathcal{F}\left(\overset{*}{E}\right)$-module of sections is the $\mathcal{F}\left(\overset{*}{E}\right)$-submodule of

$$\left(\Gamma\left(\overset{*}{\pi}{}^{*}\left(h^{*}F\right)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right),+,\cdot\right),$$

generated by the family of sections $\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{p}_{a}}\right)$.

The base sections

$$\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{p}_{a}}\right)\overset{put}{=}\left(\tilde{\partial}_{\alpha},\tilde{\partial}^{\,.\,a}\right)\tag{3.3.1.5}$$

will be called the *natural* (ρ,η)*-base.*

The matrix of coordinate transformation on $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ at a change of fibred charts is

$$\left\|\begin{array}{cc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\overset{*}{\pi} & 0\\ \left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial M_{\acute{a}}^{b}\circ\overset{*}{\pi}}{\partial x_{i}}p_{b} & M_{\acute{a}}^{a}\circ\overset{*}{\pi}\end{array}\right\|.\tag{3.3.1.6}$$

We consider the operation $[,]_{(\rho,\eta)T\overset{*}{E}}$ defined by

$$\begin{array}{l}\left[\left(\tilde{Z}_{1}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{1a}\dfrac{\partial}{\partial\tilde{p}_{a}}\right),\left(\tilde{Z}_{2}^{\beta}\dfrac{\partial}{\partial\tilde{z}^{\beta}}+Y_{2b}\dfrac{\partial}{\partial\tilde{p}_{b}}\right)\right]_{(\rho,\eta)TE^{*}}=\\ =\left[\tilde{Z}_{1}^{\alpha}T_{a},\tilde{Z}_{2}^{\beta}T_{\beta}\right]_{\overset{*}{\pi}{}^{*}(h^{*}F)}\oplus\left[\left(\rho_{\alpha}^{i}\circ h\circ\pi^{*}\right)\tilde{Z}_{1}^{\alpha}\dfrac{\partial}{\partial x^{i}}+Y_{1a}\dfrac{\partial}{\partial p_{a}},\right.\\ \qquad\qquad\left.\left(\rho_{\beta}^{i}\circ h\circ\pi^{*}\right)\tilde{Z}_{2}^{\beta}\dfrac{\partial}{\partial x^{i}}+Y_{2b}\dfrac{\partial}{\partial p_{b}}\right]_{T\overset{*}{E}},\end{array}\tag{3.3.1.7}$$

for any sections $\left(\tilde{Z}_{1}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{1a}\frac{\partial}{\partial\tilde{p}_{a}}\right)$ and $\left(\tilde{Z}_{2}^{\beta}\frac{\partial}{\partial\tilde{z}^{\beta}}+Y_{2b}\frac{\partial}{\partial\tilde{p}_{b}}\right)$.

Let $\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)$ be the $\mathbf{B}^{\mathbf{v}}$-morphism of $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ source and $\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right)$ target, where

$$\begin{array}{c}(\rho,\eta)\,T\overset{*}{E}\xrightarrow{\tilde{\rho}^{*}}T\overset{*}{E}\\ \left(\tilde{Z}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{a}\dfrac{\partial}{\partial\tilde{p}_{a}}\right)(\overset{*}{u}_{x})\longmapsto\left(\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial}{\partial x^{i}}+Y_{a}\dfrac{\partial}{\partial p_{a}}\right)(\overset{*}{u}_{x})\end{array}\tag{3.3.1.8}$$

The Lie algebroid generalized tangent bundle of the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be denoted

$$\left(\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),[,]_{(\rho,\eta)T\overset{*}{E}},\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\right).$$

3.4 (Linear) (ρ,η)-connections

The theory of (linear) connections constitutes undoubtedly one of most beautiful and most important chapter of differential geometry, which has been widely explored in the literature (see [8, 11, 14, 26, 31, 41, 42, 45, 46, 47, 51, 52, 54, 55]).

In the following, we consider the diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \left(F, [,]_{F,h}, (\rho,\eta)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E,\pi,M) \in |\mathbf{B}|$ and $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta)\right)$ is a generalized Lie algebroid.

Let

$$\left(((\rho,\eta)TE, (\rho,\eta)\tau_E, E), [,]_{(\rho,\eta)TE}, (\tilde{\rho}, Id_E)\right)$$

be the Lie algebroid generalized tangent bundle of fiber bundle (E,π,M).

We consider the $\mathbf{B^v}$-morphism

$$((\rho,\eta)\pi!, Id_E)$$

given by the commutative diagram

(3.4.1)
$$\begin{tikzcd} (\rho,\eta)TE \arrow[r, "(\rho,\eta)\pi!"] \arrow[d, "(\rho,\eta)\tau_E"'] & \pi^*(h^*F) \arrow[d, "pr_1"] \\ E \arrow[r, "id_E"] & E \end{tikzcd}$$

Using the components, this is defined as:

(3.4.2)
$$(\rho,\eta)\pi!\left(\left(\tilde{Z}^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y^a \frac{\partial}{\partial \tilde{y}^a}\right)(u_x)\right) = \left(\tilde{Z}^\alpha \tilde{T}_\alpha\right)(u_x),$$

for any $\left(\tilde{Z}^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y^a \frac{\partial}{\partial \tilde{y}^a}\right) \in \Gamma((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$.

We define the *tangent* (ρ,η)*-application* as being a $\mathbf{B^v}$-morphism

(3.4.3)
$$((\rho,\eta)T\pi, h\circ\pi) = (pr_2, h\circ\pi)\circ((\rho,\eta)\pi!, Id_E)$$

of $((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$ source and (F,ν,N) target.

Definition 3.4.1 The kernel of the tangent (ρ,η)-application

$$((\rho,\eta)T\pi, h\circ\pi)$$

is written as

$$(V(\rho,\eta)TE, (\rho,\eta)\tau_E, E)$$

and will be called *the vertical subbundle.*

The set $\left\{\frac{\partial}{\partial \tilde{y}^a},\ a\in\overline{1,r}\right\}$ is a base for the $\mathcal{F}(E)$-module

$$\left(\Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),+,\cdot\right).$$

Proposition 3.4.1 *The short sequence of vector bundles*

$$(3.4.4)\qquad \begin{tikzcd} 0 \arrow[r, hook, "i"] \arrow[d] & V(\rho,\eta)TE \arrow[r, hook, "i"] \arrow[d] & (\rho,\eta)TE \arrow[r, "(\rho,\eta)\pi!"] \arrow[d] & \pi^*\left(h^{*F}\right) \arrow[r] \arrow[d] & 0 \arrow[d] \\ E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \end{tikzcd}$$

is exact.

Definition 3.4.2 *A* **Man**-morphism $(\rho,\eta)\Gamma$ of $(\rho,\eta)TE$ source and $V(\rho,\eta)TE$ target defined by

$$(3.4.5)\qquad (\rho,\eta)\Gamma\left(\tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y^a\frac{\partial}{\partial\tilde{y}^a}\right)(u_x)=\left(Y^a+(\rho,\eta)\Gamma_\alpha^a\tilde{Z}^\alpha\right)\frac{\partial}{\partial\tilde{y}^a}(u_x),$$

such that the $\mathbf{B^v}$-morphism $((\rho,\eta)\Gamma,Id_E)$ is a split to the left in the previous exact sequence, will be called (ρ,η)*-connection for the fiber bundle* (E,π,M).

The differentiable real local functions $(\rho,\eta)\Gamma_\alpha^a$ will be called the *components of* (ρ,η)*-connection* $(\rho,\eta)\Gamma$.

The (ρ,Id_M)-connection for the fiber bundle (E,π,M) will be called ρ*-connection for the fiber bundle* (E,π,M) and will be denoted $\rho\Gamma$*.*

The (Id_{TM},Id_M)-connection for the fiber bundle (E,π,M) will be called *connection for the fiber bundle* (E,π,M) and will be denoted Γ*.*

Definition 3.4.3 If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the fiber bundle (E,π,M), then the kernel of the $\mathbf{B^v}$-morphism $((\rho,\eta)\Gamma,Id_E)$ is written as

$$\left(H\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$$

and will be called the *horizontal vector subbundle.*

Definition 3.4.4 If $(E,\pi,M)\in|\mathbf{B}|$, then the **B**-morphism (Π,π) defined by the commutative diagram

$$(3.4.6)\qquad \begin{tikzcd} V(\rho,\eta)TE \arrow[r, "\Pi"] \arrow[d, "(\rho,\eta)\tau_E"'] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"] & M \end{tikzcd}$$

such that the components of the image of vector $Y^a\frac{\partial}{\partial\tilde{y}^a}(u_x)$ are the real numbers $Y^1(u_x),...,Y^r(u_x)$ will be called the *canonical projection* **B**-*morphism.*

Let $\left\{s_a,a\in\overline{1,r}\right\}$ be the base of module of sections $\Gamma(E,\pi,M)$.

Example 3.4.1 If $(E,\pi,M)\in|\mathbf{B^v}|$, then the $\mathbf{B^v}$-morphism (Π,π) defined by the commutative diagram (3.4.6), where Π is defined by

$$(3.4.7)\qquad \Pi\left(Y^a\frac{\partial}{\partial\tilde{y}^a}(u_x)\right)=Y^a(u_x)\,s_a(x),$$

is canonical projection $\mathbf{B^v}$-morphism.

Theorem 3.4.1 *If* $(\rho,\eta)\,\Gamma$ *is a* (ρ,η)*-connection for the fiber bundle* (E,π,M)*, then its components satisfy the law of transformation*

$$(\rho,\eta)\,\Gamma_{\acute{\gamma}}^{\acute{a}}=\frac{\partial y^{\acute{a}}}{\partial y^{a}}\left[\rho_{\gamma}^{i}\circ(h\circ\pi)\frac{\partial y^{a}}{\partial x^{i}}+(\rho,\eta)\,\Gamma_{\gamma}^{a}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ(h\circ\pi)\,. \tag{3.4.8}$$

If $(\rho,\eta)\,\Gamma$ *is a* (ρ,η)*-connection for the vector bundle* (E,π,M)*, then its components satisfy the law of transformation*

$$(\rho,\eta)\,\Gamma_{\acute{\gamma}}^{\acute{a}}=M_{a}^{\acute{a}}\circ\pi\left[\rho_{\gamma}^{i}\circ(h\circ\pi)\frac{\partial M_{\acute{b}}^{a}\circ\pi}{\partial x^{i}}y^{\acute{b}}+(\rho,\eta)\Gamma_{\gamma}^{a}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ(h\circ\pi)\,. \tag{3.4.8$'$}$$

If $\rho\Gamma$ *is a* ρ*-connection for the vector bundle* (E,π,M) *and* $h=Id_M$*, then relations* $(3.4.8')$ *become*

$$\rho\Gamma_{\acute{\gamma}}^{\acute{a}}=M_{a}^{\acute{a}}\circ\pi\left[\rho_{\gamma}^{i}\circ\pi\frac{\partial M_{\acute{b}}^{a}\circ\pi}{\partial x^{i}}y^{\acute{b}}+\rho\Gamma_{\gamma}^{a}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ\pi. \tag{3.4.8$''$}$$

In particular, if $(\rho,\eta)=(Id_{TM},Id_M)$*, then the relations* $(3.4.8'')$ *become*

$$\Gamma_{\acute{k}}^{\acute{i}}=\frac{\partial x^{\acute{i}}}{\partial x^{i}}\circ\pi\left[\frac{\partial}{\partial x^{k}}\left(\frac{\partial x^{i}}{\partial x^{\acute{j}}}\circ\pi\right)y^{\acute{j}}+\Gamma_{k}^{i}\right]\frac{\partial x^{k}}{\partial x^{\acute{k}}}\circ\pi. \tag{3.4.8$'''$}$$

Proof. Let (Π,π) be the canonical projection $\mathbf{B}$-morphism.

Obviously, the components of

$$\Pi\circ(\rho,\eta)\,\Gamma\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y^{\acute{a}}\frac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x)$$

are the real numbers

$$\left(Y^{\acute{a}}+(\rho,\eta)\,\Gamma_{\acute{\gamma}}^{\acute{a}}\tilde{Z}^{\acute{\gamma}}\right)(u_x)\,.$$

Since

$$\begin{aligned}\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y^{\acute{a}}\frac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x)&=\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\frac{\partial}{\partial\tilde{z}^{\alpha}}(u_x)\\&+\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\pi\frac{\partial y^{a}}{\partial x^{\acute{i}}}+\frac{\partial y^{a}}{\partial y^{\acute{a}}}Y^{\acute{a}}\right)\frac{\partial}{\partial\tilde{y}^{a}}(u_x)\,,\end{aligned}$$

it results that the components of

$$\Pi\circ(\rho,\eta)\,\Gamma\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y^{\acute{a}}\frac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x)$$

are the real numbers

$$\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\pi\frac{\partial y^{a}}{\partial x^{\acute{i}}}+\frac{\partial y^{a}}{\partial y^{\acute{a}}}Y^{\acute{a}}+(\rho,\eta)\,\Gamma_{\alpha}^{a}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\right)(u_x)\frac{\partial y^{\acute{a}}}{\partial y^{a}},$$

where

$$\left\|\frac{\partial y^{a}}{\partial y^{\acute{a}}}\right\|=\left\|\frac{\partial y^{\acute{a}}}{\partial y^{a}}\right\|^{-1}.$$

Therefore, we have:

$$\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\pi\frac{\partial y^{a}}{\partial x^{\acute{i}}}+\frac{\partial y^{a}}{\partial y^{\acute{a}}}Y^{\acute{a}}+(\rho,\eta)\,\Gamma_{\alpha}^{a}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\right)\frac{\partial y^{\acute{a}}}{\partial y^{a}}=Y^{\acute{a}}+(\rho,\eta)\,\Gamma_{\acute{\alpha}}^{\acute{a}}\tilde{Z}^{\acute{\alpha}}.$$

After some calculations we obtain:

$$(\rho,\eta)\,\Gamma_{\acute{\alpha}}^{\acute{a}}=\frac{\partial y^{\acute{a}}}{\partial y^{a}}\left(\rho_{\alpha}^{i}\circ(h\circ\pi)\,\frac{\partial y^{a}}{\partial x^{i}}+(\rho,\eta)\,\Gamma_{\alpha}^{a}\right)\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi. \qquad q.e.d.$$

Remark 3.4.1 If we have a set of real local functions $(\rho,\eta)\,\Gamma_{\gamma}^{a}$ which satisfies the relations of passing (3.4.8) , then we have a (ρ,η)-connection $(\rho,\eta)\,\Gamma$ for the fiber bundle (E,π,M) .

Example 3.4.1 If Γ is a classical connection for the vector bundle (E,π,M) on components Γ_{k}^{a}, then the differentiable real local functions

$$(\rho,\eta)\,\Gamma_{\gamma}^{a}=\left(\rho_{\gamma}^{k}\circ h\circ\pi\right)\Gamma_{k}^{a}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\,\Gamma$ for the vector bundle (E,π,M) which will be called the *(ρ,η)-connection associated to the connection* Γ.

Definition 3.4.5 If $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M), then for any

$$z=z^{\alpha}t_{\alpha}\in\Gamma\,(F,\nu,N)$$

the application

$$\begin{array}{ccc}\Gamma\,(E,\pi,M) & \xrightarrow{(\rho,\eta)D_{v}} & \Gamma\,(E,\pi,M)\\ u=u^{a}s_{a} & \longmapsto & (\rho,\eta)\,D_{z}u\end{array} \tag{3.4.9}$$

where

$$(\rho,\eta)\,D_{z}u=z^{\alpha}\circ h\left(\rho_{\alpha}^{i}\circ h\frac{\partial u^{a}}{\partial x^{i}}+(\rho,\eta)\,\Gamma_{\alpha}^{a}\circ u\right)s_{a}$$

will be called the *covariant (ρ,η)-derivative associated to (ρ,η)-connection $(\rho,\eta)\,\Gamma$ with respect to section z.*

If $h=Id_{M}$ and $\eta=Id_{M}$, then we obtain the *covariant ρ-derivative associated to ρ-connection $\rho\Gamma$ with respect to section z.*

In addition, if $\rho=Id_{TM}$, then we obtain the *covariant derivative associated to connection Γ with respect to the vector field z.*

Remark 3.4.2 If $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M), then the operator

$$\begin{array}{ccc}\Gamma\,(F,\nu,N)\times\Gamma\,(E,\pi,M) & \xrightarrow{(\rho,\eta)D} & \Gamma\,(E,\pi,M)\\ (z,u) & \longmapsto & (\rho,\eta)\,D_{z}u\end{array}$$

satisfies the following properties:

(i) $(\rho,\eta)\,D$ is $\mathbb{R}$-bilinear;

(ii) $(\rho,\eta)\,D_{f_{1}z_{1}+f_{2}z_{2}}u=f_{1}\,(\rho,\eta)\,D_{z_{1}}u+f_{2}\,(\rho,\eta)\,D_{z_{2}}u;$

(iii) if $u \in \Gamma(E, \pi, M)$ is null on a nonempty subset of M, then $(\rho, \eta) D_z u$ is null on the same nonempty subset, for any $z \in \Gamma(F, \nu, N)$.

Definition 3.4.6 We will say that the (ρ, η)*-connection* $(\rho, \eta)\Gamma$ *is homogeneous* or *linear* if the local real functions $(\rho, \eta)\Gamma_\gamma^a$ are homogeneous or linear on the fibre of the fiber bundle (E, π, M).

Remark 3.4.3 If $(\rho, \eta)\Gamma$ is a linear (ρ, η)-connection for the fiber bundle (E, π, M), then for each local vector $(m+r)$-chart (U, s_U) and for each local vector $(n+p)$-chart (V, t_V) such that $U \cap h^{-1}(V) \neq \phi$, it exists the differentiable real functions $\rho\Gamma_{b\gamma}^a$ defined on $U \cap h^{-1}(V)$ such that

$$(\rho, \eta)\Gamma_\gamma^a \circ u = (\rho, \eta)\Gamma_{b\gamma}^a \cdot u^b, \forall u = u^b s_b \in \Gamma(E, \pi, M). \tag{3.4.10}$$

The differentiable real local functions $(\rho, \eta)\Gamma_{b\alpha}^a$ will be called the *Christoffel coefficients of linear* (ρ, η)*-connection* $(\rho, \eta)\Gamma$.

Theorem 3.4.2 *If* $(\rho, \eta)\Gamma$ *is a linear* (ρ, η)*-connection for the fiber bundle* (E, π, M), *then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma_{\acute{b}\acute{\gamma}}^{\acute{a}} = \frac{\partial y^{\acute{a}}}{\partial y^a}\left[\rho_{\gamma}^{k}\circ h\frac{\partial}{\partial x^k}\left(\frac{\partial y^a}{\partial y^{\acute{b}}}\right) + (\rho, \eta)\Gamma_{b\gamma}^a \frac{\partial y^b}{\partial y^{\acute{b}}}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ h. \tag{3.4.11}$$

If $(\rho, \eta)\Gamma$ *is a linear* (ρ, η)*-connection for the vector bundle* (E, π, M), *then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma_{\acute{b}\acute{\gamma}}^{\acute{a}} = M_a^{\acute{a}}\left[\rho_\gamma^k\circ h\frac{\partial M_{\acute{b}}^a}{\partial x^k} + (\rho,\eta)\Gamma_{b\gamma}^a M_{\acute{b}}^b\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ h. \tag{3.4.11$'$}$$

If $\rho\Gamma$ *is a* ρ*-connection for the vector bundle* (E, π, M) *and* $h = Id_M$, *then the relations* (3.4.11$'$) *become*

$$\rho\Gamma_{\acute{b}\acute{\gamma}}^{\acute{a}} = M_a^{\acute{a}}\left[\rho_\gamma^k \frac{\partial M_{\acute{b}}^a}{\partial x^k} + \rho\Gamma_{b\gamma}^a M_{\acute{b}}^b\right]\Lambda_{\acute{\gamma}}^{\gamma}. \tag{3.4.11$''$}$$

In particular, if $(\rho, \eta) = (Id_{TM}, Id_M)$, *then the relations* (3.4.11$''$) *become*

$$\Gamma_{\acute{j}\acute{k}}^{\acute{i}} = \frac{\partial x^{\acute{i}}}{\partial x^i}\left[\frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial x^{\acute{j}}}\right) + \Gamma_{jk}^i \frac{\partial x^j}{\partial x^{\acute{j}}}\right]\frac{\partial x^k}{\partial x^{\acute{k}}}. \tag{3.4.11$'''$}$$

Definition 3.4.7 We say that the (linear) (ρ, η)-connection $(\rho, \eta)\Gamma$ for the fiber bundle (E, π, M) is differentiable of C^r class, if its components are differentiable of C^r class.

Definition 3.4.8 If $(\rho, \eta)\Gamma$ is a linear (ρ, η)-connection for the vector bundle (E, π, M), then for any

$$z = z^\alpha t_\alpha \in \Gamma(F, \nu, N)$$

the application

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma(E, \pi, M) \\ u = u^a s_a & \longmapsto & (\rho, \eta)D_z u \end{array} \tag{3.4.12}$$

defined by

$$(\rho,\eta)D_z u = z^\alpha \circ h\left(\rho_\alpha^i \circ h \frac{\partial u^a}{\partial x^i} + (\rho,\eta)\Gamma_{b\alpha}^a \cdot u^b\right) s_a,$$

will be called the *covariant (ρ,η)-derivative associated to linear (ρ,η)-connection $(\rho,\eta)\Gamma$ with respect to section z.*

If $h = Id_M$ and $\eta = Id_M$, then we obtain the *covariant ρ-derivative associated to linear ρ-connection $\rho\Gamma$ with respect to section z.*

In addition, if $\rho = Id_{TM}$, then we obtain the *covariant derivative associated to linear connection Γ with respect to the vector field z.*

3.4.1 (Linear) (ρ,η)-connections for dual of vector bundles

Let (E,π,M) be a vector bundle.

We consider the following diagram:

(3.4.1.1)
$$\begin{array}{ccc} \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \overset{*}{\pi}\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array},$$

where $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let

$$\left(\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),[,]_{(\rho,\eta)T\overset{*}{E}},\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\right)$$

be the Lie algebroid generalized tangent bundle of the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

We consider the $\mathbf{B^v}$-morphism $\left((\rho,\eta)\,\overset{*}{\pi}!,Id_{\overset{*}{E}}\right)$ given by the commutative diagram

(3.4.1.2)
$$\begin{array}{ccc} (\rho,\eta)\,T\overset{*}{E} & \xrightarrow{(\rho,\eta)\overset{*}{\pi}!} & \overset{*}{\pi}{}^{*}\,(h^*F) \\ {\scriptstyle (\rho,\eta)\tau_{\overset{*}{E}}}\downarrow & & \downarrow{\scriptstyle pr_1} \\ \overset{*}{E} & \xrightarrow{id_{\overset{*}{E}}} & \overset{*}{E} \end{array}$$

Using the components, this is defined as:

(3.4.1.3)
$$(\rho,\eta)\,\overset{*}{\pi}!\left(\tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y_a\frac{\partial}{\partial\tilde{p}_a}\right)\left(\overset{*}{u}_x\right)=\left(\tilde{Z}^\alpha\tilde{T}_\alpha\right)\left(\overset{*}{u}_x\right),$$

for any $\tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y_a\frac{\partial}{\partial\tilde{p}_a}\in\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

We define the *tangent (ρ,η)-application* as being a $\mathbf{B^v}$-morphism

(3.4.1.4)
$$\left((\rho,\eta)\,T\overset{*}{\pi},h\circ\overset{*}{\pi}\right)=\left(pr_2,h\circ\overset{*}{\pi}\right)\circ\left((\rho,\eta)\,\overset{*}{\pi}!,Id_{\overset{*}{E}}\right)$$

of $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ source and (F,ν,N) target.

Definition 3.4.1.1 The kernel of the tangent (ρ,η)-application

$$\left((\rho,\eta)\,T\overset{*}{\pi},h\circ\overset{*}{\pi}\right)$$

is written as

$$\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and will be called the *vertical subbundle.*

The set $\left\{\frac{\partial}{\partial\tilde{p}_a},\ a\in\overline{1,r}\right\}$ is a base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module

$$\left(\Gamma\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right).$$

Proposition 3.4.1.1 *The short sequence of vector bundles*

$$\begin{array}{ccccccccc}
0 & \overset{i}{\hookrightarrow} & V(\rho,\eta)T\overset{*}{E} & \overset{i}{\hookrightarrow} & (\rho,\eta)T\overset{*}{E} & \xrightarrow{(\rho,\eta)\overset{*}{\pi}!} & \overset{*}{\pi}{}^{*}\,(h^{*}F) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\overset{*}{E} & \xrightarrow{Id_{\overset{*}{E}}} & \overset{*}{E} & \xrightarrow{Id_{\overset{*}{E}}} & \overset{*}{E} & \xrightarrow{Id_{\overset{*}{E}}} & \overset{*}{E} & \xrightarrow{Id_{\overset{*}{E}}} & \overset{*}{E}
\end{array}$$

is exact.

Definition 3.4.1.2 *A* **Man**-morphism $(\rho,\eta)\,\overset{*}{\Gamma}$ of $(\rho,\eta)\,T\overset{*}{E}$ source and $V\,(\rho,\eta)\,T\overset{*}{E}$ target defined by

$$(\rho,\eta)\,\Gamma\left(\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y_b\frac{\partial}{\partial\tilde{p}_b}\right)\left(\overset{*}{u}_x\right)=\left(Y_b-(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\tilde{Z}^{\alpha}\right)\frac{\partial}{\partial\tilde{p}_b}\left(\overset{*}{u}_x\right), \tag{3.4.1.5}$$

such that the $\mathbf{B^v}$-morphism $\left((\rho,\eta)\,\overset{*}{\Gamma},Id_{\overset{*}{E}}\right)$ is a split to the left in the previous exact sequence, will be called (ρ,η)*-connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

The differentiable real local functions $(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}$ will be called the *components of* (ρ,η)*-connection* $(\rho,\eta)\,\overset{*}{\Gamma}$.

The (ρ,Id_M)-connection for the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be called ρ-*connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and will be denoted $\rho\overset{*}{\Gamma}$.

The (Id_{TM},Id_M)-connection for the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be called *connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and will be denoted $\overset{*}{\Gamma}$.

Let $\left\{s^a,\ a\in\overline{1,r}\right\}$ be the dual base of the base $\left\{s_a,a\in\overline{1,r}\right\}$.

The $\mathbf{B^v}$-morphism $\left(\overset{*}{\Pi},\overset{*}{\pi}\right)$ defined by the commutative diagram

$$(3.4.1.6)\qquad \begin{array}{ccc} V(\rho,\eta)\,T\overset{*}{E} & \xrightarrow{\overset{*}{\Pi}} & \overset{*}{E} \\ {\scriptstyle (\rho,\eta)\tau_{\overset{*}{E}}}\downarrow & & \downarrow{\scriptstyle \overset{*}{\pi}} \\ \overset{*}{E} & \xrightarrow{\overset{*}{\pi}} & M \end{array},$$

where, $\overset{*}{\Pi}$ is defined by

$$(3.4.1.7)\qquad \overset{*}{\Pi}\left(Y_a\frac{\partial}{\partial\tilde{p}_a}\left(\overset{*}{u}_x\right)\right)=Y_a\left(\overset{*}{u}_x\right)s^a\left(\overset{*}{\pi}\left(\overset{*}{u}_x\right)\right),$$

is canonical projection $\mathbf{B^v}$-morphism.

Theorem 3.4.1.1 *If $(\rho,\eta)\overset{*}{\Gamma}$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then its components satisfy the law of transformation*

$$(3.4.1.8)\qquad (\rho,\eta)\overset{*}{\Gamma}_{\acute{b}\acute{\gamma}}=M_{\acute{b}}^{b}\circ\overset{*}{\pi}\left[-\rho_{\gamma}^{i}\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_{b}^{\acute{a}}\circ\overset{*}{\pi}}{\partial x^{i}}p_{\acute{a}}+(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ\left(h\circ\overset{*}{\pi}\right).$$

In particular, if $h=Id_M$, then the relations (3.4.1.8) *become*

$$(3.4.1.8')\qquad (\rho,\eta)\overset{*}{\Gamma}_{\acute{b}\acute{\gamma}}=M_{\acute{b}}^{b}\circ\overset{*}{\pi}\left[-\rho_{\gamma}^{i}\circ\overset{*}{\pi}\cdot\frac{\partial M_{b}^{\acute{a}}\circ\overset{*}{\pi}}{\partial x^{i}}p_{\acute{a}}+(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ\overset{*}{\pi}.$$

In particular, if $(\rho,\eta)=(Id_{TM},Id_M)$, then the relations $(3.4.1.8')$ *become*

$$(3.4.1.8'')\qquad \overset{*}{\Gamma}_{\acute{j}\acute{k}}=\frac{\partial x^{j}}{\partial x^{\acute{j}}}\circ\overset{*}{\pi}\left[-\frac{\partial}{\partial x^{i}}\left(\frac{\partial x^{\acute{i}}}{\partial x^{j}}\circ\overset{*}{\pi}\right)p_{\acute{i}}+\overset{*}{\Gamma}_{jk}\right]\frac{\partial x^{k}}{\partial x^{\acute{k}}}\circ\overset{*}{\pi}.$$

Proof. Let $\left(\overset{*}{\Pi},\overset{*}{\pi}\right)$ be the canonical projection $\mathbf{B}$-morphism.

Obviously, the components of

$$\Pi^{*}\circ(\rho,\eta)\overset{*}{\Gamma}\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y_{\acute{b}}\frac{\partial}{\partial\tilde{p}_{\acute{b}}}\right)\left(\overset{*}{u}_x\right)$$

are the real numbers

$$\left(Y_{\acute{b}}-(\rho,\eta)\overset{*}{\Gamma}_{\acute{b}\acute{\gamma}}\tilde{Z}^{\acute{\gamma}}\right)\left(\overset{*}{u}_x\right).$$

Since

$$\begin{aligned}\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y_{\acute{b}}\frac{\partial}{\partial\tilde{p}_{\acute{b}}}\right)\left(\overset{*}{u}_x\right)&=\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\overset{*}{\pi}\frac{\partial}{\partial\tilde{z}^{\alpha}}\left(\overset{*}{u}_x\right)\\&+\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\overset{*}{\pi}\frac{\partial M_{b}^{\acute{a}}\circ\pi}{\partial x^{\acute{i}}}p_{\acute{a}}+M_{b}^{\acute{b}}Y_{\acute{b}}\right)\frac{\partial}{\partial\tilde{p}_{b}}\left(\overset{*}{u}_x\right),\end{aligned}$$

it results that the components of

$$\Pi^{*}\circ(\rho,\eta)\overset{*}{\Gamma}\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}}+Y_{\acute{b}}\frac{\partial}{\partial\tilde{p}_{\acute{b}}}\right)\left(\overset{*}{u}_x\right)$$

are the real numbers

$$\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\overset{*}{\pi}\frac{\partial M_{b}^{\acute{a}}\circ\overset{*}{\pi}}{\partial x^{\acute{i}}}p_{\acute{a}}+M_{b}^{\acute{b}}\circ\overset{*}{\pi}Y_{\acute{b}}-(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\overset{*}{\pi}\right)M_{\acute{b}}^{b}\circ\overset{*}{\pi}\left(\overset{*}{u}_{x}\right),$$

where $\left\|M_{\acute{b}}^{b}\right\|=\left\|M_{b}^{\acute{b}}\right\|^{-1}$.

Therefore, we have:

$$\begin{aligned}&\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\overset{*}{\pi}\frac{\partial M_{b}^{\acute{a}}\circ\overset{*}{\pi}}{\partial x^{\acute{i}}}p_{\acute{a}}+M_{b}^{\acute{b}}\circ\overset{*}{\pi}Y_{\acute{b}}-(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\overset{*}{\pi}\right)M_{\acute{b}}^{b}\circ\overset{*}{\pi}\\&\qquad=Y_{\acute{b}}-(\rho,\eta)\overset{*}{\Gamma}_{\acute{b}\acute{\alpha}}\tilde{Z}^{\acute{\alpha}}.\end{aligned}$$

After some calculations we obtain:

$$(\rho,\eta)\overset{*}{\Gamma}_{\acute{b}\acute{\alpha}}=M_{\acute{b}}^{b}\circ\overset{*}{\pi}\left(-\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_{b}^{\acute{a}}\circ\overset{*}{\pi}}{\partial x^{i}}p_{\acute{a}}+(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\right)\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\overset{*}{\pi}.\qquad q.e.d.$$

Remark 3.4.1.1 If we have a set of real local functions $(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}$ which satisfies the relations of passing (3.4.1.8), then we have a (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$ for the fiber bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

Example 3.4.1.1 If $\overset{*}{\Gamma}$ is a classical connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ on components $\overset{*}{\Gamma}_{bk}$, then the differentiable real local functions

$$(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}=\left(\rho_{\gamma}^{k}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\Gamma}_{bk}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$ for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ which will be called the (ρ,η)*-connection associated to the connection* $\overset{*}{\Gamma}$.

Definition 3.4.1.3 If $(\rho,\eta)\overset{*}{\Gamma}$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then for any

$$z=z^{\alpha}t_{\alpha}\in\Gamma(F,\nu,N)$$

the application

$$\begin{array}{ccc}\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)&\xrightarrow{(\rho,\eta)D_{z}}&\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)\\\overset{*}{u}=u_{a}s^{a}&\longmapsto&(\rho,\eta)D_{z}\overset{*}{u}\end{array}\tag{3.4.1.9}$$

defined by

$$(\rho,\eta)D_{z}\overset{*}{u}=z^{\alpha}\circ h\left(\rho_{\alpha}^{i}\circ h\frac{\partial u_{b}}{\partial x^{i}}-(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\circ\overset{*}{u}\right)s^{b},$$

will be called the *covariant* (ρ,η)*-derivative associated to* (ρ,η)*-connection* $(\rho,\eta)\overset{*}{\Gamma}$ *with respect to section* z.

If $h=Id_M$ and $\eta=Id_M$, then we obtain the *covariant* ρ*-derivative associated to* ρ*-connection* $\rho\overset{*}{\Gamma}$ *with respect to section* z.

In addition, if $\rho=Id_{TM}$, then we obtain the *covariant derivative associated to connection* $\overset{*}{\Gamma}$ *with respect to the vector field* z.

Definition 3.4.1.4 We will say that the (ρ,η)*-connection* $(\rho,\eta)\overset{*}{\Gamma}$ *is homogeneous* or *linear* if the local real functions $(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}$ are homogeneous or linear on the fibre of vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ respectively.

Remark 3.4.1.2 If $(\rho,\eta)\overset{*}{\Gamma}$ is a linear (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then for each local vector $(m+r)$-chart $\left(U,\overset{*}{s}_U\right)$ and for each local vector $(n+p)$-chart (V,t_V) such that $U\cap h^{-1}(V)\neq\phi$, there exists the differentiable real functions $\rho\Gamma^{a}_{b\gamma}$ defined on $U\cap h^{-1}(V)$ such that

$$(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}\circ\overset{*}{u}=(\rho,\eta)\Gamma^{a}_{b\gamma}\cdot u_a, \tag{3.4.1.10}$$

for any $\overset{*}{u}=u_as^a\in\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

The differentiable real local functions $(\rho,\eta)\Gamma^{a}_{b\alpha}$ will be called the *Christoffel coefficients of linear* (ρ,η)*-connection* $(\rho,\eta)\Gamma$.

Theorem 3.4.1.2 *If* $(\rho,\eta)\Gamma$ *is a linear* (ρ,η)*-connection for the vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, *then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma^{\acute{a}}_{\acute{b}\acute{\gamma}}=M^{b}_{\acute{b}}\left[-\rho^{i}_{\gamma}\circ h\frac{\partial M^{\acute{a}}_{b}}{\partial x^{i}}+(\rho,\eta)\Gamma^{a}_{b\gamma}M^{\acute{a}}_{a}\right]\Lambda^{\gamma}_{\acute{\gamma}}\circ h. \tag{3.4.1.11}$$

In particular, if $(\rho,\eta)=(Id_{TM},Id_M)$ *and* $h=Id_M$, *then the relations* (3.4.1.11) *become*

$$\Gamma^{\acute{i}}_{\acute{j}\acute{k}}=\frac{\partial x^{j}}{\partial x^{\acute{j}}}\left[-\frac{\partial}{\partial x^{i}}\left(\frac{\partial x^{\acute{i}}}{\partial x^{j}}\right)+\Gamma^{i}_{jk}\frac{\partial x^{\acute{i}}}{\partial x^{i}}\right]\frac{\partial x^{k}}{\partial x^{\acute{k}}}. \tag{3.4.1.11$'$}$$

Remark 3.4.1.3 Since

$$\frac{\partial M^{\acute{a}}_{b}}{\partial x^{i}}M^{b}_{\acute{b}}+\frac{\partial M^{b}_{\acute{b}}}{\partial x^{i}}M^{\acute{a}}_{b}=0,$$

it results that the relations (3.4.11) are equivalent with the relations (3.4.1.11′).

Definition 3.4.1.5 If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then for any

$$z=z^{\alpha}t_{\alpha}\in\Gamma(F,\nu,N)$$

the application

$$\begin{array}{lrcl} (3.4.1.12) & \Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right) & \xrightarrow{(\rho,\eta)D_z} & \Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right) \\ & \overset{*}{u}=u_a s^a & \longmapsto & (\rho,\eta)D_z\overset{*}{u} \end{array}$$

defined by

$$(\rho,\eta)D_z\overset{*}{u}=z^\alpha\circ h\left(\rho^i_\alpha\circ h\frac{\partial u_b}{\partial x^i}-(\rho,\eta)\Gamma^a_{b\alpha}\cdot u_a\right)s^b$$

will be called the *covariant* (ρ,η)*-derivative associated to linear* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$ *with respect to section* z.

If $h=Id_M$ and $\eta=Id_M$, then we obtain the *covariant* ρ*-derivative associated to linear* ρ*-connection* $\rho\Gamma$ *with respect to section* z.

In addition, if $\rho=Id_{TM}$, then we obtain the *covariant derivative associated to linear connection* Γ *with respect to vector field* z.

Note. In the next we use the same notation $(\rho,\eta)\,\Gamma$ for the linear (ρ,η)-connection for the vector bundle (E,π,M) or for its dual $\left(\overset{*}{E},\overset{*}{\pi},M\right)$

Remark 3.4.1.4 If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M) or for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ then, the tensor fields algebra

$$\left(\mathcal{T}\left(E,\pi,M\right),+,\cdot,\otimes\right)$$

is endowed with the (ρ,η)-derivative

$$\begin{array}{lrcl} (3.4.1.13) & \Gamma\left(F,\nu,N\right)\times\mathcal{T}\left(E,\pi,M\right) & \xrightarrow{(\rho,\eta)D} & \mathcal{T}\left(E,\pi,M\right) \\ & (z,T) & \longmapsto & (\rho,\eta)\,D_zT \end{array}$$

defined for a tensor field $T\in\mathcal{T}^p_q\left(E,\pi,M\right)$ by the relation:

$$\begin{array}{ll} & (\rho,\eta)\,D_zT\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)=\Gamma\left(\rho,\eta\right)(z)\left(T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)\right) \\ (3.4.1.14) & -T\left((\rho,\eta)\,D_z\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)-...-T\left(\overset{*}{u}_1,...,(\rho,\eta)\,D_z\overset{*}{u}_p,u_1,...,u_q\right) \\ & -T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,(\rho,\eta)\,D_zu_1,...,u_q\right)-...-T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,(\rho,\eta)\,D_zu_q\right). \end{array}$$

Moreover, it satisfies the condition

$$(3.4.1.15)\qquad (\rho,\eta)\,D_{f_1z_1+f_2z_2}T=f_1\,(\rho,\eta)\,D_{z_1}T+f_2\,(\rho,\eta)\,D_{z_2}T.$$

Consequently, if the tensor algebra $\left(\mathcal{T}\left(E,\pi,M\right),+,\cdot,\otimes\right)$ is endowed with a (ρ,η)-derivative (3.4.1.13) defined for a tensor field $T\in\mathcal{T}{}^p_q\left(E,\pi,M\right)$ by (3.4.1.14) which satisfies the condition (3.4.1.15), then we can endowed (E,π,M) with a linear (ρ,η)-connection $(\rho,\eta)\,\Gamma$ such that its components are defined by the equality:

$$(\rho,\eta)\,D_{t_\alpha}s_b=(\rho,\eta)\,\Gamma^a_{b\alpha}s_a$$

or

$$(\rho,\eta)\,D_{t_\alpha}s^a=-\,(\rho,\eta)\,\Gamma^a_{b\alpha}s^b.$$

The (ρ,η)-derivative (3.4.1.13) will be called the *covariant* (ρ,η)*-derivative.*

After some calculations, we obtain:

$$\begin{array}{l}(\rho,\eta)\,D_z\left(T_{b_1,\dots,b_q}^{a_1,\dots,a_p}s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\right)\\ =z^\alpha\circ h\left(\rho_\alpha^i\circ h\dfrac{\partial T_{b_1,\dots,b_q}^{a_1,\dots,a_p}}{\partial x^i}+(\rho,\eta)\,\Gamma_{a\alpha}^{a_1}T_{b_1,\dots,b_q}^{a,a_2\dots,a_p}\right.\\ +(\rho,\eta)\,\Gamma_{a\alpha}^{a_2}T_{b_1,\dots,b_q}^{a_1,a\dots,a_p}+\dots+(\rho,\eta)\,\Gamma_{a\alpha}^{a_p}T_{b_1,\dots,b_q}^{a_1,a_2\dots,a}-\dots\\ -(\rho,\eta)\,\Gamma_{b_1\alpha}^{b}T_{b,b_2,\dots,b_q}^{a_1,a_2\dots,a_p}-(\rho,\eta)\,\Gamma_{b_2\alpha}^{b}T_{b_1,b,\dots,b_q}^{a_1,a_2\dots,a_p}-\dots\\ \left.-(\rho,\eta)\,\Gamma_{b_q\alpha}^{b}T_{b_1,b_2,\dots,b}^{a_1,a_2\dots,a_p}\right)s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\\ \overset{put}{=}z^\alpha\circ hT_{b_1,\dots,b_q|\alpha}^{a_1,\dots,a_p}s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}.\end{array}\tag{3.4.1.16}$$

If $(\rho,\eta)\,\Gamma$ is the linear (ρ,η)-connection associated to linear connection Γ, namely $(\rho,\eta)\,\Gamma_{b\alpha}^a=\left(\rho_\alpha^k\circ h\right)\Gamma_{bk}^a$, then

$$T_{b_1,\dots,b_q|\alpha}^{a_1,\dots,a_p}=\left(\rho_\alpha^k\circ h\right)T_{b_1,\dots,b_q|k}^{a_1,\dots,a_p}.\tag{3.4.1.17}$$

In particular, if $h=Id_M$, then we obtain the formula:

$$\begin{array}{l}(\rho,\eta)\,D_z\left(T_{b_1,\dots,b_q}^{a_1,\dots,a_p}s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\right)\\ =z^\alpha\left(\rho_\alpha^i\dfrac{\partial T_{b_1,\dots,b_q}^{a_1,\dots,a_p}}{\partial x^i}+(\rho,\eta)\,\Gamma_{a\alpha}^{a_1}T_{b_1,\dots,b_q}^{a,a_2\dots,a_p}\right.\\ +(\rho,\eta)\,\Gamma_{a\alpha}^{a_2}T_{b_1,\dots,b_q}^{a_1,a\dots,a_p}+\dots+(\rho,\eta)\,\Gamma_{a\alpha}^{a_p}T_{b_1,\dots,b_q}^{a_1,a_2\dots,a}-\dots\\ -(\rho,\eta)\,\Gamma_{b_1\alpha}^{b}T_{b,b_2,\dots,b_q}^{a_1,a_2\dots,a_p}-(\rho,\eta)\,\Gamma_{b_2\alpha}^{b}T_{b_1,b,\dots,b_q}^{a_1,a_2\dots,a_p}-\dots\\ \left.-(\rho,\eta)\,\Gamma_{b_q\alpha}^{b}T_{b_1,b_2,\dots,b}^{a_1,a_2\dots,a_p}\right)s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\\ \overset{put}{=}z^\alpha T_{b_1,\dots,b_q|\alpha}^{a_1,\dots,a_p}s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}.\end{array}\tag{3.4.1.18}$$

4 The geometry of base of the Lie algebroid generalized tangent bundle for a vector bundle

In this section, we present new applications of generalized Lie algebroids in the study of the geometry of vector bundles using the theory of generalized linear connections.

4.1 Torsion and curvature. Formulas of Ricci type

We apply the theory for the diagram:

$$\begin{array}{ccc}E & & \left(F,[,]_{F,h},(\rho,Id_M)\right)\\ \downarrow\pi & & \downarrow\nu\\ M & \xrightarrow{h} & M\end{array},\tag{4.1.1}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,M),[,]_{F,h},(\rho,Id_M)\right)\in|\mathbf{GLA}|$.

Let $\rho\Gamma$ be a linear ρ-connection for the vector bundle (E,π,M) by components $\rho\Gamma_{b\alpha}^{a}$.

Using the components of the linear ρ-connection $\rho\Gamma$, then we obtain a linear ρ-connection $\rho\dot{\Gamma}$ for the vector bundle (E,π,M) given by the diagram:

$$\begin{array}{ccc} E & & \left(h^{*}F,[,]_{h^{*}F},\left(\overset{h^{*}F}{\rho},Id_{M}\right)\right) \\ \pi\downarrow & & \downarrow h^{*}\nu \\ M & \xrightarrow{Id_{M}} & M \end{array} \tag{4.1.2}$$

If $(E,\pi,M)=(F,\nu,N)$, then, using the components of the linear ρ-connection $\rho\Gamma$, we can consider a linear ρ-connection $\rho\ddot{\Gamma}$ for the vector bundle $(h^{*}E,h^{*}\pi,M)$ given by the diagram:

$$\begin{array}{ccc} h^{*}E & & \left(h^{*}E,[,]_{h^{*}E},\left(\overset{h^{*}E}{\rho},Id_{M}\right)\right) \\ h^{*}\pi\downarrow & & \downarrow h^{*}\pi \\ M & \xrightarrow{Id_{M}} & M \end{array}, \tag{4.1.3}$$

In the following, we will use the exterior differentiation operators d, d^{E} and $d^{h^{*}E}$ respectively for the exterior differential $\mathcal{F}(M)$-algebras $(\Lambda(TM,\tau_{M},M),+,\cdot,\wedge)$, $(\Lambda(E,\pi,M),+,\cdot,\wedge)$ and $((h^{*}E,h^{*}\pi,M),+,\cdot,\wedge)$ respectively.

Definition 4.1.1 If $(E,\pi,M)=(F,\nu,N)$, then the application

$$\begin{array}{ccc} \Gamma(h^{*}E,h^{*}\pi,M)^{2} & \xrightarrow{(\rho,h)\mathbb{T}} & \Gamma(h^{*}E,h^{*}\pi,M) \\ (U,V) & \longrightarrow & \rho\mathbb{T}(U,V) \end{array} \tag{4.1.4}$$

defined by:

$$(\rho,h)\mathbb{T}(U,V)=\rho\ddot{D}_{U}V-\rho\ddot{D}_{V}U-[U,V]_{h^{*}E}, \tag{4.1.5}$$

for any $U,V\in\Gamma(h^{*}E,h^{*}\pi,M)$, will be called *$(\rho,h)$-torsion associated to linear ρ-connection $\rho\Gamma$.*

Remark 4.1.1 In particular, if $h=Id_{M}$, then we obtain the application

$$\begin{array}{ccc} \Gamma(E,\pi,M)^{2} & \xrightarrow{\rho\mathbb{T}} & \Gamma(E,\pi,M) \\ (u,v) & \longrightarrow & \rho\mathbb{T}(u,v) \end{array} \tag{4.1.4'}$$

defined by:

$$\rho\mathbb{T}(u,v)=\rho D_{u}v-\rho D_{v}u-[u,v]_{E}, \tag{4.1.5'}$$

for any $u,v\in\Gamma(E,\pi,M)$, which will be called *ρ-torsion associated to linear ρ-connection $\rho\Gamma$.*

Moreover, if $\rho=Id_{TM}$, then we obtain the torsion $\mathbb{T}$ associated to linear connection Γ.

Proposition 4.1.1 *The (ρ,h)-torsion $(\rho,h)\mathbb{T}$ associated to linear ρ-connection $\rho\Gamma$ is $\mathbb{R}$-bilinear and antisymmetric.*

If

$$(\rho,h)\mathbb{T}(S_{a},S_{b})\overset{put}{=}(\rho,h)\mathbb{T}^{c}{}_{ab}S_{c}$$

then

$$(4.1.6)\qquad (\rho,h)\,\mathbb{T}^{c}{}_{ab}=\rho\Gamma^{c}_{ab}-\rho\Gamma^{c}_{ba}-L^{c}_{ab}\circ h.$$

In particular, if $h=Id_M$ *and* $\rho\mathbb{T}\left(s_a,s_b\right)\overset{put}{=}\rho\mathbb{T}^{c}_{ab}s_c$*, then*

$$(4.1.6')\qquad \rho\mathbb{T}^{c}{}_{ab}=\rho\Gamma^{c}_{ab}-\rho\Gamma^{c}_{ba}-L^{c}_{ab}.$$

Moreover, if $\rho=Id_{TM}$*, then the equality* $(4.1.6')$ *becomes:*

$$(4.1.6'')\qquad \mathbb{T}^{i}{}_{jk}=\Gamma^{i}_{jk}-\Gamma^{i}_{kj}.$$

Definition 4.1.2 The application

$$(4.1.7)\qquad \begin{array}{ccc}\left(\Gamma\left(h^{*}F,h^{*}\nu,M\right)^{2}\times\Gamma(E,\pi,M)\right. & \xrightarrow{(\rho,h)\mathbb{R}} & \Gamma(E,\pi,M)\\ ((Z,V),u) & \longrightarrow & \rho\mathbb{R}(Z,V)u\end{array}$$

defined by

$$(4.1.8)\qquad (\rho,h)\,\mathbb{R}\left(Z,V\right)u=\rho\dot{D}_{Z}\left(\rho\dot{D}_{V}u\right)-\rho\dot{D}_{V}\left(\rho\dot{D}_{Z}u\right)-\rho\dot{D}_{[Z,V]_{h^{*}F}}u,$$

for any $Z,V\in\Gamma\left(h^{*}F,h^{*}\nu,M\right),\ u\in\Gamma\left(E,\pi,M\right)$, will be called (ρ,h)*-curvature associated to linear* ρ*-connection* $\rho\Gamma$.

Remark 4.1.1 In particular, if $h=Id_M$, then we obtain the application

$$(4.1.7')\qquad \begin{array}{ccc}\Gamma\left(F,\nu,M\right)^{2}\times\Gamma(E,\pi,M) & \xrightarrow{\rho\mathbb{R}} & \Gamma(E,\pi,M)\\ ((z,v),u) & \longrightarrow & \rho\mathbb{R}(z,v)u\end{array}$$

defined by

$$(4.1.8')\qquad \rho\mathbb{R}\left(z,v\right)u=\rho D_{z}\left(\rho D_{v}u\right)-\rho D_{v}\left(\rho D_{z}u\right)-\rho D_{[z,v]_{F}}u,$$

for any $z,v\in\Gamma\left(F,\nu,M\right),\ u\in\Gamma\left(E,\pi,M\right)$, which will be called ρ*-curvature associated to linear* ρ*-connection* $\rho\Gamma$.

Moreover, if $\rho=Id_{TM}$, then we obtain the curvature $\mathbb{R}$ associated to linear connection Γ.

Proposition 4.1.2 *The* (ρ,h)*-curvature* $(\rho,h)\,\mathbb{R}$ *associated to linear* ρ*-connection* $\rho\Gamma$*, is* $\mathbb{R}$*-linear in each argument and antisymmetric in the first two arguments.*

If

$$(\rho,h)\,\mathbb{R}\left(T_{\beta},T_{\alpha}\right)s_{b}\overset{put}{=}(\rho,h)\,\mathbb{R}^{a}_{b\ \alpha\beta}s_{a},$$

then

$$(4.1.9)\quad (\rho,h)\,\mathbb{R}^{a}_{b\ \alpha\beta}=\rho^{j}_{\beta}\circ h\frac{\partial\rho\Gamma^{a}_{b\alpha}}{\partial x^{j}}+\rho\Gamma^{a}_{e\beta}\rho\Gamma^{e}_{b\alpha}-\rho^{i}_{\alpha}\circ h\frac{\partial\rho\Gamma^{a}_{b\beta}}{\partial x^{i}}-\rho\Gamma^{a}_{e\alpha}\rho\Gamma^{e}_{b\beta}+\rho\Gamma^{a}_{b\gamma}L^{\gamma}_{\alpha\beta}\circ h.$$

In particular, if $h=Id_M$ *and* $\rho\mathbb{R}\left(t_{\beta},t_{\alpha}\right)s_{b}\overset{put}{=}\rho\mathbb{R}^{a}_{b\ \alpha\beta}s_{a}$*, then*

$$(4.1.9')\qquad \rho\mathbb{R}^{a}_{b\ \alpha\beta}=\rho^{j}_{\beta}\frac{\partial\rho\Gamma^{a}_{b\alpha}}{\partial x^{j}}+\rho\Gamma^{a}_{e\beta}\rho\Gamma^{e}_{b\alpha}-\rho^{i}_{\alpha}\frac{\partial\rho\Gamma^{a}_{b\beta}}{\partial x^{i}}-\rho\Gamma^{a}_{e\alpha}\rho\Gamma^{e}_{b\beta}+\rho\Gamma^{a}_{b\gamma}L^{\gamma}_{\alpha\beta}.$$

Moreover, if $\rho = Id_{TM}$*, then equality* $(4.1.9')$ *becomes:*

$$\mathbb{R}_{b\ hk}^{a} = \frac{\partial \Gamma_{bh}^{a}}{\partial x^{k}} + \Gamma_{ek}^{a}\Gamma_{bh}^{e} - \frac{\partial \Gamma_{bk}^{a}}{\partial x^{h}} - \Gamma_{eh}^{a}\Gamma_{bk}^{e}. \tag{4.1.9''}$$

Theorem 4.1.1 *For any* $u^{a}s_{a} \in \Gamma(E, \pi, M)$ *we shall use the notation*

$$u^{a}{}_{|\alpha\beta} = \rho_{\beta}^{j} \circ h\frac{\partial}{\partial x^{j}}\left(u^{a_1}{}_{|\alpha}\right) + \rho\Gamma_{b_{\beta}}^{a_1}u^{b}{}_{|\alpha}, \tag{4.1.10}$$

and we verify the formulas:

$$u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} = u^{a}(\rho, h)\mathbb{R}_{a\ \alpha\beta}^{a_1} - u^{a_1}{}_{|\gamma}L_{\alpha\beta}^{\gamma} \circ h. \tag{4.1.11}$$

After some calculations, we obtain

$$(\rho, h)\mathbb{R}_{a\ \alpha\beta}^{a_1} = u_{a}\left(u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} + u^{a_1}{}_{|\gamma}L_{\alpha\beta}^{\gamma} \circ h\right), \tag{4.1.12}$$

where $u_{a}s^{a} \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ *such that* $u_{a}u^{b} = \delta_{a}^{b}$*.*

In particular, if $h = Id_{M}$*, then the relations* $(4.1.12)$ *become*

$$\rho\mathbb{R}_{a\ \alpha\beta}^{a_1} = u_{a}\left(u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} + u^{a_1}{}_{|\gamma}L_{\alpha\beta}^{\gamma}\right). \tag{4.1.12'}$$

Moreover, if $\rho = id_{TM}$*, then the relations* $(4.1.12')$ *become*

$$\mathbb{R}_{a\ ij}^{a_1} = u_{a}\left(u^{a_1}{}_{|ij} - u^{a_1}{}_{|ji}\right). \tag{4.1.12''}$$

Proof. Since

$$\begin{aligned}
u^{a_1}{}_{|\alpha\beta} &= \rho_{\beta}^{j} \circ h\left(\frac{\partial}{\partial x^{j}}\left(\rho_{\alpha}^{i} \circ h\frac{\partial u^{a_1}}{\partial x^{i}} + \rho\Gamma_{a\alpha}^{a_1}u^{a}\right)\right) \\
&+ \rho\Gamma_{b\beta}^{a_1}\left(\rho_{\alpha}^{i} \circ h\frac{\partial u^{b}}{\partial x^{i}} + \rho\Gamma_{a\alpha}^{b}u^{a}\right) \\
&= \rho_{\beta}^{j} \circ h\frac{\partial \rho_{\alpha}^{i} \circ h}{\partial x^{j}}\frac{\partial u^{a_1}}{\partial x^{i}} + \rho_{\beta}^{j} \circ h\rho_{\alpha}^{i} \circ h\frac{\partial}{\partial x^{j}}\left(\frac{\partial u^{a_1}}{\partial x^{i}}\right) \\
&+ \rho_{\beta}^{j} \circ h\frac{\partial \rho\Gamma_{a\alpha}^{a_1}}{\partial x^{j}}u^{a} + \rho_{\beta}^{j} \circ h\rho\Gamma_{a\alpha}^{a_1}\frac{\partial u^{a}}{\partial x^{j}} \\
&+ \rho_{\alpha}^{i} \circ h\rho\Gamma_{b\beta}^{a_1}\frac{\partial u^{b}}{\partial x^{i}} + \rho\Gamma_{b\beta}^{a_1}\rho\Gamma_{a\alpha}^{b}u^{a}
\end{aligned}$$

and

$$\begin{aligned}
u^{a_1}{}_{|\beta\alpha} &= \rho_{\alpha}^{i} \circ h\left(\frac{\partial}{\partial x^{i}}\left(\rho_{\beta}^{j} \circ h\frac{\partial u^{a_1}}{\partial x^{j}} + \rho\Gamma_{a\beta}^{a_1}u^{a}\right)\right) \\
&+ \rho\Gamma_{b\alpha}^{a_1}\left(\rho_{\beta}^{j} \circ h\frac{\partial u^{b}}{\partial x^{j}} + \rho\Gamma_{a\beta}^{b}u^{a}\right) \\
&= \rho_{\alpha}^{i} \circ h\frac{\partial \rho_{\beta}^{j} \circ h}{\partial x^{i}}\frac{\partial u^{a_1}}{\partial x^{j}} + \rho_{\beta}^{j} \circ h\rho_{\alpha}^{i} \circ h\frac{\partial}{\partial x^{i}}\left(\frac{\partial u^{a_1}}{\partial x^{j}}\right) \\
&+ \rho_{\alpha}^{i} \circ h\frac{\partial \rho\Gamma_{a\beta}^{a_1}}{\partial x^{i}}u^{a} + \rho_{\alpha}^{i} \circ h\rho\Gamma_{a\beta}^{a_1}\frac{\partial u^{a}}{\partial x^{i}} \\
&+ \rho_{\beta}^{j} \circ h\rho\Gamma_{b\alpha}^{a_1}\frac{\partial u^{b}}{\partial x^{j}} + \rho\Gamma_{b\alpha}^{a_1}\rho\Gamma_{a\beta}^{b}u^{a},
\end{aligned}$$

it results that

$$\begin{aligned} u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} &= \rho^j_\beta \circ h \frac{\partial \rho^i_\alpha \circ h}{\partial x^j} \frac{\partial u^{a_1}}{\partial x^i} - \rho^i_\alpha \circ h \frac{\partial \rho^j_\beta \circ h}{\partial x^i} \frac{\partial u^{a_1}}{\partial x^j} \\ &+ \left(\rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial^2 u^{a_1}}{\partial x^i \partial x^j} - \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial^2 u^{a_1}}{\partial x^j \partial x^i} \right) \\ &+ \left(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} u^a \right) \\ &+ \left(\rho^j_\beta \circ h \rho\Gamma^{a_1}_{a\alpha} \frac{\partial u^a}{\partial x^j} - \rho^j_\beta \circ h \rho\Gamma^{a_1}_{b\alpha} \frac{\partial u^b}{\partial x^j} \right) \\ &+ \left(\rho^i_\alpha \circ h \rho\Gamma^{a_1}_{b\beta} \frac{\partial u^b}{\partial x^i} - \rho^i_\alpha \circ h \rho\Gamma^{a_1}_{a\beta} \frac{\partial u^a}{\partial x^i} \right) \\ &+ \rho\Gamma^{a_1}_{b\beta} \rho\Gamma^b_{a\alpha} u^a - \rho\Gamma^{a_1}_{b\alpha} \rho\Gamma^b_{a\beta} u^a. \end{aligned}$$

After some calculations, we obtain:

$$\begin{aligned} u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} &= L^\gamma_{\beta\alpha} \circ h \rho^k_\gamma \circ h \frac{\partial u^{a_1}}{\partial x^k} \\ &+ \left(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} u^a \right) \\ &+ \rho\Gamma^{a_1}_{b\beta} \rho\Gamma^b_{a\alpha} u^a - \rho\Gamma^{a_1}_{b\alpha} \rho\Gamma^b_{a\beta} u^a. \end{aligned}$$

Since

$$\begin{aligned} u^a (\rho, h) \mathbb{R}^{a_1}_{a\ \alpha\beta} \quad = \quad & u^a \left(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} + \rho\Gamma^{a_1}_{e\beta} \rho\Gamma^e_{a\alpha} - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} \right. \\ & \left. - \rho\Gamma^{a_1}_{e\alpha} \rho\Gamma^e_{a\beta} - \rho\Gamma^{a_1}_{a\gamma} L^\gamma_{\beta\alpha} \circ h \right). \end{aligned}$$

and

$$u^{a_1}{}_{|\gamma} L^\gamma_{\alpha\beta} \circ h = \left(\rho^k_\gamma \circ h \frac{\partial u^{a_1}}{\partial x^k} + \rho\Gamma^{a_1}_{a\gamma} u^a \right) L^\gamma_{\alpha\beta} \circ h$$

it results that

$$\begin{aligned} u^a (\rho, h) \mathbb{R}^{a_1}_{a\ \alpha\beta} - u^{a_1}{}_{|\gamma} L^\gamma_{\alpha\beta} \circ h &= -L^\gamma_{\alpha\beta} \circ h \rho^k_\gamma \circ h \frac{\partial u^{a_1}}{\partial x^k} \\ &+ \left(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} u^a \right) \\ &+ \rho\Gamma^{a_1}_{b\beta} \rho\Gamma^b_{a\alpha} u^a - \rho\Gamma^{a_1}_{b\alpha} \rho\Gamma^b_{a\beta} u^a. \end{aligned}$$

q.e.d.

Lemma 4.1.1 *If* $(E, \pi, M) = (F, \nu, N)$*, then, for any*

$$u^a s_a \in \Gamma(E, \pi, M),$$

we have that $u^a{}_{|c}$, $a, c \in \overline{1, n}$ are the *components of a tensor field of* $(1,1)$ *type.*

Proof. Let U and U' be two vector local $(m+n)$-charts such that $U \cap U' \neq \phi$.
Since $u^{a'}(x) = M_a^{a'}(x)\, u^a(x)$, for any $x \in U \cap U'$, it results that

$$\rho_{c'}^{k'} \circ h(x) \frac{\partial u^{a'}(x)}{\partial x^{k'}} = \rho_{c'}^{k'} \circ h(x) \frac{\partial}{\partial x^{k'}} \left(M_a^{a'}(x) \right) u^a(x) + M_a^{a'}(x)\, \rho_{c'}^{k'} \circ h(x) \frac{\partial u^a(x)}{\partial x^{k'}}. \quad (1)$$

Since , for any $x \in U \cap U'$, we have

$$\rho\Gamma_{b'c'}^{a'}(x) = M_a^{a'}(x) \left(\rho_c^k \circ h(x) \frac{\partial}{\partial x^k}(M_{b'}^a(x)) + \rho\Gamma_{bc}^a(x) M_{b'}^b(x) \right) M_{c'}^c(x), \quad (2)$$

and

$$0 = \frac{\partial}{\partial x^{k'}} \left(M_a^{a'}(x)\, M_{b'}^a(x) \right) = \frac{\partial}{\partial x^{k'}} \left(M_a^{a'}(x) \right) M_{b'}^a(x) + M_a^{a'}(x) \frac{\partial}{\partial x^{k'}} \left(M_{b'}^a(x) \right) \quad (3)$$

it results that

$$\begin{aligned} \rho\Gamma_{b'c'}^{a'}(x)\, u^{b'}(x) = & -\rho_{c'}^{k'} \circ h(x) \frac{\partial}{\partial x^{k'}} \left(M_a^{a'}(x) \right) u^a(x) \\ & + M_a^{a'}(x)\, \rho\Gamma_{bc}^a(x)\, u^b(x)\, M_{c'}^c(x). \end{aligned} \quad (4)$$

Summing the equalities (1) and (4), it results the conclusion of lemma. *q.e.d.*

Theorem 4.1.2 *If* $(E, \pi, M) = (F, \nu, N)$, *then, for any*

$$u^a s_a \in \Gamma(E, \pi, M),$$

we shall use the notation

(4.1.13) $$u^{a_1}{}_{|a|b} = u^{a_1}{}_{|ab} - \rho\Gamma_{ab}^d u^{a_1}{}_{|d}$$

and we verify the formulas of Ricci type

(4.1.14) $$u^{a_1}{}_{|a|b} - u^{a_1}{}_{|b|a} + (\rho, h)\, \mathbb{T}^d{}_{ab} u^{a_1}{}_{|d} = u^d (\rho, h)\, \mathbb{R}_{d\ ab}^{a_1} - u^{a_1}{}_{|c} L_{ab}^c \circ h$$

In particular, if $h = Id_M$, *then the relations* (4.1.14) *become*

(4.1.14′) $$u^{a_1}{}_{|a|b} - u^{a_1}{}_{|b|a} + \rho\mathbb{T}^d{}_{ab} u^{a_1}{}_{|d} = u^d \rho\mathbb{R}_{d\ ab}^{a_1} - u^{a_1}{}_{|c} L_{ab}^c$$

Moreover, if $\rho = id_{TM}$, *then the relations* (4.1.14′) *become*

(4.1.14″) $$u^{i_1}{}_{|i|j} - u^{i_1}{}_{|i|j} + \mathbb{T}^k{}_{ij} u^{i_1}{}_{|k} = u^k \mathbb{R}_{k\ ij}^{i_1}$$

Theorem 4.1.3 *For any* $u_a s^a \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ *we shall use the notation*

(4.1.15) $$u_{b_1|\alpha\beta} = \rho_\beta^j \circ h \frac{\partial}{\partial x^j} \left(u_{b_1|\alpha} \right) - \rho\Gamma_{b_1\beta}^b u_{b|\alpha}$$

and we verify the formulas:

(4.1.16) $$u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} = -u_b (\rho, h)\, \mathbb{R}_{b_1\ \alpha\beta}^b - u_{b_1|\gamma} L_{\alpha\beta}^\gamma \circ h$$

After some calculations, we obtain

$$(4.1.17) \qquad (\rho, h)\, \mathbb{R}^{b}_{b_1\ \alpha\beta} = u^b \left(-u_{b_1|\alpha\beta} + u_{b_1|\beta\alpha} - u_{b_1|\gamma} L^{\gamma}_{\alpha\beta} \circ h \right),$$

where $u^a s_a \in \Gamma(E, \pi, M)$ *such that* $u_a u^b = \delta^b_a$.

In particular, if $h = Id_M$, *then the relations* (4.1.17) *become*

$$(4.1.17') \qquad \rho \mathbb{R}^{b}_{b_1\ \alpha\beta} = u^b \left(-u_{b_1|\alpha\beta} + u_{b_1|\beta\alpha} - u_{b_1|\gamma} L^{\gamma}_{\alpha\beta} \right).$$

Moreover, if $\rho = id_{TM}$ *then the relations* (4.1.17′) *become*

$$(4.1.17'') \qquad \mathbb{R}^{b}_{b_1\ ij} = u^b \left(-u_{b_1|ij} + u_{b_1|ji} \right).$$

Proof. Since

$$\begin{aligned} u_{b_1|\alpha\beta} &= \rho^j_\beta \circ h \left(\frac{\partial}{\partial x^j} \left(\rho^i_\alpha \circ h \frac{\partial u_{b_1}}{\partial x^i} - \rho \Gamma^b_{b_1\alpha} u_b \right) \right) \\ &- \rho \Gamma^b_{b_1\beta} \left(\rho^i_\alpha \circ h \frac{\partial u_b}{\partial x^i} - \rho \Gamma^a_{b\alpha} u_a \right) \\ &= \rho^j_\beta \circ h \frac{\partial \rho^i_\alpha \circ h}{\partial x^j} \frac{\partial u_{b_1}}{\partial x^i} + \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial}{\partial x^j} \left(\frac{\partial u_{b_1}}{\partial x^i} \right) \\ &- \rho^j_\beta \circ h \frac{\partial \rho \Gamma^b_{b_1\alpha}}{\partial x^j} u_b - \rho^j_\beta \circ h \rho \Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^j} \\ &- \rho^i_\alpha \circ h \rho \Gamma^b_{b_1\beta} \frac{\partial u_b}{\partial x^i} + \rho \Gamma^b_{b_1\beta} \rho \Gamma^a_{b\alpha} u_a \end{aligned}$$

and

$$\begin{aligned} u_{b_1|\beta\alpha} &= \rho^i_\alpha \circ h \left(\frac{\partial}{\partial x^i} \left(\rho^j_\beta \circ h \frac{\partial u_{b_1}}{\partial x^j} - \rho \Gamma^b_{b_1\beta} u_b \right) \right) \\ &- \rho \Gamma^b_{b_1\alpha} \left(\rho^j_\beta \circ h \frac{\partial u_b}{\partial x^j} - \rho \Gamma^a_{b\beta} u_a \right) \\ &= \rho^i_\alpha \circ h \frac{\partial \rho^j_\beta \circ h}{\partial x^i} \frac{\partial u_{b_1}}{\partial x^i} + \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial}{\partial x^i} \left(\frac{\partial u_{b_1}}{\partial x^j} \right) \\ &- \rho^i_\alpha \circ h \frac{\partial \rho \Gamma^b_{b_1\beta}}{\partial x^i} u_b - \rho^i_\alpha \circ h \rho \Gamma^b_{b_1\beta} \frac{\partial u_b}{\partial x^i} \\ &- \rho^j_\beta \circ h \rho \Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^i} + \rho \Gamma^b_{b_1\alpha} \rho \Gamma^a_{b\beta} u_a \end{aligned}$$

it results that

$$\begin{aligned} u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} &= \rho^j_\beta \circ h \frac{\partial \rho^i_\alpha \circ h}{\partial x^j} \frac{\partial u_{b_1}}{\partial x^i} - \rho^i_\alpha \circ h \frac{\partial \rho^j_\beta \circ h}{\partial x^i} \frac{\partial u_{b_1}}{\partial x^j} \\ &+ \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial}{\partial x^j} \left(\frac{\partial u_{b_1}}{\partial x^i} \right) - \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial}{\partial x^i} \left(\frac{\partial u_{b_1}}{\partial x^j} \right) \\ &+ \rho^i_\alpha \circ h \frac{\partial \rho \Gamma^b_{b_1\beta}}{\partial x^i} u_b - \rho^j_\beta \circ h \frac{\partial \rho \Gamma^b_{b_1\alpha}}{\partial x^j} u_b \\ &+ \rho^j_\beta \circ h \rho \Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^j} - \rho^j_\beta \circ h \rho \Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^j} \\ &+ \rho^i_\alpha \circ h \rho \Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^i} - \rho^i_\alpha \circ h \rho \Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^i} \\ &+ \rho \Gamma^b_{b_1\beta} \rho \Gamma^a_{b\alpha} u_a - \rho \Gamma^b_{b_1\alpha} \rho \Gamma^a_{b\beta} u_a. \end{aligned}$$

After some calculations, we obtain:

$$\begin{aligned} u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} &= L^{\gamma}_{\beta\alpha} \circ h\rho^k_{\gamma} \circ h\frac{\partial u_{b_1}}{\partial x^k} \\ &+ \left(\rho^i_{\alpha} \circ h\frac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} u_b - \rho^j_{\beta} \circ h\frac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} u_b \right) \\ &+ \rho\Gamma^b_{b_1\beta}\rho\Gamma^a_{b\alpha}u_a - \rho\Gamma^b_{b_1\alpha}\rho\Gamma^a_{b\beta}u_a. \end{aligned}$$

Since

$$\begin{aligned} u_b\left(\rho,h\right)\mathbb{R}^b_{b_1\alpha\beta} = u_b \Bigg(& \rho^j_{\beta} \circ h\frac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} + \rho\Gamma^b_{e\beta}\rho\Gamma^e_{b_1\alpha} \\ & - \rho^i_{\alpha} \circ h\frac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} - \rho\Gamma^b_{e\alpha}\rho\Gamma^e_{b_1\beta} - \rho\Gamma^b_{b_1\gamma}L^{\gamma}_{\beta\alpha} \circ h \Bigg) \end{aligned}$$

and

$$u_{b_1|\gamma}L^{\gamma}_{\alpha\beta} \circ h = \left(\rho^k_{\gamma} \circ h\frac{\partial u_{b_1}}{\partial x^k} - \rho\Gamma^b_{b_1\gamma}u_b \right) L^{\gamma}_{\alpha\beta} \circ h$$

it results that

$$\begin{aligned} -u_b\left(\rho,h\right)\mathbb{R}^b_{b_1,\alpha\beta} - u_{b_1|\gamma}L^{\gamma}_{\alpha\beta} \circ h = & -L^{\gamma}_{\alpha\beta} \circ h\rho^k_{\gamma} \circ h\frac{\partial u_{b_1}}{\partial x^k} \\ & + \left(\rho^i_{\alpha} \circ h\frac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} u_b - \rho^j_{\beta} \circ h\frac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} u_b \right) \\ & + \rho\Gamma^b_{b_1\beta}\rho\Gamma^a_{b\alpha}u_a - \rho\Gamma^b_{b_1\alpha}\rho\Gamma^a_{b\beta}u_a. \end{aligned}$$

q.e.d.

Lemma 4.1.2 *If* $(E,\pi,M) = (F,\nu,N)$*, then, for any*

$$u_b s^b \in \Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right),$$

we have that $u_{b\ |c}$, $b,c \in \overline{1,n}$ are the *components of a tensor field of* $(0,2)$ *type.*

Proof. Let U and U' be two vector local $(m+n)$-charts such that $U \cap U' \neq \phi$.
Since $u_{b'}\left(x\right) = M^b_{b'}\left(x\right)u_b\left(x\right),$ for any $x \in U \cap U'$, it results that

$$(1)\qquad \begin{aligned} \rho^{k'}_{c'} \circ h\left(x\right)\frac{\partial u_{b'}\left(x\right)}{\partial x^{k'}} &= \rho^{k'}_{c'} \circ h\left(x\right)\frac{\partial}{\partial x^{k'}}\left(M^b_{b'}\left(x\right)\right)u_b\left(x\right) \\ &+ M^b_{b'}\left(x\right)\rho^{k'}_{c'} \circ h\left(x\right)\frac{\partial u_b\left(x\right)}{\partial x^{k'}}. \end{aligned}$$

Since, for any $x \in U \cap U'$, we have

$$(2)\qquad \begin{aligned} \rho\Gamma^{a'}_{b'c'}\left(x\right) &= M^{a'}_a\left(x\right)\left(\rho^k_c \circ h\left(x\right)\frac{\partial}{\partial x^k}\left(M^a_{b'}\left(x\right)\right) \right. \\ &\left. + \rho\Gamma^a_{bc}\left(x\right)M^b_{b'}\left(x\right)\right)M^c_{c'}\left(x\right), \end{aligned}$$

and

$$
(3) \qquad \begin{aligned} 0 &= \frac{\partial}{\partial x^{k'}}\left(M_a^{a'}(x)\, M_{b'}^a(x)\right) \\ &= \frac{\partial}{\partial x^{k'}}\left(M_a^{a'}(x)\right) M_{b'}^a(x) + M_a^{a'}(x)\, \frac{\partial}{\partial x^{k'}}\left(M_{b'}^a(x)\right) \end{aligned}
$$

it results that

$$
(4) \qquad \begin{aligned} \rho\Gamma_{b'c'}^{a'}(x)\, u_{a'}(x) = -\rho_{c'}^{k'} \circ h(x)\, \frac{\partial}{\partial x^{k'}}\left(M_{b'}^b(x)\right) u_b(x) \\ +M_{b'}^b(x)\, \rho\Gamma_{bc}^a(x)\, u_a(x)\, M_{c'}^c(x)\,. \end{aligned}
$$

Summing the equalities (1) and (4), it results the conclusion of lemma. *q.e.d.*

Theorem 4.1.4 *If* $(E,\pi,M)=(F,\nu,N)$*, then, for any*

$$
u_b s^b \in \Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right),
$$

we shall use the notation

$$
(4.1.18) \qquad u_{b_1\ |a|b} = u_{b_1\ |ab} - \rho\Gamma_{ab}^d u_{b_1\ |d}
$$

and we verify the formulas of Ricci type

$$
(4.1.19) \qquad u_{b_1\ |a|b} - u_{b_1\ |b|a} + (\rho,h)\,\mathbb{T}_{\ ab}^{d} u_{b_1\ |d} = -u_d\,(\rho,h)\,\mathbb{R}_{b_1\ ab}^{d} - u_{b_1\ |d} L_{ab}^d \circ h
$$

In particular, if $h=Id_M$*, then the relations* (4.1.19) *become*

$$
(4.1.19') \qquad u_{b_1\ |a|b} - u_{b_1\ |b|a} + \rho\mathbb{T}_{\ ab}^{d} u_{b_1\ |d} = -u_d\rho\mathbb{R}_{b_1\ ab}^{d} - u_{b_1\ |d} L_{ab}^d.
$$

Moreover, if $\rho = id_{TM}$ *then the relations* (4.1.19′) *become*

$$
(4.1.19'') \qquad u_{j_1\ |i|j} - u_{j_1\ |j|i} + \mathbb{T}_{\ ij}^{h} u_{j_1\ |h} = u_h\mathbb{R}_{j_1\ ij}^{h}.
$$

Theorem 4.1.5 *For any tensor field*

$$
T_{b_1,\ldots,b_q}^{a_1,\ldots,a_p} s_{a_1} \otimes \ldots \otimes s_{a_p} \otimes s^{b_1} \otimes \ldots \otimes s^{b_q} \ ,
$$

we verify the equality:

$$
(4.1.20) \qquad \begin{aligned} & T_{b_1,\ldots,b_q|\alpha\beta}^{a_1,\ldots,a_p} - T_{b_1,\ldots,b_q|\beta\alpha}^{a_1,\ldots,a_p} = T_{b_1,\ldots,b_q}^{aa_2,\ldots,a_p}\,(\rho,h)\,\mathbb{R}_{a\ \alpha\beta}^{a_1} + \ldots \\ & +T_{b_1,\ldots,b_q}^{a_1,\ldots,a_{p-1}a}\,(\rho,h)\,\mathbb{R}_{a\ \alpha\beta}^{a_p} - T_{b,b_2,\ldots,b_q}^{a_1,\ldots,a_p}\,(\rho,h)\,\mathbb{R}_{b_1\ \alpha\beta}^{b} - \ldots \\ & -T_{b_1,\ldots,b_{q-1}b}^{a_1,\ldots,a_p}\,(\rho,h)\,\mathbb{R}_{b_q\ \alpha\beta}^{b} - T_{b_1,\ldots,b_q|\gamma}^{a_1,\ldots,a_p} L_{\alpha\beta}^{\gamma} \circ h. \end{aligned}
$$

In particular, if $h=Id_M$*, then the relations* (4.1.20) *become*

$$
(4.1.20') \qquad \begin{aligned} & T_{b_1,\ldots,b_q|\alpha\beta}^{a_1,\ldots,a_p} - T_{b_1,\ldots,b_q|\beta\alpha}^{a_1,\ldots,a_p} = T_{b_1,\ldots,b_q}^{aa_2,\ldots,a_p}\rho\mathbb{R}_{a\ \alpha\beta}^{a_1} + \ldots \\ & +T_{b_1,\ldots,b_q}^{a_1,\ldots,a_{p-1}a}\rho\mathbb{R}_{a\ \alpha\beta}^{a_p} - T_{b,b_2,\ldots,b_q}^{a_1,\ldots,a_p}\rho\mathbb{R}_{b_1\ \alpha\beta}^{b} - \ldots \\ & -T_{b_1,\ldots,b_{q-1}b}^{a_1,\ldots,a_p}\rho\mathbb{R}_{b_q\ \alpha\beta}^{b} - T_{b_1,\ldots,b_q|\gamma}^{a_1,\ldots,a_p} L_{\alpha\beta}^{\gamma}. \end{aligned}
$$

Theorem 4.1.6 *If* $(E,\pi,M)=(F,\nu,N)$*, then we obtain the following formulas of Ricci type:*

$$\begin{array}{ll}(4.1.21) & \begin{array}{l}T_{b_1,\ldots,b_q\ |b|c}^{a_1,\ldots,a_p}-T_{b_1,\ldots,b_q|c|b}^{a_1,\ldots,a_p}+(\rho,h)\,\mathbb{T}_{\ bc}^{d}T_{b_1,\ldots,b_q|d}^{a_1,\ldots,a_p}\\ =T_{b_1,\ldots,b_q}^{aa_2,\ldots,a_p}\,(\rho,h)\,\mathbb{R}_{a\ bc}^{a_1}+\ldots+T_{b_1,\ldots,b_q}^{a_1,\ldots,a_{p-1}a}\,(\rho,h)\,\mathbb{R}_{a\ bc}^{a_p}\\ -T_{b,b_2,\ldots,b_q}^{a_1,\ldots,a_p}\,(\rho,h)\,\mathbb{R}_{b_1\ bc}^{b}-\ldots-T_{b_1,\ldots,b_{q-1}b}^{a_1,\ldots,a_p}\,(\rho,h)\,\mathbb{R}_{b_q\ bc}^{b}-T_{b_1,\ldots,b_q|d}^{a_1,\ldots,a_p}L_{bc}^{d}\circ h.\end{array}\end{array}$$

In particular, if $h=Id_M$*, then the relations* (4.1.21) *become*

$$\begin{array}{ll}(4.1.21') & \begin{array}{l}T_{b_1,\ldots,b_q\ |b|c}^{a_1,\ldots,a_p}-T_{b_1,\ldots,b_q|c|b}^{a_1,\ldots,a_p}+\rho\mathbb{T}_{\ bc}^{d}T_{b_1,\ldots,b_q|d}^{a_1,\ldots,a_p}\\ =T_{b_1,\ldots,b_q}^{aa_2,\ldots,a_p}\rho\mathbb{R}_{a\ bc}^{a_1}+\ldots+T_{b_1,\ldots,b_q}^{a_1,\ldots,a_{p-1}a}\rho\mathbb{R}_{a\ bc}^{a_p}\\ -T_{b,b_2,\ldots,b_q}^{a_1,\ldots,a_p}\rho\mathbb{R}_{b_1\ bc}^{b}-\ldots-T_{b_1,\ldots,b_{q-1}b}^{a_1,\ldots,a_p}\rho\mathbb{R}_{b_q\ bc}^{b}-T_{b_1,\ldots,b_q|d}^{a_1,\ldots,a_p}L_{bc}^{d}.\end{array}\end{array}$$

We observe that if the structure functions of generalized Lie algebroid

$$\left((F,\nu,M)\,,[,]_{F,h}\,,(\rho,Id_M)\right),$$

the (ρ,h)-torsion associated to linear ρ-connection $\rho\Gamma$ and the (ρ,h)-curvature associated to linear ρ-connection $\rho\Gamma$ are null, then we have the equality:

$$(4.1.22)\qquad T_{b_1,\ldots,b_q|b|c}^{a_1,\ldots,a_p}=T_{b_1,\ldots,b_q|c|b}^{a_1,\ldots,a_p},$$

which generalizes the Schwartz equality.

4.2 Torsion and curvature forms. Identities of Cartan and Bianchi type

We apply the theory of the generalized linear connections for the diagram:

$$(4.2.1)\qquad \begin{array}{ccc} E & & \left(F,[,]_{F,h}\,,(\rho,Id_M)\right)\\ \pi\downarrow & & \downarrow\nu\\ M & \xrightarrow{\ h\ } & M\end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,M)\,,[,]_{F,h}\,,(\rho,Id_M)\right)\in|\mathbf{GLA}|$.

Let $\rho\Gamma$ be a linear ρ-connection for the vector bundle (E,π,M).

Definition 4.2.1 For each $a,b\in\overline{1,n}$, we obtain the scalar 1-forms

$$(4.2.2)\qquad \Omega_b^a=\rho\Gamma_{b\alpha}^aT^\alpha$$

and

$$(4.2.2')\qquad \omega_b^a=\rho\Gamma_{b\alpha}^at^\alpha$$

which will be called the *form of linear* ρ*-connection* $\rho\dot{\Gamma}$ *and* $\rho\Gamma$ *respectively.*

Definition 4.2.2 If $(E,\pi,M)=(F,\nu,M)$, then the vector valued 2-form

$$(4.2.3)\qquad (\rho,h)\,\mathbb{T}=((\rho,h)\,\mathbb{T}_{\ ab}^{c}S_c)\,S^a\wedge S^b$$

will be called the *vector valued form of* (ρ, h)*-torsion* $(\rho, h)\,\mathbb{T}$.

In particular, if $h = Id_M$, then the vector valued 2-form

$$\rho\mathbb{T} = \left(\rho\mathbb{T}^c{}_{ab}s_c\right) s^a \wedge s^b \tag{4.2.3'}$$

will be called the *vector form of* ρ*-torsion* $\rho\mathbb{T}$.

Moreover, if $\rho = Id_{TM}$, then the vector valued form (4.2.3′) becomes:

$$\mathbb{T} = \left(\mathbb{T}^i{}_{jk}\frac{\partial}{\partial x^i}\right) dx^j \wedge dx^k. \tag{4.2.3''}$$

Definition 4.2.3 For each $c \in \overline{1, n}$ we obtain the *scalar* 2*-form of* (ρ, h)*-torsion* $(\rho, h)\,\mathbb{T}$

$$(\rho, h)\,\mathbb{T}^c = (\rho, h)\,\mathbb{T}^c{}_{ab}S^a \wedge S^b. \tag{4.2.4}$$

In particular, if $h = Id_M$, then, for each $c \in \overline{1, n}$, we obtain the *scalar* 2*-form of* ρ*-torsion* $\rho\mathbb{T}$

$$\rho\mathbb{T}^c = \rho\mathbb{T}^c{}_{ab}s^a \wedge s^b. \tag{4.2.4'}$$

Moreover, if $\rho = Id_{TM}$, then the scalar 2-form (4.2.4′) becomes:

$$\mathbb{T}^i = \mathbb{T}^i{}_{jk}dx^j \wedge dx^k. \tag{4.2.4''}$$

Definition 4.2.3 The vector mixed form

$$(\rho, h)\,\mathbb{R} = \left(\left((\rho, h)\,\mathbb{R}^a_{b\ \alpha\beta}s_a\right) T^\alpha \wedge T^\beta\right) s^b \tag{4.2.5}$$

will be called the *vector valued form of* (ρ, h)*-curvature* $(\rho, h)\,\mathbb{R}$.

In particular, if $h = Id_M$, then the vector mixed form

$$\rho\mathbb{R} = \left(\left(\rho\mathbb{R}^a_{b\ \alpha\beta}s_a\right) t^\alpha \wedge t^\beta\right) s^b \tag{4.2.5'}$$

will be called the *vector valued form of* ρ*-curvature* $\rho\mathbb{R}$.

Moreover, if $\rho = Id_{TM}$, then the vector form (4.2.5′) becomes:

$$\mathbb{R} = \left(\left(\mathbb{R}^a_{b\ hk}s_a\right) dx^h \wedge dx^k\right) s^b. \tag{4.2.5''}$$

Definition 4.2.4 For each $a, b \in \overline{1, n}$ we obtain the *scalar* 2*-form of* (ρ, h)*-curvature* $(\rho, h)\,\mathbb{R}$

$$(\rho, h)\,\mathbb{R}^a_b = (\rho, h)\,\mathbb{R}^a_{b\ \alpha\beta}T^\alpha \wedge T^\beta. \tag{4.2.6}$$

In particular, if $h = Id_M$, then, for each $a, b \in \overline{1, n}$, we obtain the *scalar* 2*-form of* ρ*-curvature* $\rho\mathbb{R}$

$$\rho\mathbb{R}^a_b = \rho\mathbb{R}^a_{b\ \alpha\beta}t^\alpha \wedge t^\beta. \tag{4.2.6'}$$

Moreover, if $\rho = Id_{TM}$, then the scalar form (4.2.6′) becomes:

$$\mathbb{R}^a_b = \mathbb{R}^a_{b\ hk}dx^h \wedge dx^k. \tag{4.2.6''}$$

Theorem 4.2.1 *The identities*

$$(C_1) \qquad (\rho,h)\,\mathbb{T}^a = d^{h^*F}S^a + \Omega^a_b \wedge S^b,$$

and

$$(C_2) \qquad (\rho,h)\,\mathbb{R}^a_b = d^{h^*F}\Omega^a_b + \Omega^a_c \wedge \Omega^c_b$$

hold good. These will be called the first respectively the second identity of Cartan type.

Proof. To prove the first identity we consider that $(E,\pi,M) = (F,\nu,M)$. Therefore, $\Omega^a_b = \rho\Gamma^a_{bc}S^c$. Since

$$\begin{array}{l} d^{h^*F}S^a(U,V)S_a = ((\Gamma(\overset{h^*F}{\rho}, Id_M)U)S^a(V) \\ \quad -(\Gamma(\overset{h^*F}{\rho}, Id_M)V)(S^a(U)) - S^a([U,V]_{h^*F}))S_a \\ \quad = (\Gamma(\overset{h^*F}{\rho}, Id_M)U)(V^a) - (\Gamma(\overset{h^*F}{\rho}, Id_M)V)(U^a) - S^a([U,V]_{h^*F})S_a \\ \quad = \rho\ddot{D}_UV - V^b\rho\ddot{D}_US_b - \rho\ddot{D}_VU - U^b\rho\ddot{D}_VS_b - [U,V]_{h^*F} \\ \quad = (\rho,h)\,\mathbb{T}(U,V) - (\rho\Gamma^a_{bc}V^bU^c - \rho\Gamma^a_{bc}U^bV^c)S_a \\ \quad = ((\rho,h)\,\mathbb{T}^a(U,V) - \Omega^a_b \wedge S^b(U,V))S_a, \end{array}$$

it results the first identity.

To prove the second identity, we consider that $(E,\pi,M) \neq (F,\nu,M)$. Since

$$\begin{array}{l} (\rho,h)\,\mathbb{R}^a_b\,(Z,W)\,s_a = (\rho,h)\,\mathbb{R}\,((W,Z)\,,s_b) \\ \quad = \rho\dot{D}_Z\left(\rho\dot{D}_Ws_b\right) - \rho\dot{D}_W\left(\rho\dot{D}_Zs_b\right) - \rho\dot{D}_{[Z,W]_{h^*F}}s_b \\ \quad = \rho\dot{D}_Z\left(\Omega^a_b\,(W)\,s_a\right) - \rho\dot{D}_W\left(\Omega^a_b\,(Z)\,s_a\right) - \Omega^a_b\left([Z,W]_{h^*F}\right)s_a \\ \quad + \left(\Omega^a_c\,(Z)\,\Omega^c_b\,(W) - \Omega^a_c\,(W)\,\Omega^c_b\,(Z)\right)s_a \\ \quad = \left(d^{h^*F}\Omega^a_b\,(Z,W) + \Omega^a_c \wedge \Omega^c_b\,(Z,W)\right)s_a \end{array}$$

it results the second identity.

Corollary 4.2.1 *In particular, if $h = Id_M$, then the identities (C_1) and (C_2) become*

$$(C_1') \qquad \rho\mathbb{T}^a = d^Fs^a + \omega^a_b \wedge s^b,$$

and

$$(C_2') \qquad \rho\mathbb{R}^a_b = d^F\omega^a_b + \omega^a_c \wedge \omega^c_b$$

respectively.

Moreover, if $\rho = Id_{TM}$, then the identities (C_1') and (C_2') become:

$$(C_1'') \qquad \mathbb{T}^i = ddx^i + \omega^i_j \wedge dx^j = \omega^i_j \wedge dx^j$$

and

$$(C_2'') \qquad \mathbb{R}^i_j = d\omega^i_j + \omega^i_h \wedge \omega^h_j,$$

respectively. *q.e.d.*

Theorem 4.2.2 *The identities*

$$(B_1) \qquad d^{h^*F}(\rho,h)\,\mathbb{T}^a = (\rho,h)\,\mathbb{R}^a_b \wedge S^b - \Omega^a_c \wedge (\rho,h)\,\mathbb{T}^c$$

and

$$(B_2) \qquad d^{h^*F}(\rho,h)\,\mathbb{R}^a_b = (\rho,h)\,\mathbb{R}^a_c \wedge \Omega^c_b - \Omega^a_c \wedge (\rho,h)\,\mathbb{R}^c_b,$$

hold good. We will called these the first respectively the second identity of Bianchi type.

If the (ρ,h)-torsion is null, then the first identity of Bianchi type becomes:

$$(\tilde{B}_1) \qquad (\rho,h)\,\mathbb{R}^a_b \wedge s^b = 0.$$

Proof. We consider $(E,\pi,M) = (F,\nu,M)$. Using the first identity of Cartan type and the equality $d^{h^*F} \circ d^{h^*F} = 0$, we obtain:

$$d^{h^*F}(\rho,h)\,\mathbb{T}^a = d^{h^*F}\Omega^a_b \wedge S^b - \Omega^a_c \wedge d^{h^*F}S^c.$$

Using the second identity of Cartan type and the previous identity, we obtain:

$$d^{h^*F}(\rho,h)\,\mathbb{T}^a = ((\rho,h)\,\mathbb{R}^a_b - \Omega^a_c \wedge \Omega^c_b) \wedge S^b - \Omega^a_c \wedge \Big((\rho,h)\,\mathbb{T}^c - \Omega^c_b \wedge S^b\Big).$$

After some calculations, we obtain the first identity of Bianchi type.

Using the second identity of Cartan type and the equality $d^{h^*F} \circ d^{h^*F} = 0$, we obtain:

$$d^{h^*F}\Omega^a_c \wedge \Omega^c_b - \Omega^a_c \wedge d^{h^*F}\Omega^c_b = d^{h^*F}(\rho,h)\,\mathbb{R}^a_b.$$

Using the second of Cartan type and the previous identity, we obtain:

$$d^{h^*F}(\rho,h)\,\mathbb{R}^a_b = ((\rho,h)\,\mathbb{R}^a_c - \Omega^a_e \wedge \Omega^e_c) \wedge \Omega^c_b - \Omega^a_c \wedge ((\rho,h)\,\mathbb{R}^c_b - \Omega^c_e \wedge \Omega^e_b).$$

After some calculations, we obtain the second identity of Bianchi type. *q.e.d.*

Corollary 4.2.2 *In particular, if $h = Id_M$, then the identities (B_1) and (B_2) become*

$$(B_1') \qquad d^F\rho\mathbb{T}^a = \rho\mathbb{R}^a_b \wedge s^b - \omega^a_c \wedge \rho\mathbb{T}^c$$

and

$$(B_2') \qquad d^F\rho\mathbb{R}^a_b = \rho\mathbb{R}^a_c \wedge \omega^c_b - \omega^a_c \wedge \rho\mathbb{R}^c_b,$$

respectively.

Moreover, if $\rho = Id_{TM}$, then the identities (B_1') and (B_2') become:

$$(B_1'') \qquad d\mathbb{T}^i = \mathbb{R}^i_j \wedge dx^j - \omega^i_k \wedge \mathbb{T}^k$$

and

$$(B_2'') \qquad d\mathbb{R}^i_j = \mathbb{R}^i_h \wedge \omega^h_j - \omega^i_h \wedge \mathbb{R}^h_j,$$

respectively.

Theorem 4.2.3 *If* (E,π,M)=(F,ν,M), *then the following relations hold good*

$$(\tilde{B}_1)\qquad \sum_{cyclic(u_1,u_2,u_3)}\Big\{\rho\ddot{D}_{U_1}\left(\left(\rho,h\right)\mathbb{T}\left(U_2,U_3\right)\right)-\left(\rho,h\right)\mathbb{R}\left(U_1,U_2\right)U_3 +\left(\rho,h\right)\mathbb{T}\left(\left(\rho,h\right)\ \mathbb{T}\left(U_1,U_2\right),U_3\right)\right\}=0,$$

and

$$(\tilde{B}_2)\qquad \sum_{cyclic(u_1,u_2,u_3,u)}\Big\{\rho\ddot{D}_{U_1}\left(\left(\rho,h\right)\mathbb{R}\left(U_2,U_3\right)U\right)-\left(\rho,h\right)\mathbb{R}\left(\left(\rho,h\right)\mathbb{T}\left(U_1,U_2\right),U_3\right)U\Big\}=0.$$

respectively. This identities will be called the first respectively the second identity of Bianchi type.

In particular, if $h=Id_M$, *then the identities* $(\tilde{B}_1)$ *and* $(\tilde{B}_2)$ *become*

$$(\tilde{B}_1')\qquad \sum_{cyclic(u_1,u_2,u_3)}\left\{\rho D_{u_1}\left(\rho\mathbb{T}\left(u_2,u_3\right)\right)-\rho\mathbb{R}\left(u_1,u_2\right)u_3+\rho\mathbb{T}\left(\rho\ \mathbb{T}\left(u_1,u_2\right),u_3\right)\right\}=0,$$

$$(\tilde{B}_2')\qquad \sum_{cyclic(u_1,u_2,u_3,u)}\left\{\rho D_{u_1}\left(\rho\mathbb{R}\left(u_2,u_3\right)u\right)-\rho\mathbb{R}\left(\rho\mathbb{T}\left(u_1,u_2\right),u_3\right)u\right\}=0.$$

which will be called the first respectively the second identity of Bianchi type.

Remark 4.2.1 On components, the identities of Bianchi type $(\tilde{B}_1)$ and $(\tilde{B}_2)$ become:

$$(\tilde{B}_1'')\qquad \sum_{cyclic(a_1,a_2,a_3)}\Big\{\left(\rho,h\right)\mathbb{T}^a{}_{a_2a_3|a_1}+\left(\rho,h\right)\mathbb{T}^a{}_{ga_3}\cdot\left(\rho,h\right)\mathbb{T}^g{}_{a_1a_2}\Big\} =\sum_{cyclic(a_1,a_2,a_3)}\left(\rho,h\right)\mathbb{R}^a_{a_3\ a_1a_2}$$

and

$$(\tilde{B}_2'')\qquad \sum_{cyclic(a_1,a_2,a_3)}\Big\{\left(\rho,h\right)\mathbb{R}^a_{b\ a_2a_3|a_1}+\left(\rho,h\right)\mathbb{R}^a_{b\ ga_3}\cdot\left(\rho,h\right)\mathbb{T}^g{}_{a_1a_2}\Big\}=0.$$

If the (ρ,h)-torsion is null, then the identities of Bianchi type become:

$$(\tilde{B}_1''')\qquad \sum_{cyclic(a_1\ a_2,a_3)}\left(\rho,h\right)\mathbb{R}^a_{a_3,a_1a_2}=0$$

and

$$(\tilde{B}_2''')\qquad \sum_{cyclic(a_1,a_2,a_3)}\left(\rho,h\right)\mathbb{R}^a_{b\ a_2a_3|_{a_1}}=0.$$

4.3 (Pseudo)metrizable vector bundles

We will apply our theory for the diagram:

$$(4.3.1)\qquad \begin{array}{ccc} E & & \left(F,[,]_{F,h},(\rho,Id_M)\right) \\ \pi\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & M \end{array},$$

where $(E,\pi,M) \in |\mathbf{B^v}|$ and $\left((F,\nu,M), [,]_{F,h}, (\rho, Id_M)\right) \in |\mathbf{GLA}|$.

Definition 4.3.1 We will say that the *vector bundle* (E,π,M) *is endowed with a pseudometrical structure* if it exists

$$g = g_{ab}s^a \otimes s^b \in \mathcal{T}\,{}^0_2\,(E,\pi,M)$$

such that for each $x \in M$, the matrix $\|g_{ab}(x)\|$ is nondegenerate and symmetric.

Moreover, if for each $x \in M$ the matrix $\|g_{ab}(x)\|$ has constant signature, then we will say that *the vector bundle* (E,π,M) *is endowed with a metrical structure.*

If

$$g = g_{ab}s^a \otimes s^b \in \mathcal{T}\,{}^0_2\,(E,\pi,M)$$

is a (pseudo) metrical structure, then, for any $a, b \in \overline{1,r}$ and for any vector local $(m+r)$-chart (U, s_U) of (E,π,M), we consider the real functions

$$U \xrightarrow{\tilde{g}^{ba}} \mathbb{R}$$

such that

$$\left\|\tilde{g}^{ba}(x)\right\| = \|g_{ab}(x)\|^{-1},$$

for any $\forall x \in U$.

Definition 4.3.2 Let (E,π,M) be a vector bundle endowed with a (pseudo)metrical structure g and with a linear ρ-connection $\rho\Gamma$.

We will say that the *linear ρ-connection $\rho\Gamma$ is compatible with the (pseudo)metrical structure g* if

$$\rho D_z g = 0,\ \forall z \in \Gamma(F,\nu,M). \tag{4.3.2}$$

Definition 4.3.3 We will say that the vector bundle (E,π,M) is ρ*-(pseudo)metrizable,* if it exists a (pseudo)metrical structure

$$g \in \mathcal{T}_2^0(E,\pi,M)$$

and a linear ρ-connection $\rho\Gamma$ for (E,π,M) compatible with g. The id_{TM}-(pseudo)metrizable vector bundles will be called *(pseudo)metrizable vector bundles.*

In particular, if (TM,τ_M,M) is a (pseudo)metrizable vector bundle, then we will say that (TM,τ_M,M) *is a* (pseudo)Riemannian space*,* and the manifold M will be called the *(pseudo)Riemannian manifold.*

The linear connection of a (pseudo)Riemannian space will be called *(pseudo)Riemannian linear connection.*

Theorem 4.3.1 *If* $(E,\pi,M) = (F,\nu,M)$ *and* $g \in \mathcal{T}_2^0(h^*E, h^*\pi, M)$ *is a (pseudo)metrical structure, then the local real functions*

$$\begin{array}{rl}\rho\Gamma^a_{bc} & = \dfrac{1}{2}\tilde{g}^{ad}\left(\rho^k_c \circ h\dfrac{\partial g_{bd}}{\partial x^k} + \rho^j_b \circ h\dfrac{\partial g_{dc}}{\partial x^j} - \rho^h_d \circ h\dfrac{\partial g_{bc}}{\partial x^h}\right. \\ & \left. +g_{ec}L^e_{bd}\circ h + g_{be}L^e_{dc}\circ h - g_{de}L^e_{bc}\circ h\right).\end{array} \tag{4.3.3}$$

are the components of a linear ρ-connection $\rho\Gamma$ for the vector bundle $(h^*E, h^*\pi, M)$ *compatible with g such that* $(\rho, h)\,\mathbb{T} = 0$.

Therefore, the vector bundle $(h^*E, h^*\pi, M)$ becomes ρ-(pseudo)metrizable.

The linear ρ-connection $\rho\Gamma$ will be called the *linear ρ-connection of Levi-Civita type.*

Proof. Since

$$\begin{array}{ll}\left(\rho\ddot{D}_U g\right) V\otimes Z & =\Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)\left((g(V\otimes Z))-g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right)\right.\\ & \left.-g\left(V\otimes\left(\rho\ddot{D}_U Z\right)\right)\right),\ \forall U,V,Z\in\Gamma(h^*E,h^*\pi,M).\end{array}$$

it results that, for any $U,V,Z\in\Gamma(h^*E,h^*\pi,M)$, we obtain the equalities:

$$(1)\qquad \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)(g(V\otimes Z))=g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right)+g\left(V\otimes\left(\rho\ddot{D}_U Z\right)\right),$$

$$(2)\qquad \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)(g(U\otimes V))=g\left(\left(\rho\ddot{D}_Z U\right)\otimes V\right)+g\left(U\otimes\left(\rho\ddot{D}_Z V\right)\right),$$

$$(3)\qquad \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)(g(Z\otimes U))=g\left(\left(\rho\ddot{D}_V Z\right)\otimes U\right)+g\left(Z\otimes\left(\rho\ddot{D}_V U\right)\right).$$

We observe that $(1)+(3)-(2)$ is equivalent with the equality:

$$\begin{array}{l}g\left(\left(\rho\ddot{D}_U V+\rho\ddot{D}_V U\right)\otimes Z\right)+g\left(\left(\rho\ddot{D}_V Z-\rho\ddot{D}_Z V\right)\otimes U\right)\\ \quad+g\left(\left(\rho\ddot{D}_U Z-\rho\ddot{D}_Z U\right)\otimes V\right)=\Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)(g(V\otimes Z))\\ \quad+\Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)(g(Z\otimes U))-\ \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)(g(U\otimes V)).\end{array}$$

Using the condition $(\rho,h)\mathbb{T}=0$, which is equivalent with the equality

$$\rho\ddot{D}_U V-\rho\ddot{D}_V U-[U,V]_{h^*E}=0,$$

we obtain:

$$\begin{array}{ll}2g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right) & =\Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)\cdot(g(V\otimes Z))\\ & +\Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)(g(Z\otimes U))-\Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)(g(U\otimes V))\\ & +g\left([U,V]_{h^*E}\otimes Z\right)-g\left([U,Z]_{h^*E}\otimes V\right)\\ & -g\left([V,Z]_{h^*E}\otimes U\right),\ \forall U,V,Z\in\Gamma(h^*E,h^*\pi,M).\end{array}$$

Therefore, we obtain the equality:

$$\begin{array}{l}2g\left(\left(\rho\Gamma_{ba}^d S_d\right)\otimes S_c\right)=\rho_a^i\circ h\dfrac{\partial g(S_b\otimes S_c)}{\partial x^i}+\ \rho_b^j\circ h\dfrac{\partial g(S_c\otimes S_a)}{\partial x^j}-\rho_c^k\circ h\dfrac{\partial g(S_a\otimes S_b)}{\partial x^k}\\ \qquad+g\left(\left(L_{ab}^d\circ h\right)S_d\otimes S_c\right)-g\left(\left(L_{ac}^d\circ h\right)S_d\otimes S_b\right)-g\left(\left(L_{bc}^d\circ h\right)S_d\otimes S_a\right),\end{array}$$

which is equivalent with:

$$\begin{array}{r}2g_{dc}\rho\Gamma_{ba}^d=\rho_a^i\circ h\dfrac{\partial g_{bc}}{\partial x^i}+\rho_b^j\circ h\dfrac{\partial g_{ca}}{\partial x^j}-\rho_c^k\circ h\dfrac{\partial g_{ab}}{\partial x^k}+\left(L_{ab}^d\circ h\right)g_{dc}\\ -\left(L_{ac}^d\circ h\right)g_{db}-\left(L_{bc}^d\circ h\right)g_{da}.\end{array}$$

Finally, we obtain:

$$\begin{array}{ll}\rho\Gamma_{ba}^d & =\dfrac{1}{2}\tilde{g}^{dc}\left(\rho_a^i\circ h\dfrac{\partial g_{bc}}{\partial x^i}+\rho_b^j\circ h\dfrac{\partial g_{ca}}{\partial x^j}-\rho_c^k\circ h\dfrac{\partial g_{ab}}{\partial x^k}\right.\\ & \left.+\left(L_{ab}^d\circ h\right)g_{dc}-\left(L_{ac}^d\circ h\right)g_{db}-\left(L_{bc}^d\circ h\right)g_{da}\right),\end{array}$$

where $\left\|g^{dc}(x)\right\| = \|g_{cd}(x)\|^{-1}$, for any $x \in M$. *q.e.d.*

Corollary 4.3.1 *In particular, if $h = Id_M$, $(E,\pi,M) = (F,\nu,M)$ and $g \in \mathcal{T}{}^{0}_{2}(E,\pi,M)$ is a (pseudo)metrical structure, then the local real functions*

$$\rho\Gamma^a_{bc} = \frac{1}{2}\tilde{g}^{ad}\left(\rho^k_c\frac{\partial g_{bd}}{\partial x^k} + \rho^j_b\frac{\partial g_{dc}}{\partial x^j} - \rho^h_d\frac{\partial g_{bc}}{\partial x^h} + g_{ec}L^e_{bd} + g_{be}L^e_{dc} - g_{de}L^e_{bc}\right). \tag{4.3.3$'$}$$

are the components of a linear ρ-connection $\rho\Gamma$ for the vector bundle (E,π,M) compatible with g such that $\rho\mathbb{T} = 0$.

Therefore, the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

The linear ρ-connection $\rho\Gamma$ will be called the *linear ρ-connection of Levi-Civita type.*

In particular, if $\rho = Id_{TM}$, we obtain the classical Levi-Civita connection.

Theorem 4.3.2. *If $(E,\pi,M) = (F,\nu,M)$, $g \in \mathcal{T}{}^{0}_{2}(h^*E,h^*\pi,M)$ is a pseudo(metrical) structure and $\mathbb{T} \in \mathcal{T}{}^{1}_{2}(h^*E,h^*\pi,M)$ such that its components are skew symmetric in the lover indices, then the local real functions*

$$\rho\overset{\circ}{\Gamma}{}^a_{bc} = \rho\Gamma^a_{bc} + \frac{1}{2}\tilde{g}^{ad}\left(g_{de}\mathbb{T}^e_{bc} - g_{be}\mathbb{T}^e_{dc} + g_{ec}\mathbb{T}^e_{bd}\right), \tag{4.3.4}$$

are the components of a linear ρ-connection compatible with the (pseudo) metrical structure $g,$ where $\rho\Gamma^a_{bc}$ are the components of linear ρ-connection of Levi-Civita type. Therefore, the vector bundle $(h^*E,h^*\pi,M)$ becomes ρ-(pseudo)metrizable.

In addition, the tensor field $\mathbb{T}$ is the (ρ,h)-torsion tensor field.

Corollary 4.3.2 *In particular, if $h = Id_M$, $(E,\pi,M) = (F,\nu,M)$, $g \in \mathcal{T}{}^{0}_{2}(E,\pi,M)$ is a pseudo(metrical) structure and $T \in \mathcal{T}{}^{1}_{2}(E,\pi,M)$ such that its components are skew symmetric in the lover indices, then the local real functions*

$$\rho\overset{\circ}{\Gamma}{}^a_{bc} = \rho\Gamma^a_{bc} + \frac{1}{2}\tilde{g}^{ad}\left(g_{de}\mathbb{T}^e_{bc} - g_{be}\mathbb{T}^e_{dc} + g_{ec}\mathbb{T}^e_{bd}\right), \tag{4.3.4$'$}$$

are the components of a linear ρ-connection compatible with the (pseudo)metrical structure $g,$ where $\rho\Gamma^a_{bc}$ are the components of linear ρ-connection of Levi-Civita type. Therefore, the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

In addition, the tensor field $\mathbb{T}$ is the ρ-torsion tensor field.

Theorem 4.3.3 *If $g \in \mathcal{T}{}^{0}_{2}(E,\pi,M)$ is a pseudo (metrical) structure and $\rho\overset{\circ}{\Gamma}$ is a linear ρ-connection for the vector bundle (E,π,M), then the local real functions*

$$\rho\overset{k}{\Gamma}{}^{a}_{b\alpha} = \rho\overset{\circ}{\Gamma}{}^a_{b\alpha} + \tfrac{1}{2}\tilde{g}^{ac}g_{\underset{cb|\alpha}{\circ}} \tag{4.3.5}$$

are the components of a linear ρ-connection compatible with the (pseudo) metrical structure g. Therefore, the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

Theorem 4.3.4 *If $g \in \mathcal{T}{}^{0}_{2}(E,\pi,M)$ is a pseudo (metrical) structure, $\rho\overset{\circ}{\Gamma}$ is a linear ρ-connection for the vector bundle (E,π,M) and $T = T^d_{c\alpha}s_d \otimes s^c \otimes t^\alpha$, then the local real functions*

$$\rho\Gamma^a_{b\alpha} = \rho\overset{k}{\Gamma}{}^{a}_{b\alpha} + \tfrac{1}{2}O^{ca}_{bd}T^d_{c\alpha}, \tag{4.3.6}$$

are the components of a linear ρ-connection compatible with (pseudo) metrical structure g, where

$$O_{bd}^{ca} = \frac{1}{2}\left(\delta_b^c \delta_d^a - g_{bd}\tilde{g}^{ca}\right) \tag{4.3.7}$$

is the Obata operator.

Therefore, the vector bundle (E, π, M) becomes ρ-(pseudo)metrizable.

4.4 Lifts of differentiable curves

In this section we extend the notion of lift of a curve c at the total space of a vector bundle using the new notion of *locally invertible* $\mathbf{B}^{\mathbf{v}}$*-morphism.*

4.4.1 The lift of a differentiable curve at the total space of a vector bundle

We consider the following diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & (F, [,]_{F,h}, (\rho, \eta)) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd} \tag{4.4.1.1}$$

where $(E, \pi, M) \in |\mathbf{B}^{\mathbf{v}}|$ and $\Big((F, \nu, M), [,]_{F,h}, (\rho, \eta)\Big) \in |\mathbf{GLA}|$.

We admit that $(\rho, \eta)\,\Gamma$ is a (ρ, η)-connection for the vector bundle (E, π, M).

Let

$$I \xrightarrow{c} M$$

be a differentiable curve.

We say that

$$\left(E_{|\,\mathrm{Im}(\eta\circ h\circ c)}, \pi_{|\,\mathrm{Im}(\eta\circ h\circ c)}, \mathrm{Im}\,(\eta \circ h \circ c)\right)$$

is a vector subbundle of the vector bundle (E, π, M).

Definition 4.4.1.1 Let

$$\begin{array}{ccc} I & \xrightarrow{\dot{c}} & E_{|\,\mathrm{Im}(\eta\circ h\circ c)} \\ t & \longmapsto & y^a(t)\, s_a(\eta \circ h \circ c(t)) \end{array} \tag{4.4.1.2}$$

be a differentiable curve.

If there exists $g \in \mathbf{Man}(E, F)$ such that the following conditions are satisfied:

1. $(g, h) \in \mathbf{B}^v((E, \pi, M), (F, \nu, N))$ and
2. $\rho \circ g \circ \dot{c}(t) = \dfrac{d(\eta \circ h \circ c)^i(t)}{dt} \dfrac{\partial}{\partial x^i}((\eta \circ h \circ c)(t))$, for any $t \in I$, then we will say that $\dot{c}$ *is the* (g, h)*-lift of the differentiable curve* c.

Remark 4.4.1.1 Condition 2 is equivalent with the following affirmation:

$$\rho_\alpha^i(\eta \circ h \circ c(t)) \cdot g_a^\alpha(h \circ c(t)) \cdot y^a(t) = \frac{d(\eta \circ h \circ c)^i(t)}{dt},\ i \in \overline{1, m}. \tag{4.4.1.3}$$

Definition 4.4.1.2 If

$$I \overset{\dot{c}}{\to} E_{|\,\mathrm{Im}(\eta\circ h\circ c)}$$

is a differentiable (g,h)-lift of the differentiable curve c, then the section

$$\begin{array}{ccc} \mathrm{Im}\left(\eta\circ h\circ c\right) & \overset{u(c,\dot{c})}{\longrightarrow} & E_{|\,\mathrm{Im}(\eta\circ h\circ c)} \\ \eta\circ h\circ c\left(t\right) & \longmapsto & \dot{c}\left(t\right) \end{array} \tag{4.4.1.4}$$

will be called the *canonical section associated to the couple* $(c,\dot{c})$.

We will denote by $\left(T^{E}\left(c,\dot{c}\right),\tau,\mathrm{Im}\left(\eta\circ h\circ c\right)\right)$ the vector subbundle with minimal dimension such that

$$u\left(c,\dot{c}\right)\in\Gamma\left(T^{E}\left(c,\dot{c}\right),\tau,\mathrm{Im}\left(\eta\circ h\circ c\right)\right) \tag{4.4.1.5}$$

and will denoted by $\left(S^{E}\left(c,\dot{c}\right),\sigma,\mathrm{Im}\left(\eta\circ h\circ c\right)\right)$ the vector subbundle such that

$$T^{E}\left(c,\dot{c}\right)\oplus S^{E}\left(c,\dot{c}\right)=E_{|\,\mathrm{Im}(\eta\circ h\circ c)}.$$

Definition 4.4.1.3 If $(g,h)\in\mathbf{B^{v}}\left(\left(E,\pi,M\right),\left(F,\nu,N\right)\right)$ has the components

$$g_{a}^{\alpha};a\in\overline{1,r},\ \ \alpha\in\overline{1,p}$$

such that for any local vector $(n+p)$-chart (V,t_V) of (F,ν,N) there exists the real functions

$$V\ \ \xrightarrow{\tilde{g}_{\alpha}^{a}}\ \ \mathbb{R}\ ;\ a\in\overline{1,r},\ \ \alpha\in\overline{1,p}$$

such that

$$\tilde{g}_{\alpha}^{b}\left(\varkappa\right)\cdot g_{a}^{\alpha}\left(\varkappa\right)=\delta_{a}^{b},$$

for any $\varkappa\in V$, then we will say that *the* $\mathbf{B^{v}}$*-morphism* (g,h) *is locally invertible.*

Remark 4.4.2.2 In particular, if $(Id_{TM},Id_M,Id_M)=(\rho,\eta,h)$ and the $\mathbf{B^{v}}$ morphism (g,Id_M) is locally invertible, then we have the differentiable (g,Id_M)-lift

$$\begin{array}{ccc} I & \overset{\dot{c}}{\longrightarrow} & TM \\ t & \longmapsto & \tilde{g}_{j}^{i}\left(c\left(t\right)\right)\dfrac{dc^{j}\left(t\right)}{dt}\dfrac{\partial}{\partial x^{i}}\left(c\left(t\right)\right) \end{array}. \tag{4.4.1.6}$$

Moreover, if $g=Id_{TM}$, then we obtain the usual lift of tangent vectors

$$\begin{array}{ccc} I & \overset{\dot{c}}{\longrightarrow} & TM \\ t & \longmapsto & \dfrac{dc^{i}\left(t\right)}{dt}\dfrac{\partial}{\partial x^{i}}\left(c\left(t\right)\right) \end{array}. \tag{4.4.1.6$'$}$$

Definition 4.4.1.4 If

$$I\ \overset{\dot{c}}{\longrightarrow}\ E_{|\,\mathrm{Im}(\eta\circ h\circ c)} \tag{4.4.1.7}$$

is a differentiable (g,h)-lift of differentiable curve c, such that its components functions $\left(y^{a},\ a\in\overline{1,n}\right)$ are solutions for the differentiable system of equations:

$$\frac{du^{a}}{dt}+\left(\rho,\eta\right)\Gamma_{\alpha}^{a}\circ u\left(c,\dot{c}\right)\circ\left(\eta\circ h\circ c\right)\cdot g_{b}^{\alpha}\circ h\circ c\cdot u^{b}=0, \tag{4.4.1.8}$$

then we will say that *the* (g,h)*-lift* $\dot{c}$ *is parallel with respect to the* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$.

Remark 4.4.1.3 In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$ and the $\mathbf{B^v}$ morphism (g,Id_M) is locally invertible, then the differentiable (g,Id_{TM})-lift

$$\begin{array}{rcl} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \left(\tilde{g}^i_j\circ c\cdot\dfrac{dc^j}{dt}\right)\dfrac{\partial}{\partial x^i}\left(c\left(t\right)\right), \end{array} \tag{4.4.1.9}$$

is parallel with respect to the connection Γ if the component functions

$$\left(\tilde{g}^i_j\circ c\cdot\frac{dc^j}{dt},\ i\in\overline{1,n}\right)$$

are solutions for the differentiable system of equations

$$\frac{du^i}{dt}+\Gamma^i_k\circ u\left(c,\dot{c}\right)\circ c\cdot g^k_h\circ c\cdot u^h=0, \tag{4.4.1.10}$$

namely

$$\begin{array}{l} \dfrac{d}{dt}\left(\tilde{g}^i_j\left(c\left(t\right)\right)\cdot\dfrac{dc^j\left(t\right)}{dt}\right) \\ \qquad +\Gamma^i_k\left(c\left(t\right),\left(\tilde{g}^i_j\left(c\left(t\right)\right)\cdot\dfrac{dc^j\left(t\right)}{dt}\right)\cdot\dfrac{\partial}{\partial x^i}\left(c\left(t\right)\right)\right)\cdot\dfrac{dc^k\left(t\right)}{dt}=0. \end{array} \tag{4.4.1.10$'$}$$

Moreover, if $g=Id_{TM}$, then the usual lift of tangent vectors $(4.4.1.6)'$ is parallel with respect to the connection Γ if the component functions $\left(\frac{dc^j}{dt},\ j\in\overline{1,n}\right)$ are solutions for the differentiable system of equations

$$\frac{du^i}{dt}+\Gamma^i_k\circ u\left(c,\dot{c}\right)\circ c\cdot u^k=0, \tag{4.4.1.10$''$}$$

namely

$$\frac{d}{dt}\left(\frac{dc^j\left(t\right)}{dt}\right)+\Gamma^i_k\left(c\left(t\right),\frac{dc^j\left(t\right)}{dt}\cdot\frac{\partial}{\partial x^i}\left(c\left(t\right)\right)\right)\cdot\frac{dc^k\left(t\right)}{dt}=0. \tag{4.4.1.10$'''$}$$

4.4.2 The lift of a differentiable curve at the total space of dual vector bundle

We consider the following diagram:

$$\begin{array}{ccc} \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ {\scriptstyle\overset{*}{\pi}}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow{h} & N \end{array} \tag{4.4.2.1}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left(\left(F,\nu,N\right),[,]_{F,h},(\rho,\eta)\right)\in|\mathbf{GLA}|$.

We admit that $(\rho,\eta)\overset{*}{\Gamma}$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

Let

$$I \xrightarrow{c} M$$

be a differentiable curve. We say that

$$\left(\overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)},\overset{*}{\pi}_{|\operatorname{Im}(\eta\circ h\circ c)},\operatorname{Im}(\eta\circ h\circ c)\right)$$

is a vector subbundle of the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

Definition 4.4.2.1 Let

$$\begin{array}{ccc} I & \xrightarrow{\dot{c}} & \overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)} \\ t & \longmapsto & p_a(t)\,s^a(\eta\circ h\circ c(t)) \end{array} \tag{4.4.2.2}$$

be a differentiable curve.

If there exists $g\in\mathbf{Man}\left(\overset{*}{E},F\right)$ such that the following conditions are satisfied:

1. $(g,h)\in\mathbf{B}^v\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),(F,\nu,N)\right)$ and
2. $\rho\circ g\circ\dot{c}(t)=\dfrac{d(\eta\circ h\circ c)^i(t)}{dt}\dfrac{\partial}{\partial x^i}((\eta\circ h\circ c)(t))$, for any $t\in I$,then we will say that $\dot{c}$ *is the* (g,h)*-lift of the differentiable curve* c.

Remark 4.4.2.1 Condition 2 is equivalent with the following affirmation:

$$\rho_\alpha^i(\eta\circ h\circ c(t))\,g^{\alpha a}(h\circ c(t))\,p_a(t)=\frac{d(\eta\circ h\circ c)^i(t)}{dt},\ \ i\in\overline{1,m}. \tag{4.4.2.3}$$

Definition 4.4.2.2 If

$$I \xrightarrow{\dot{c}} \overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)}$$

is a differentiable (g,h)-lift of the differentiable curve c, then the section

$$\begin{array}{ccc} \operatorname{Im}(\eta\circ h\circ c) & \xrightarrow{\overset{*}{u}(c,\dot{c})} & \overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)} \\ \eta\circ h\circ c(t) & \longmapsto & \dot{c}(t) \end{array} \tag{4.4.2.4}$$

will be called the *canonical section associated to the couple* $(c,\dot{c})$.

We will denote by $\left(T^{\overset{*}{E}}(c,\dot{c}),\tau,\operatorname{Im}(\eta\circ h\circ c)\right)$ the vector subbundle with minimal dimension such that

$$\overset{*}{u}(c,\dot{c})\in\Gamma\left(T^{\overset{*}{E}}(c,\dot{c}),\tau,\operatorname{Im}(\eta\circ h\circ c)\right) \tag{4.4.2.5}$$

and will denoted by $\left(S^{\overset{*}{E}}(c,\dot{c}),\sigma,\operatorname{Im}(\eta\circ h\circ c)\right)$ the vector subbundle such that

$$T^{\overset{*}{E}}(c,\dot{c})\oplus S^{\overset{*}{E}}(c,\dot{c})=\overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)}.$$

Definition 4.4.2.3 If $(g,h) \in \mathbf{B^v}\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),(F,\nu,N)\right)$ has the components

$$g^{\alpha a}; a \in \overline{1,r},\ \alpha \in \overline{1,p}$$

such that for any vector local $(n+p)$-chart (V,t_V) of (F,ν,N) there exists the real functions

$$V \xrightarrow{\tilde{g}_{a\alpha}} \mathbb{R}\ ;\ a \in \overline{1,r},\ \alpha \in \overline{1,p}$$

such that

$$\tilde{g}_{a\alpha}(\varkappa) \cdot g^{\alpha b}(\varkappa) = \delta_a^b,\ \forall \varkappa \in V,$$

then we will say that *the* $\mathbf{B^v}$*-morphism* (g,h) *is locally invertible.*

Remark 4.4.2.2 In particular, if $(Id_{TM}, Id_M, Id_M) = (\rho,\eta,h)$ and the $\mathbf{B^v}$ morphism (g, Id_M) is locally invertible, then we have the differentiable (g, Id_M)-lift

$$\begin{array}{rcl} I & \xrightarrow{\dot{c}} & \overset{*}{T}M \\ t & \longmapsto & \tilde{g}_{ji}(c(t))\dfrac{dc^j(t)}{dt}dx^i(c(t)) \end{array}. \tag{4.4.2.6}$$

Definition 4.4.2.4 If

$$I \xrightarrow{\dot{c}} \overset{*}{E}_{|\,\mathrm{Im}(\eta\circ h\circ c)} \tag{4.4.2.7}$$

is a differentiable (g,h)-lift for the curve c such that its components functions $\left(p_b,\ b \in \overline{1,r}\right)$ are solutions for the differentiable system of equations:

$$\frac{du_b}{dt} + (\rho,\eta)\overset{*}{\Gamma}_{b\alpha} \circ \overset{*}{u}(c,\dot{c}) \circ (\eta\circ h\circ c) \cdot g^{a\alpha} \circ h \circ c \cdot u_a = 0, \tag{4.4.2.8}$$

then we will say that *the* (g,h)*-lift* $\dot{c}$ *is parallel with respect to the* (ρ,η)*-connection* $(\rho,\eta)\overset{*}{\Gamma}$.

Remark 4.4.2.3 In particular, if $(Id_{TM}, Id_M, Id_M) = (\rho,\eta,h)$ and the $\mathbf{B^v}$ morphism (g, Id_M) is locally invertible, then the differentiable (g, Id_M)-lift (4.4.2.6) is parallel with respect to the connection Γ if the component functions $\left(\tilde{g}_{ji} \circ c \cdot \dfrac{dc^j}{dt},\ i \in \overline{1,m}\right)$ are solutions for the differentiable system of equations

$$\frac{du_j}{dt} + \Gamma_{jk} \circ \overset{*}{u}(c,\dot{c}) \circ c \cdot g^{kh} \circ c \cdot u_h = 0, \tag{4.4.2.9}$$

namely

$$\begin{array}{l} \dfrac{d}{dt}\left(\tilde{g}_{ji} \circ c(t) \cdot \dfrac{dc^j(t)}{dt}\right) \\ \qquad +\Gamma_{jk}\left(c(t), \left(\tilde{g}_{ji} \circ c(t) \cdot \dfrac{dc^j(t)}{dt}\right) \cdot dx^i(c(t))\right) \cdot \dfrac{dc^k(t)}{dt} = 0, \end{array} \tag{4.4.2.9'}$$

4.5 Parallel transport

We consider the following diagram:

$$\begin{array}{ccccc} & & E & \xrightarrow{g} & (F,[,]_F,(\rho,Id_M)) \\ (4.5.1) & & \downarrow\pi & & \downarrow\nu \\ I & \xrightarrow{c} & M & \xrightarrow{Id_M} & M \end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$, $((F,\nu,M),[,]_F,(\rho,Id_M))\in|\mathbf{LA}|$, (g,Id_M) is a $\mathbf{B^v}$-morphism and c is a differentiable curve.

Let $\dot{c}$ be a (g,Id_M)-lift of the curve c.

We admit that $\rho\Gamma$ is a linear ρ-connection for the vector bundle (E,π,M).

Definition 4.5.1 We will called *parallel transport of tensor fields of (r,s) type along a curve c* any family

$$\mathcal{P}_c=\left\{P_{t_1,t_2}\in Izo\left(\mathcal{T}_q^p(E,\pi,M)_{c(t_1)},\mathcal{T}_q^p(E,\pi,M)_{c(t_2)}\right),\ t_1,t_2\in I\right\}$$

with the following properties:

1. For any $t_1,t_2\in I$ it exists a unique isomorphism $P_{t_1,t_2}\in\mathcal{P}_c$ such that $(P_{t_1,t_2})^{-1}=P_{t_2,t_1}$.
2. For any $t_1,t_2,t_3\in I$ we have that $P_{t_2,t_3}\circ P_{t_1,t_2}=P_{t_1,t_3}$.

Theorem 4.5.1 *If $t_0,t\in I$ and U is a local vector $(m+n)$-chart such that $c(t_0),c(t)\in U$, then it exists an unique isomorphism*

$$P_{t_0,t}\in Izo\left(\mathcal{T}_q^p(E,\pi,M)_{c(t_0)},\mathcal{T}_q^p(E,\pi,M)_{c(t)}\right)$$

such that $(P_{t_0,t})^{-1}=P_{t,t_0}$ which not depend on the local vector chart used.

Proof. Let $T_{c(t_0)}\in\mathcal{T}_q^p(E,\pi,M)_{c(t_0)}$ be. We admit that

$$T_{\pi\circ c(t_0)}=\left(T_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}s_{a_1}\otimes\ldots\otimes s_{a_p}\otimes s^{b_1}\otimes\ldots\otimes s^{b_q}\right)(c(t_0))$$

and

$$\begin{array}{rl} P_{t_0,t}\left(T_{c(t_0)}\right) & =T_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}(c(t_0))\,A_{a_1}^{\tilde{a}_1}(t_0,t)\cdot\ldots\cdot A_{a_p}^{\tilde{a}_p}(t_0,t)\cdot B_{\tilde{b}_1}^{b_1}(t_0,t)\cdot \\ & \ldots\cdot B_{\tilde{b}_q}^{b_q}(t_0,t)\cdot\cdot\left(s_{\tilde{a}_1}\otimes\ldots\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes\ldots\otimes s^{\tilde{b}_q}\right)(c(t_0)), \end{array}$$

where the matrices

$$\left\|A_{a_1}^{\tilde{a}_1}(t_0,t)\right\|,\ldots\left\|A_{a_p}^{\tilde{a}_p}(t_0,t)\right\|,\left\|B_{\tilde{b}_1}^{b_1}(t_0,t)\right\|,\ldots,\left\|B_{\tilde{b}_q}^{b_q}(t_0,t)\right\|$$

are the matrices used for base transformation. Using the equality

$$
\begin{aligned}
0=\frac{d}{dt}\left(T^{a_1,\dots,a_p}_{b_1,\dots,b_q}c\left(t_0\right)\right) &= \frac{d}{dt}\left(T^{a_1,\dots,a_p}_{b_1,\dots,b_q}c\left(t_0\right)A^{\tilde{a}_1}_{a_1}\left(t_0,t\right)\cdot\ldots\cdot A^{\tilde{a}_p}_{a_p}\left(t_0,t\right)\right.\\
&\left.\cdot B^{b_1}_{\tilde{b}_1}\left(t_0,t\right)\ldots\cdot B^{b_q}_{\tilde{b}_q}\left(t_0,t\right)\right)\cdot A^{a_1}_{\tilde{a}_1}\left(t,t_0\right)\cdot\ldots\\
&\cdot A^{a_p}_{\tilde{a}_p}\left(t,t_0\right)\cdot B^{\tilde{b}_1}_{b_1}\left(t,t_0\right)\ldots\cdot B^{\tilde{b}_q}_{b_q}\left(t,t_0\right)+T^{a_1,\dots,a_p}_{b_1,\dots,b_q}c\left(t_0\right)\\
&\cdot A^{\tilde{a}_1}_{a_1}\left(t_0,t\right)\cdot\ldots\cdot A^{\tilde{a}_p}_{a_p}\left(t_0,t\right)\cdot B^{b_1}_{\tilde{b}_1}\left(t_0,t\right)\cdot\ldots\\
&\cdot B^{b_q}_{\tilde{b}_q}\left(t_0,t\right)\cdot\frac{d}{dt}\left(A^{a_1}_{\tilde{a}_1}\left(t,t_0\right)\cdot\ldots\cdot A^{a_p}_{\tilde{a}_p}\left(t,t_0\right)\right.\\
&\left.\cdot B^{\tilde{b}_1}_{b_1}\left(t,t_0\right)\ldots\cdot B^{\tilde{b}_q}_{b_q}\left(t,t_0\right)\right)
\end{aligned}
$$

and the notation

$$
\tilde{T}^{a_1,\dots,a_p}_{b_1,\dots,b_q}c\left(t\right) = T^{a_1,\dots,a_p}_{b_1,\dots,b_q}\left(\pi\circ c\left(t_0\right)\right)A^{\tilde{a}_1}_{a_1}\left(t_0,t\right)\cdot\ldots\cdot A^{\tilde{a}_p}_{a_p}\left(t_0,t\right)\cdot B^{b_1}_{\tilde{b}_1}\left(t_0,t\right)\ldots\cdot B^{b_q}_{\tilde{b}_q}\left(t_0,t\right)
$$

we obtain the equality

$$
\begin{aligned}
-\frac{d}{dt}\tilde{T}^{\tilde{a}_1,\dots,\tilde{a}_p}_{\tilde{b}_1,\dots,\tilde{b}_q}c\left(t\right) &= A^{\tilde{a}_1}_{a_1}\left(t_0,t\right)\frac{d}{dt}A^{a_1}_{a}\left(t,t_0\right)T^{a\tilde{a}_2,\dots,\tilde{a}_p}_{\tilde{b}_1,\dots,\tilde{b}_q}c\left(t\right)+\ldots\\
&+A^{\tilde{a}_p}_{a_p}\left(t_0,t\right)\frac{d}{dt}A^{a_p}_{a}\left(t,t_0\right)T^{\tilde{a}_1,\dots,\tilde{a}_{p-1}a}_{\tilde{b}_1,\dots,\tilde{b}_q}c\left(t\right)\\
&+B^{b_1}_{\tilde{b}_1}\left(t_0,t\right)\frac{d}{dt}B^{b}_{b_1}\left(t,t_0\right)T^{\tilde{a}_1,\dots,\tilde{a}_p}_{b\tilde{b}_2,\dots,\tilde{b}_q}c\left(t\right)+\ldots\\
&+B^{b_q}_{\tilde{b}_q}\left(t_0,t\right)\frac{d}{dt}B^{b}_{b_q}\left(t,t_0\right)T^{\tilde{a}_1,\dots,\tilde{a}_p}_{\tilde{b}_1,\dots,\tilde{b}_{q-1}b}c\left(t\right).
\end{aligned}
$$

Since the differentiable equations:

$$
\begin{aligned}
A^{\tilde{a}_1}_{a_1}\left(t_0,t\right)\frac{d}{dt}A^{a_1}_{a}\left(t,t_0\right) &= \rho\Gamma^{\tilde{a}_1}_{a\alpha}c\left(t\right)g^{\alpha}_{c}\left(x\left(t\right)\right)y^{c}\left(t\right)\\
A^{a_1}_{\tilde{a}_1}\left(t_0,t_0\right) &= \delta^{a_1}_{\tilde{a}_1}\\
&\ldots\\
A^{\tilde{a}_p}_{a_p}\left(t_0,t\right)\frac{d}{dt}A^{a_p}_{a}\left(t,t_0\right) &= \rho\Gamma^{\tilde{a}_p}_{a\alpha}c\left(t\right)g^{\alpha}_{c}\left(x\left(t\right)\right)y^{c}\left(t\right)\\
A^{a_p}_{\tilde{a}_p}\left(t_0,t_0\right) &= \delta^{a_p}_{\tilde{a}_p}\\
B^{b_1}_{\tilde{b}_1}\left(t_0,t\right)\frac{d}{dt}B^{b}_{b_1}\left(t,t_0\right) &= -\rho\Gamma^{b}_{\tilde{b}_1\alpha}c\left(t\right)g^{\alpha}_{c}\left(x\left(t\right)\right)y^{c}\left(t\right)\\
B^{\tilde{b}_1}_{b_1}\left(t_0,t_0\right) &= \delta^{\tilde{b}_1}_{b_1}\\
&\ldots\\
B^{b_q}_{\tilde{b}_q}\left(t_0,t\right)\frac{d}{dt}B^{b}_{b_q}\left(t,t_0\right) &= -\rho\Gamma^{b}_{\tilde{b}_q\alpha}c\left(t\right)g^{\alpha}_{c}\left(x\left(t\right)\right)y^{c}\left(t\right)\\
B^{\tilde{b}_q}_{b_q}\left(t_0,t_0\right) &= \delta^{\tilde{b}_q}_{b_q}
\end{aligned}
$$

are equivalent with the following differentiable equations

$$\begin{array}{rl}
\dfrac{d}{dt}A_{\tilde{a}_1}^{a_1}(t,t_0) & = A_a^{a_1}(t,t_0)\,\rho\Gamma_{\tilde{a}_1\alpha}^{a}(\pi\circ c(t))\,g_c^{\alpha}(x(t))\,y^c(t)\\
A_{\tilde{a}_1}^{a_1}(t_0,t_0) & = \delta_{\tilde{a}_1}^{a_1}\\
 & \ldots\\
\dfrac{d}{dt}A_{\tilde{a}_p}^{a_p}(t,t_0) & = A_a^{a_p}(t,t_0)\,\rho\Gamma_{\tilde{a}_p\alpha}^{a}(\pi\circ c(t))\,g_c^{\alpha}(x(t))\,y^c(t)\\
A_{\tilde{a}_p}^{a_p}(t_0,t_0) & = \delta_{\tilde{a}_p}^{a_p}\\
\dfrac{d}{dt}B_{b_1}^{\tilde{b}_1}(t,t_0) & = -B_{b_1}^{b}(t,t_0)\,\rho\Gamma_{b\alpha}^{\tilde{b}_1}(\pi\circ c(t))\,g_c^{\alpha}(x(t))\,y^c(t)\\
B_{b_1}^{\tilde{b}_1}(t_0,t_0) & = \delta_{b_1}^{\tilde{b}_1}\\
 & \ldots\\
\dfrac{d}{dt}B_{b_q}^{\tilde{b}_q}(t,t_0) & = -B_{b_q}^{b}(t,t_0)\,\rho\Gamma_{b\alpha}^{\tilde{b}_q}(\pi\circ c(t))\,g_c^{\alpha}(x(t))\,y^c(t)\\
B_{b_q}^{\tilde{b}_q}(t_0,t_0) & = \delta_{b_q}^{\tilde{b}_q}
\end{array}$$

which has unique solutions which not depend on the local vector chart used, it results the conclusion of the theorem. *q.e.d.*

Corollary 4.5.1 *For any $p,q\in\mathbb{N}$, it exists a parallel transport $\mathcal{P}_c$ between the tensors of (p,q) type.*

This parallel transport will be called the *parallel transport along the curve c associated to linear ρ-connection $\rho\Gamma$.*

Proof. Let $p,q\in\mathbb{N}$ and $t_0,t\in I$ be. Without restricting the generality, we admit that not exists a vector local $m+r$-chart U which contain the points $c(t_0)$ and $c(t)$.

Since I is a conex manifold, it results that it exist a finite numbers of real numbers $t_1,t_2,...,t_r=t$ such that for each $j\in\overline{1,r}$, the points $c(t_{j-1})$ and $c(t_j)$ belongs to the same vector local $m+r$-chart.

Using the previous theorem, we build the linear isomorphisms $P_{t_0,t_1},\ P_{t_1,t_2},\ \ldots,P_{t_{r-1},t}$.

The linear isomorphism $P_{t_{r-1},t}\circ\ldots\circ P_{t_1,t_2}\circ P_{t_0,t_1}=P_{t_0,t}$ not depend on the vector local $m+r$-charts used. *q.e.d.*

Remark 4.5.1 Using the notations of the previous theorem we obtain:

$$\begin{array}{rl}
-\dfrac{d}{dt}\tilde{T}_{\tilde{b}_1,\ldots,\tilde{b}_q}^{\tilde{a}_1,\ldots,\tilde{a}_p}c(t) & = T_{\tilde{b}_1,\ldots,\tilde{b}_q}^{a\tilde{a}_2,\ldots,\tilde{a}_p}c(t)\,\rho\Gamma_{a\alpha}^{\tilde{a}_1}c(t)\,g_c^{\alpha}(x(t))\,y^c(t)+\ldots\\
 & +T_{\tilde{b}_1,\ldots,\tilde{b}_q}^{\tilde{a}_1,\ldots,\tilde{a}_{p-1}a}c(t)\,\rho\Gamma_{a\alpha}^{\tilde{a}_p}c(t)\,g_c^{\alpha}(x(t))\,y^c(t)+\\
 & -T_{b\tilde{b}_2,\ldots,\tilde{b}_q}^{\tilde{a}_1,\ldots,\tilde{a}_p}c(t)\,\rho\Gamma_{\tilde{b}_1\alpha}^{b}c(t)\,g_c^{\alpha}(x(t))\,y^c(t)-\ldots\\
 & -T_{\tilde{b}_1,\ldots,\tilde{b}_{q-1}b}^{\tilde{a}_1,\ldots,\tilde{a}_p}c(t)\,\rho\Gamma_{\tilde{b}_q\alpha}^{b}c(t)\,g_c^{\alpha}(x(t))\,y^c(t).
\end{array}\tag{4.5.2}$$

Theorem 4.5.2 *If $\mathcal{P}_c$ is the parallel transport along the curve c associated to linear ρ-connection $\rho\Gamma$, then, for any $t\in I$ we obtain:*

$$\lim_{h\longrightarrow 0}\frac{P_{t+h,t}\left(T_{c(t+h)}\right)-T_{c(t)}}{h}=\left(\rho D_{u(c,\dot{c})}T\right)c(t),\tag{4.5.3}$$

for any $T \in \mathcal{T}\,{}^{p}_{q}(E,\pi,M)$.

Proof. Let be $T \in \mathcal{T}\,{}^{p}_{q}(E,\pi,M)$. Let $t \in I$ and $h>0$ be such that $]t-h,t+h[\,\subset I$.

For any $\tilde{a}_1,...,\tilde{a}_p,\tilde{b}_1,...,\tilde{b}_q \in \overline{1,n}$ we build the following application

$$\begin{array}{ccc} [t,t+h] & \xrightarrow{z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}} & \mathbb{R} \\ \theta & \longmapsto & z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(\theta) \end{array}$$

defined by

$$z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(\theta) = T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q} c(t+h)\, A^{\tilde{a}_1}_{a_1}(t+h,\theta)\cdot ...\cdot A^{\tilde{a}_p}_{a_p}(t+h,\theta)\cdot B^{b_1}_{\tilde{b}_1}(t+h,\theta)...\cdot B^{b_q}_{\tilde{b}_q}(t+h,\theta)$$

Using the main theorem, it exists a unique real number

$$\xi^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q} \in \,]t,t+h[$$

such that

$$z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(t+h) = z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(t) + h\left(z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\right)'\left(\xi^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\right).$$

Since the component by indices ${}^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}$ of a tensor $P_{t+h,t}\left(T_{c(t+h)}\right)$ is $z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(t)$, it results that

$$\begin{aligned}
&\lim_{h\longrightarrow 0}\frac{P_{t+h,t}\left(T_{\pi\circ c(t+h)}\right)-T_{\pi\circ c(t)}}{h} = \\
&= \lim_{h\longrightarrow 0}\frac{z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(t)-T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)}{h}\cdot\left(s_{\tilde{a}_1}\otimes...\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes...\otimes s^{\tilde{b}_q}\right)c(t)\\
&= \lim_{h\longrightarrow 0}\frac{T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t+h)-T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)}{h}\cdot\left(s_{\tilde{a}_1}\otimes...\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes...\otimes s^{\tilde{b}_q}\right)c(t)\\
&-\lim_{h\longrightarrow 0}\frac{h\left(z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\right)'\left(\xi^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\right)}{h}\cdot\left(s_{\tilde{a}_1}\otimes...\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes...\otimes s^{\tilde{b}_q}\right)c(t)\\
&= \frac{d}{dt}T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)\cdot\left(s_{\tilde{a}_1}\otimes...\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes...\otimes s^{\tilde{b}_q}\right)c(t)\\
&-\frac{d}{dt}\tilde{T}^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)\cdot\left(s_{\tilde{a}_1}\otimes...\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes...\otimes s^{\tilde{b}_q}\right)c(t).
\end{aligned}$$

Using Remark 4.5.1 and the equality

$$\frac{d}{dt}T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t) = \frac{dx^i}{dt}\frac{\partial T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)}{\partial x^i} = g^{\alpha}_{c}(x(t))\,y^c(t)\,\rho^i_{\alpha}\frac{\partial T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)}{\partial x^i},$$

it results the conclusion of theorem. *q.e.d.*

Definition 4.5.2 *The tensor field* $T \in \mathcal{T}^p_q(E,\pi,M)$ *is parallel along the curve* c with respect to the linear ρ-connection $\rho\Gamma$ if for any $t_1,t_2 \in I$ it results that

(4.5.4) $$P_{t_1,t_2}\left(T_{c(t_1)}\right) = T_{c(t_2)}.$$

Theorem 4.5.3 The tensor field $T \in \mathcal{T}^p_q(E,\pi,M)$ is parallel along the curve c with respect to linear ρ-connection $\rho\Gamma$ if and only if

(4.5.5) $$\left(\rho D_{u(c,\dot{c})}T\right)c(t) = 0, \forall t \in I.$$

Corollary 4.5.2 The tensor field $T \in \mathcal{T}^p_q(E, \pi, M)$ is parallel along the curve c with respect to linear ρ-connection $\rho\Gamma$ if and only if

$$\begin{array}{c}\frac{d}{dt}T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c\left(t\right)+g^{\alpha}_{c}\left(x\left(t\right)\right)y^{c}\left(t\right)\left(T^{a\tilde{a}_2,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\rho\Gamma^{\tilde{a}_1}_{a\alpha}+...+T^{\tilde{a}_1,...,\tilde{a}_{p-1}a}_{\tilde{b}_1,...,\tilde{b}_q}\rho\Gamma^{\tilde{a}_p}_{a\alpha}\right.\\ \left.-T^{\tilde{a}_1,...,\tilde{a}_p}_{b\tilde{b}_2,...,\tilde{b}_q}\rho\Gamma^{b}_{\tilde{b}_1\alpha}-T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_{q-1}b}\rho\Gamma^{b}_{\tilde{b}_q\alpha}\right)c\left(t\right)=0,\quad \forall t\in I.\end{array}$$

4.6 Formulas of Gauss-Weingarten type

Using the main ideas of the theory of Myller configurations, introduced by R. Miron in [37] and applied to Finsler spaces by O. Constantinescu in [13] and his Ph.D. Thesis, we present the Gauss-Weingarten formulas for generalized Lie algebroids.

Let

$$\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$$

be a generalized Lie algebroid given by the diagram:

$$\begin{tikzcd} & \left(F,\left[,\right]_{F,h},\left(\rho,\eta\right)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd} \tag{4.6.1}$$

The geometry of the couple (M, h) is the geometry of the pull-back vector bundle $(h^*F, h^*\nu, M)$ using the diagram

$$\begin{tikzcd} h^*F \arrow[d, "h^*\nu"'] & \left(h^*F,\left[,\right]_{h^*F},\left(\overset{h^*F}{\rho},Id_M\right)\right) \arrow[d, "h^*\nu"] \\ M \arrow[r, "Id_M"] & M \end{tikzcd} \tag{4.6.2}$$

Let $I\xrightarrow{c}M$ be a differentiable curve and let $M' = Im(\eta \circ h \circ c)$ be.
Let $I\xrightarrow{\dot{c}}h^*F_{|M'}$ be the (Id_{h^*F}, Id_M)-lift of the curve c.
Let $\{T_\alpha, \alpha \in \overline{1,p}\}$, $\{s_a, a \in \overline{1,q}\}$ and $\{\chi_i, i \in \overline{1,s}\}$ be the base for

$$\Gamma\left(h^*F_{|M'},h^*\nu_{|M'},M'\right),\ \Gamma\left(T^{h^*F}\left(c,\dot{c}\right),\tau,M'\right)\text{ and }\Gamma\left(S^{h^*F}\left(c,\dot{c}\right),\sigma,M'\right),$$

respectively.

The dimension of type fibre of the vector bundle $\left(h^*F_{|M'}, h^*\nu_{|M'}, M'\right)$ is $q + s = p$.

Consequently, for any $a \in \overline{1,q}$ we have

$$s_a = \Lambda^\alpha_a T_\alpha \tag{4.6.3}$$

and for any $i \in \overline{1,s}$ we have

$$\chi_i = \Lambda^\alpha_i T_\alpha. \tag{4.6.4}$$

Let $g = g_{\alpha\beta}t^\alpha \otimes t^\beta \in \mathcal{T}\,^0_2(F, \nu, N)$ be a (pseudo)metrical structure.

Remark 4.6.1 The following affirmations are satisfied:

1. The section $\overset{h^*F}{g} = \overset{h^*F}{g}_{\alpha\beta} T^\alpha \otimes T^\beta$ defined by

$$\overset{h^*F}{g}_{\alpha\beta}(x) = g_{\alpha\beta}(h(x)) \tag{4.6.5}$$

is a (pseudo)metrical structure.

2. The section $\overset{T(M')}{g} = \overset{T(M')}{g}_{ab} s^a \otimes s^b$ defined by

$$\overset{T(M')}{g}_{ab}(x) = \Lambda_a^\alpha g_{\alpha\beta}(h(x)) \Lambda_b^\beta \tag{4.6.6}$$

is a (pseudo)metrical structure.

3. The section $\overset{S(M')}{g} = \overset{S(M')}{g}_{ij} \chi^i \otimes \chi^j$ defined by

$$\overset{S(M')}{g}_{ij}(x) = \Lambda_i^\alpha g_{\alpha\beta}(h(x)) \Lambda_j^\beta \tag{4.6.7}$$

is a (pseudo)metrical structure.

Remark 4.6.2 Using the diagram (4.6.2) we can construct the linear ρ-connection of Levi Civita type $\overset{h^*F}{\rho}\Gamma$ of components $\overset{h^*F}{\rho}\Gamma^\alpha_{\beta\gamma}$.

We have the covariant ρ-derivative defined by

$$\overset{h^*F}{\rho} D_z w = z^\gamma \left(\left(\overset{h^*F}{\rho} \right)^k_\gamma \frac{\partial w^\alpha}{\partial x^k} + \overset{h^*F}{\rho}\Gamma^\alpha_{\beta\gamma} w^\beta \right) T_\alpha, \tag{4.6.8}$$

where $\left(\overset{h^*F}{\rho} \right)^k_\gamma$ are the components of the map $\overset{h^*F}{\rho}$.

Definition 4.6.1 If we can defined

$$\overset{h^*F}{\rho} D_{v\oplus 0}(u \oplus 0) = v^c \left(\left(\overset{h^*F}{\rho} \right)^k_c \frac{\partial u^a}{\partial x^k} + \overset{h^*F}{\rho}\Gamma^a_{bc} u^b \right) s_a, \tag{4.6.9}$$

$$\overset{h^*F}{\rho} D_{v\oplus 0}(0 \oplus \xi) = v^c \left(\left(\overset{h^*F}{\rho} \right)^k_c \frac{\partial \xi^i}{\partial x^k} + \overset{h^*F}{\rho}\Gamma^i_{jc} \xi^j \right) \chi_i, \tag{4.6.10}$$

$$\overset{h^*F}{\rho} D_{0\oplus \eta}(u \oplus 0) = \eta^h \left(\overset{h^*F}{\rho}{}^k_h \frac{\partial u^a}{\partial x^k} + \overset{h^*F}{\rho}\Gamma^a_{bh} u^b \right) s_a, \tag{4.6.11}$$

$$\overset{h^*F}{\rho} D_{0\oplus \eta}(0 \oplus \xi) = \eta^h \left(\left(\overset{h^*F}{\rho} \right)^k_h \frac{\partial \xi^i}{\partial x^k} + \overset{h^*F}{\rho}\Gamma^i_{jh} \xi^j \right) \chi_i. \tag{4.6.12}$$

and we can consider the bilinear applications

$$\Gamma\left(T^{h^*F_{|M'}}(c,\dot{c}), \tau, M'\right) \times \Gamma\left(T^{h^*F_{|M'}}(c,\dot{c}), \tau, M'\right) \xrightarrow{H} \Gamma\left(S^{h^*F_{|M'}}(c,\dot{c}), \sigma, M'\right)$$

and

$$\Gamma\left(S^{h^*F_{|M'}}(c,\dot{c}), \sigma, M'\right) \times \Gamma\left(T^{h^*F_{|M'}}(c,\dot{c}), \tau, M'\right) \xrightarrow{A} \Gamma\left(T^{h^*F_{|M'}}(c,\dot{c}), \tau, M'\right)$$

which satisfy the following relations

$$(4.6.13) \qquad \overset{h^*F}{\rho}D_{v^c\Lambda_c^\gamma T_\gamma}\left(u^b\Lambda_b^\beta T_\beta\right)=\overset{h^*F}{\rho}D_{v\oplus 0}\left(u\oplus 0\right)\oplus H\left(u,v\right),$$

$$(4.6.14) \qquad \overset{h^*F}{\rho}D_{v^c\Lambda_c^\gamma T_\gamma}\left(\xi^j\Lambda_j^\beta T_\beta\right)=-A_\xi\left(v\right)\oplus\overset{h^*F}{\rho}D_{v\oplus 0}\left(0\oplus\xi\right)$$

and

$$(4.6.15) \qquad \overset{S(M')}{g}\left(H\left(u,v\right),\xi\right)=\overset{T(M')}{g}\left(A_\xi\left(v\right),u\right),$$

then we will say that the *relations* (4.6.13), (4.6.14) *and* (4.6.15) *are formulas of Gauss-Weingarten type associated to differentiable curve c, metrical structure g and bilinear applications H and A.*

The bilinear application H will be called *the second fundamental form of differentiable curve c.*

Remark 4.6.2 Using the base sections, then the formulas of Gauss-Weingarten type become:

$$(4.6.13') \qquad \Lambda_c^\gamma\left(\left(\overset{h^*F}{\rho}\right)_\gamma^k\frac{\partial\Lambda_b^\alpha}{\partial x^k}+\overset{h^*F}{\rho}\Gamma_{\beta\gamma}^\alpha\Lambda_b^\beta\right)=\overset{h^*F}{\rho}\Gamma_{bc}^a\Lambda_a^\alpha+H_{bc}^i\Lambda_i^\alpha,$$

$$(4.6.14') \qquad \Lambda_c^\gamma\left(\left(\overset{h^*F}{\rho}\right)_\gamma^h\frac{\partial\Lambda_j^\alpha}{\partial x^h}+\overset{h^*F}{\rho}\Gamma_{\beta\gamma}^\alpha\Lambda_j^\beta\right)=-A_{jc}^a\Lambda_a^\alpha+\overset{h^*F}{\rho}\Gamma_{jc}^i\Lambda_i^\alpha$$

$$(4.6.15') \qquad \overset{S(M')}{g}{}_{ij}H_{bc}^i=\overset{T(M')}{g}{}_{ab}A_{jc}^a.$$

5 The geometry of total space of the Lie algebroid generalized tangent bundle for a vector bundle

5.1 Adapted (ρ,η)-basis and adapted dual (ρ,η)-basis

In the following, we consider the following diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \left(F,[,]_{F,h},(\rho,\eta)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E,\pi,M)\in|\mathbf{B}^{\mathbf{v}}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M).

If we put the problem of finding a base for the $\mathcal{F}(E)$-module

$$\left(\Gamma\left(H\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),+,\cdot\right)$$

of the type

$$\frac{\delta}{\delta\tilde{z}^\alpha}=\tilde{Z}_\alpha^\beta\frac{\partial}{\partial\tilde{z}^\alpha}+Y_\alpha^a\frac{\partial}{\partial\tilde{y}^a},\alpha\in\overline{1,r}$$

which satisfies the following conditions:

$$\begin{array}{rcl}\Gamma\left(\left(\rho,\eta\right)\pi!,Id_E\right)\left(\dfrac{\delta}{\delta\tilde{z}^\alpha}\right) & = & \tilde{T}_\alpha\\ \Gamma\left(\left(\rho,\eta\right)\Gamma,Id_E\right)\left(\dfrac{\delta}{\delta\tilde{z}^\alpha}\right) & = & 0,\end{array}\tag{5.1.1}$$

then we obtain the sections

$$\frac{\delta}{\delta\tilde{z}^\alpha}=\frac{\partial}{\partial\tilde{z}^\alpha}-\left(\rho,\eta\right)\Gamma_\alpha^a\frac{\partial}{\partial\tilde{y}^a}.\tag{5.1.2}$$

We observe that their law of change is a tensorial law under a change of vector fiber charts.

Definition 5.1.1 The base

$$\left(\frac{\delta}{\delta\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{y}^a}\right)\overset{put}{=}\left(\tilde{\delta}_\alpha,\dot{\tilde{\partial}}_a\right)$$

will be called the *adapted* (ρ,η)*-base.*

The following equality holds good

$$\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\alpha\right)=\left(\rho_\alpha^i\circ h\circ\pi\right)\partial_i-\left(\rho,\eta\right)\Gamma_\alpha^a\dot{\partial}_a,\tag{5.1.3}$$

where $\left(\partial_i,\dot{\partial}_a\right)$ is the natural base for the $\mathcal{F}(E)$-module $\left(\Gamma\left(TE,\tau_E,E\right),+,\cdot\right)$.

Moreover, if $\rho\Gamma$ is the ρ-connection associated to the connection Γ, then we obtain

$$\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\alpha\right)=\left(\rho_\alpha^i\circ h\circ\pi\right)\delta_i,\tag{5.1.4}$$

where $\left(\delta_i,\dot{\partial}_a\right)$ is the adapted base for the $\mathcal{F}(E)$-module $\left(\Gamma\left(TE,\tau_E,E\right),+,\cdot\right)$.

Theorem 5.1.1 *The following equality holds good*

$$\left[\tilde{\delta}_\alpha,\tilde{\delta}_\beta\right]_{(\rho,\eta)TE}=L_{\alpha\beta}^\gamma\circ\left(h\circ\pi\right)\tilde{\delta}_\gamma+\left(\rho,\eta,h\right)\mathbb{R}^a{}_{\alpha\beta}\dot{\tilde{\partial}}_a,\tag{5.1.5}$$

where

$$\begin{array}{c}\left(\rho,\eta,h\right)\mathbb{R}^a{}_{\alpha\beta}=\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\beta\right)\left(\left(\rho,\eta\right)\Gamma_\alpha^a\right)\\ -\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\alpha\right)\left(\left(\rho,\eta\right)\Gamma_\beta^a\right)+\left(L_{\alpha\beta}^\gamma\circ h\circ\pi\right)\left(\rho,\eta\right)\Gamma_\gamma^a,\end{array}\tag{5.1.6}$$

Moreover, we have:

$$\left[\tilde{\delta}_\alpha,\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE}=\Gamma\left(\tilde{\rho},Id_E\right)\left(\dot{\tilde{\partial}}_b\right)\left(\left(\rho,\eta\right)\Gamma_\alpha^a\right)\dot{\tilde{\partial}}_a,\tag{5.1.7}$$

and

$$\Gamma\left(\tilde{\rho},Id_E\right)\left[\tilde{\delta}_\alpha,\tilde{\delta}_\beta\right]_{(\rho,\eta)TE}=\left[\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\alpha\right),\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\beta\right)\right]_{TE}.\tag{5.1.8}$$

If we consider the problem of finding a base for the $\mathcal{F}(E)$-module

$$\left(\Gamma\left(\left(V\left(\rho,\eta\right)TE\right)^{*},\left(\left(\rho,\eta\right)\tau_{E}\right)^{*},E\right),+,\cdot\right)$$

of the type

$$\delta\tilde{y}^{a}=\theta_{\alpha}^{a}d\tilde{z}^{\alpha}+\omega_{b}^{a}d\tilde{y}^{b},\ a\in\overline{1,n}$$

which satisfies the following conditions:

$$\left\langle \delta\tilde{y}^{a},\dot{\tilde{\partial}}_{a}\right\rangle =1\wedge\left\langle \delta\tilde{y}^{a},\tilde{\delta}_{\alpha}\right\rangle =0, \tag{5.1.9}$$

then we obtain the sections

$$\delta\tilde{y}^{a}=\left(\rho,\eta\right)\Gamma_{\alpha}^{a}d\tilde{z}^{\alpha}+d\tilde{y}^{a},a\in\overline{1,n}. \tag{5.1.10}$$

We observe that their changing rule is tensorial under a change of vector fiber charts.

Definition 5.1.2 The base $(d\tilde{z}^{\alpha},\delta\tilde{y}^{a})$ will be called the *adapted dual* (ρ,η)*-base.*

5.2 Remarkable Mod-endomorphisms

In the following we consider the diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \left(F,\left[,\right]_{F,h},\left(\rho,\eta\right)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E,\pi,M)\in|\mathbf{B}^{\mathbf{v}}|$ and $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$ is a generalized Lie algebroid.

Definition 5.2.1 For any **Mod**-endomorphism e of

$$\left(\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right),+,\cdot\right)$$

we define the application of Nijenhuis type

$$\Gamma((\rho,\eta)TE,(\rho,\eta)\tau_{E},E)^{2}\ \xrightarrow{N_{e}}\ \Gamma((\rho,\eta)TE,(\rho,\eta)\tau_{E},E)$$

defined by

$$N_{e}\left(X,Y\right)=\left[eX,eY\right]_{\rho TE}+e^{2}\left[X,Y\right]_{\rho TE}-e\left[eX,Y\right]_{\rho TE}-e\left[X,eY\right]_{\rho TE},$$

for any $X,Y\in\Gamma((\rho,\eta)TE,(\rho,\eta)\tau_{E},E)$.

Remark 5.2.1 The vertical and the horizontal vector subbundles are interior differential systems for the Lie algebroid generalized tangent bundle

$$\left(((\rho,\eta)TE,(\rho,\eta)\tau_{E},E),\left[,\right]_{(\rho,\eta)TE},(\tilde{\rho},Id_{E})\right).$$

These interior differential systems will be called *vertical* and *horizontal interior differential systems.*

5.2.1 Projectors

Definition 5.2.1.1 Any **Mod**-endomorphism e of

$$\Gamma\left((\rho,\eta)TE,(\rho,\eta\ )\tau_E,E\right)$$

with the property

$$e^2=e \tag{5.2.1.1}$$

will be called *projector.*

Example 5.2.1.1 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right) & \stackrel{\mathcal{V}}{\longrightarrow} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\\ \tilde{Z}^\alpha\tilde{\delta}_\alpha+Y^a\dot{\tilde{\partial}}_a & \longmapsto & Y^a\dot{\tilde{\partial}}_a\end{array}$$

is a projector which will be called *the vertical projector.*

Remark 5.2.1.1 We have $\mathcal{V}\left(\tilde{\delta}_\alpha\right)=0$ and $\mathcal{V}\left(\dot{\tilde{\partial}}_a\right)=\dot{\tilde{\partial}}_a$. Therefore, it follows

$$\mathcal{V}\left(\tilde{\partial}_\alpha\right)=(\rho,\eta)\,\Gamma^a_\alpha\dot{\tilde{\partial}}_a.$$

Theorem 5.2.1.1 *A (ρ,η)-connection for the vector bundle (E,π,M) is characterized by the existence of a* **Mod**-*endomorphism $\mathcal{V}$ of*

$$\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\mathcal{V}\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\right)\subset\Gamma\left(V\left(\rho,\eta\right)TE,(\rho,\eta)\,\tau_E,E\right)\\ \mathcal{V}\left(X\right)=X\Longleftrightarrow X\in\Gamma\left(V\left(\rho,\eta\right)TE,(\rho,\eta)\,\tau_E,E\right)\end{array} \tag{5.2.1.2}$$

Example 5.2.1.2 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right) & \stackrel{\mathcal{H}}{\longrightarrow} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\\ \tilde{Z}^\alpha\tilde{\delta}_\alpha+Y^a\dot{\tilde{\partial}}_a & \longmapsto & \tilde{Z}^\alpha\tilde{\delta}_\alpha\end{array}$$

is a projector which will be called the *horizontal projector.*

Remark 5.2.1.2 We have $\mathcal{H}\left(\tilde{\delta}_\alpha\right)$=$\tilde{\delta}_\alpha$ and $\mathcal{H}(\dot{\tilde{\partial}}_a)$=0. Therefore, we obtain $\mathcal{H}\left(\tilde{\partial}_\alpha\right)$=$\tilde{\delta}_\alpha$.

Theorem 5.2.1.2 *A (ρ,η)-connection for the vector bundle (E,π,M) is characterized by the existence of a* **Mod**-*endomorphism $\mathcal{H}$ of*

$$\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\mathcal{H}\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\right)\subset\Gamma\left(H\left(\rho,\eta\right)TE,(\rho,\eta)\,\tau_E,E\right)\\ \mathcal{H}\left(X\right)=X\Longleftrightarrow X\in\Gamma\left(H\left(\rho,\eta\right)TE,(\rho,\eta)\,\tau_E,E\right).\end{array} \tag{5.2.1.3}$$

Corollary 5.2.1.1 *A (ρ,η)-connection for the vector bundle (E,π,M) is characterized by the existence of a* **Mod**-*endomorphism $\mathcal{H}$ of*

$$\left(\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\mathcal{H}^{2}=\mathcal{H}\\ Ker\left(\mathcal{H}\right)=\left(\Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right),+,\cdot\right).\end{array}\tag{5.2.1.4}$$

Remark 5.2.1.3 For any

$$X\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$$

we obtain the following unique decomposition

$$X=\mathcal{H}X+\mathcal{V}X.$$

Proposition 5.2.1.1 *After some calculations we obtain*

$$N_{\mathcal{V}}\left(X,Y\right)=\mathcal{V}\left[\mathcal{H}X,\mathcal{H}Y\right]_{\left(\rho,\eta\right)TE}=N_{\mathcal{H}}\left(X,Y\right),\tag{5.2.1.5}$$

for any $X,Y\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$.

Corollary 5.2.1.2 *The horizontal interior differential system*

$$\left(H\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$$

is involutive if and only if $N_{\mathcal{V}}=0$ or $N_{\mathcal{H}}=0$.

5.2.2 The almost product structure

Definition 5.2.2.1 Any **Mod**-endomorphism e of

$$\left(\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right),+,\cdot\right)$$

with the property

$$e^{2}=Id\tag{5.2.2.1}$$

will be called the *almost product structure.*

Example 5.2.2.1 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right) & \xrightarrow{\mathcal{P}} & \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)\\ \tilde{Z}^{\alpha}\tilde{\delta}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a} & \longmapsto & \tilde{Z}^{\alpha}\tilde{\delta}_{\alpha}-Y^{a}\dot{\tilde{\partial}}_{a}\end{array}$$

is an almost product structure.

Remark 5.2.2.1 The previous almost product structure has the properties:

$$\begin{array}{l}\mathcal{P}=2\mathcal{H}-Id;\\ \mathcal{P}=Id-2\mathcal{V};\\ \mathcal{P}=\mathcal{H}-\mathcal{V}.\end{array}\tag{5.2.2.2}$$

Remark 5.2.2.2 We obtain that $\mathcal{P}\left(\tilde{\delta}_\alpha\right) = \tilde{\delta}_\alpha$ and $\mathcal{P}\left(\dot{\tilde{\partial}}_a\right) = -\dot{\tilde{\partial}}_a$. Therefore, it follows
$$\mathcal{P}\left(\tilde{\partial}_\alpha\right) = \tilde{\delta}_\alpha - \rho\Gamma^a_\alpha \dot{\tilde{\partial}}_a.$$

Theorem 5.2.2.1 *A (ρ, η)-connection for the vector bundle (E, π, M) is characterized by the existence of a* **Mod***-endomorphism $\mathcal{P}$ of*
$$\left(\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),+,\cdot\right)$$
with the following property:
$$\mathcal{P}\left(X\right) = -X \Longleftrightarrow X \in \Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right). \tag{5.2.2.3}$$

Proposition 5.2.2.1 *After some calculations, we obtain*
$$N_{\mathcal{P}}\left(X,Y\right) = 4\mathcal{V}\left[\mathcal{H}X,\mathcal{H}Y\right],$$
for any $X, Y \in \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$.

Corollary 5.2.2.1 *The horizontal interior differential system*
$$\left(H\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$$
is involutive if and only if $N_{\mathcal{P}} = 0$.

5.2.3 The almost tangent structure

Definition 5.2.3.1 Any **Mod**-endomorphism e of
$$(\Gamma((\rho,\eta)TE,(\rho,\eta)\tau_E,E),+,\cdot)$$
with the property
$$e^2 = 0 \tag{5.2.3.1}$$
will be called the *almost tangent structure.*

Example 5.2.3.1 If $(E,\pi,M) = (F,\nu,N)$, $g \in \mathbf{Man}\left(E,E\right)$ such that (g,h) is a $\mathbf{B^v}$-morphism locally invertible, then the **Mod**-endomorphism
$$\begin{array}{ccc} \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right) & \xrightarrow{\mathcal{J}_{(g,h)}} & \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right) \\ \tilde{Z}^a\tilde{\partial}_a + \tilde{Y}^b\dot{\tilde{\partial}}_b & \longmapsto & \left(\tilde{g}^b_a \circ h \circ \pi\right)\tilde{Z}^a\dot{\tilde{\partial}}_b \end{array}$$
is an almost tangent structure which will be called the *almost tangent structure associated to* $\mathbf{B^v}$*-morphism* (g,h). (See: **Definition 4.4.1.3)**

Remark 5.2.3.1 We obtain that
$$\mathcal{J}_{(g,h)}\left(\tilde{\delta}_a\right) = \mathcal{J}_{(g,h)}\left(\tilde{\partial}_a\right) = \left(\tilde{g}^b_a \circ h \circ \pi\right)\dot{\tilde{\partial}}_b \text{ and } \mathcal{J}_{(g,h)}\left(\dot{\tilde{\partial}}_b\right) = 0.$$

Remark 5.2.3.2 The previous almost tangent structure has the following properties:

$$(5.2.3.2)\qquad \begin{array}{rcl} \mathcal{J}_{(g,h)}\circ\mathcal{P} & = & \mathcal{J}_{(g,h)};\\ \mathcal{P}\circ\mathcal{J}_{(g,h)} & = & -\mathcal{J}_{(g,h)};\\ \mathcal{J}_{(g,h)}\circ\mathcal{H} & = & \mathcal{J}_{(g,h)};\\ \mathcal{H}\circ\mathcal{J}_{(g,h)} & = & 0;\\ \mathcal{J}_{(g,h)}\circ\mathcal{V} & = & 0;\\ \mathcal{V}\circ\mathcal{J}_{(g,h)} & = & \mathcal{J}_{(g,h)};\\ N_{\mathcal{J}_{(g,h)}} & = & 0. \end{array}$$

5.2.4 The almost complex structure

Let us consider in the case $(E,\pi,M)=(F,\nu,N)$.

Definition 5.2.4.1 Any **Mod**-endomorphism e of

$$(\Gamma((\rho,\eta)\,TE,(\rho,\eta)\tau_E,E),+,\cdot)$$

with the property

$$(5.3.4.1)\qquad e^2=-Id$$

will be called the *almost complex structure*.

Example 5.2.4.1 If (g,h) is a $\mathbf{B^v}$-morphism of (E,π,M) source and target locally invertible, then the **Mod**-endomorphism

$$\begin{array}{ccc} \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right) & \xrightarrow{\mathcal{F}\ \ (g,h)} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\\ \tilde{Z}^a\tilde{\delta}_a+Y^b\dot{\tilde{\partial}}_b & \longmapsto & \left(g_b^a\circ h\circ\pi\right)Y^b\tilde{\delta}_a-\left(\tilde{g}_a^b\circ h\circ\pi\right)\tilde{Z}^a\dot{\tilde{\partial}}_b \end{array}$$

is an almost complex structure.

Remark 5.2.4.1 We have

$$\mathcal{F}_{(g,h)}\left(\tilde{\delta}_a\right)=-\left(\tilde{g}_a^b\circ h\circ\pi\right)\dot{\tilde{\partial}}_b$$

and

$$\mathcal{F}_{(g,h)}\left(\dot{\tilde{\partial}}_b\right)=\left(g_b^a\circ h\circ\pi\right)\tilde{\delta}_a.$$

Therefore, we obtain:

$$\mathcal{F}_{(g,h)}\left(\tilde{\partial}_c\right)=-\left(\rho,\eta\right)\Gamma_c^a\left(g_b^a\circ h\circ\pi\right)\tilde{\delta}_a-\left(\tilde{g}_c^b\circ h\circ\pi\right)\dot{\tilde{\partial}}_b.$$

Remark 5.2.4.2 The previous almost complex structure has the following properties:

$$(5.2.4.2)\qquad \begin{array}{rcl} \mathcal{F}_{(g,h)}\circ\mathcal{J}_{(g,h)} & = & \mathcal{H};\\ \mathcal{F}_{(g,h)}\circ\mathcal{H} & = & -\mathcal{J}_{(g,h)};\\ \mathcal{J}_{(g,h)}\circ\mathcal{F}_{(g,h)} & = & \mathcal{V}. \end{array}$$

5.2.5 The (ρ,η)-tension endomorphism

Since
$$\frac{\partial\left(\rho,\eta\right)\Gamma_{\acute{\alpha}}^{\acute{a}}}{\partial y^{\acute{b}}}=M_{a}^{\acute{a}}\left(\rho_{\alpha}^{i}\frac{\partial M_{\acute{b}}^{a}}{\partial x^{i}}+\frac{\partial\left(\rho,\eta\right)\Gamma_{c}^{a}}{\partial y^{b}}M_{\acute{b}}^{b}\right)\Lambda_{\acute{\alpha}}^{\alpha},$$
it results that
$$\left(\rho,\eta\right)\Gamma_{\acute{\alpha}}^{\acute{a}}-y^{\acute{b}}\frac{\partial\left(\rho,\eta\right)\Gamma_{\acute{\alpha}}^{\acute{a}}}{\partial y^{\acute{b}}}=M_{a}^{\acute{a}}\left(\left(\rho,\eta\right)\Gamma_{\alpha}^{a}-y^{b}\frac{\partial\left(\rho,\eta\right)\Gamma_{\alpha}^{a}}{\partial y^{b}}\right)\Lambda_{\acute{\alpha}}^{\alpha},$$
Therefore, we can introduce the following

Definition 5.2.5.1 The **Mod**-endomorphism
$$\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)\ \xrightarrow{(\rho,\eta)\mathbb{H}}\ \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$$
defined by
$$(5.2.5.1)\qquad \begin{array}{rcl}\left(\rho,\eta\right)\mathbb{H}\left(\tilde{\delta}_{\alpha}\right)&=&\left(\left(\rho,\eta\right)\Gamma_{\alpha}^{a}-y^{b}\dfrac{\partial\left(\rho,\eta\right)\Gamma_{\alpha}^{a}}{\partial y^{b}}\right)\dot{\tilde{\partial}}_{a},\\ \left(\rho,\eta\right)\mathbb{H}\left(\dot{\tilde{\partial}}_{a}\right)&=&0_{(\rho,\eta)TE}\end{array}$$
will be called the (ρ,η)*-tension of* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$.

In particular, if $h=Id_{M}$ and $(\rho,\eta)=(Id_{TM},Id_{M})$, then we obtain the *tension of connection* Γ.

Proposition 5.2.5.1 *We obtain the following equalities*
$$\mathcal{J}_{(Id_{E},Id_{M})}\circ\left(\rho,\eta\right)\mathbb{H}=0=\left(\rho,\eta\right)\mathbb{H}\circ\mathcal{J}_{(Id_{E},Id_{M})}.$$

5.3 The (ρ,η,h)-torsion and the (ρ,η,h)-curvature of a (ρ,η)-connection

We consider the following diagram:
$$\begin{tikzcd} E \arrow[d, "\pi"'] & \left(F,[,]_{F,h},(\rho,\eta)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$
where $(E,\pi,M)\in|\mathbf{B}^{\mathbf{v}}|$ and $\left(\left(F,\nu,N\right),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Definition 5.3.1 If $(E,\pi,M)=(F,\nu,N)$, then the $\mathcal{F}(E)$-bilinear application
$$\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)^{2}\ \xrightarrow{(\rho,\eta,h)\mathbb{T}}\ \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$$
defined by
$$(5.3.1)\qquad \begin{array}{rl}\left(\rho,\eta,h\right)\mathbb{T}\left(\tilde{\delta}_{b},\tilde{\delta}_{c}\right)&=\left(\dfrac{\partial\left(\rho,\eta\right)\Gamma_{c}^{a}}{\partial y^{b}}-\dfrac{\partial\left(\rho,\eta\right)\Gamma_{b}^{a}}{\partial y^{c}}-L_{bc}^{a}\circ h\circ\pi\right)\dot{\tilde{\partial}}_{a};\\ \left(\rho,\eta,h\right)\mathbb{T}\left(\tilde{\delta}_{b},\dot{\tilde{\partial}}_{c}\right)&=0=\left(\rho,\eta,h\right)\mathbb{T}\left(\dot{\tilde{\partial}}_{c},\tilde{\delta}_{b}\right);\\ \left(\rho,\eta,h\right)\mathbb{T}\left(\dot{\tilde{\partial}}_{b},\dot{\tilde{\partial}}_{c}\right)&=0;\end{array}$$

will be called the (ρ,η,h)-*torsion associated to* (ρ,η)-*connection* $(\rho,\eta)\,\Gamma$.

In particular, if $h = Id_M$, then we obtain the (ρ,η)-*torsion associated to* (ρ,η)-*connection* $(\rho,\eta)\,\Gamma$.

Moreover, if $(\rho,\eta) = (Id_{TM}, Id_M)$, then we obtain the *torsion associated to connection* Γ.

Remark 5.3.1 If $(\rho,\eta,h)\,\mathbb{T}$ is the (ρ,η,h)-torsion associated to (ρ,η)-connection $(\rho,\eta)\,\Gamma$, then

$$(\rho,\eta,h)\,\mathbb{T}(X,Y) = -\,(\rho,\eta,h)\,\mathbb{T}(Y,X), \tag{5.3.2}$$

for any $X,Y \in \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$.

Definition 5.3.2 If we consider the notation

$$(\rho,\eta,h)\,\mathbb{T}^a{}_{bc} \overset{put}{=} \frac{\partial\,(\rho,\eta)\,\Gamma^a_c}{\partial y^b} - \frac{\partial\,(\rho,\eta)\,\Gamma^a_b}{\partial y^c} - L^a_{bc}\circ h\circ\pi \tag{5.3.3}$$

then the tensor field

$$(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}\frac{\delta}{\delta\tilde{z}^a}\otimes d\tilde{z}^b\otimes d\tilde{z}^c \tag{5.3.4}$$

will be called the (ρ,η,h)-*torsion tensor field associated to* (ρ,η)-*connection* $(\rho,\eta)\,\Gamma$.

Proposition 5.3.1 *We obtain*

$$\mathcal{J}_{(Id_E,Id_M)}\circ(\rho,\eta)\,\mathbb{T} = 0$$

and

$$\begin{array}{rcl}(\rho,\eta,h)\,\mathbb{T}\left(\mathcal{J}_{(Id_E,Id_M)}X,Y\right) & = & (\rho,\eta)\,\mathbb{T}\left(\mathcal{J}_{(Id_E,Id_M)}X,\mathcal{J}_{(Id_E,Id_M)}Y\right)\\ & = & (\rho,\eta)\,\mathbb{T}\left(X,\mathcal{J}_{(Id_E,Id_M)}Y\right),\end{array}$$

for any $X,Y \in \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$.

Theorem 5.3.1 *Using the* (ρ,η)-*tension tensor field*

$$(\rho,\eta)\,\mathbb{H}^a_b\frac{\partial}{\partial\tilde{y}^a}\otimes d\tilde{z}^b = \left((\rho,\eta)\,\Gamma^a_b - y^c\frac{\partial\,(\rho,\eta)\,\Gamma^a_b}{\partial y^c}\right)\frac{\partial}{\partial\tilde{y}^a}\otimes d\tilde{z}^b, \tag{5.3.5}$$

and the (ρ,η,h)-*deflection of the* (ρ,η)-*connection* $(\rho,\eta)\,\Gamma$

$$(\rho,\eta,h)\,\mathbb{D}^a_b = -\,(\rho,\eta)\,\Gamma^a_b + y^c\frac{\partial\,(\rho,\eta)\,\Gamma^a_c}{\partial y^b} - y^c L^a_{bc}\circ h\circ\pi, \tag{5.3.6}$$

we obtain that $(\rho,\eta,h)\mathbb{D}^a_b{=}0$ *if and only if* $(\rho,\eta)\mathbb{H}^a_b{=}0$ *and* $(\rho,\eta,h)\mathbb{T}^a{}_{bc}{=}0$.

Proof. If $(\rho,\eta,h)\,\mathbb{D}^a_b{=}0$, then deriving with respect to y^c, we obtain:

$$-\frac{\partial\,(\rho,\eta)\,\Gamma^a_b}{\partial y^c} + \frac{\partial\,(\rho,\eta)\,\Gamma^a_c}{\partial y^b} - L^a_{bc}\circ h\circ\pi = 0 \Longleftrightarrow (\rho,\eta,h)\,\mathbb{T}^a{}_{bc} = 0.$$

The equality $(\rho,\eta,h)\,\mathbb{D}_b^a$=0 implies:

$$(\rho,\eta)\,\Gamma_b^a = y^c\frac{\partial\,(\rho,\eta)\,\Gamma_c^a}{\partial y^b} - y^c L_{bc}^a\circ h\circ\pi. \tag{1}$$

Since

$$\begin{array}{ll}(\rho,\eta)\,\mathbb{H}_b^a & = (\rho,\eta)\,\Gamma_b^a - y^c\dfrac{\partial\,(\rho,\eta)\,\Gamma_b^a}{\partial y^c} = \\ & = y^c\dfrac{\partial\,(\rho,\eta)\,\Gamma_c^a}{\partial y^b} - y^c L_{bc}^a\circ h\circ\pi - y^c\dfrac{\partial\,(\rho,\eta)\,\Gamma_b^a}{\partial y^c} = y^c\,(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}\end{array}$$

it results the equality $(\rho,\eta)\,\mathbb{H}_b^a$=0.

Conservely, if $(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}$=0, then, multiplying with y^c, we obtain:

$$\frac{\partial\,(\rho,\eta)\,\Gamma_c^a}{\partial y^b}y^c - \frac{\partial\,(\rho,\eta)\,\Gamma_b^a}{\partial y^c}y^c - y^c L_{bc}^a\circ h\circ\pi = 0. \tag{2}$$

The equality $(\rho,\eta)\,\mathbb{H}_b^a$=0 is equivalent with:

$$(\rho,\eta)\,\Gamma_b^a = y^c\frac{\partial\,(\rho,\eta)\,\Gamma_b^a}{\partial y^c}. \tag{3}$$

Using (2) and (3), it results the equality $(\rho,\eta,h)\,\mathbb{D}_b^a = 0$. *q.e.d.*

Definition 5.3.3 The $\mathcal{F}(E)$-bilinear application

$$\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)^2 \xrightarrow{(\rho,\eta,h)\mathbb{R}} \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$$

defined by

$$\begin{array}{l}(\rho,\eta,h)\,\mathbb{R}\left(\tilde{\delta}_\alpha,\tilde{\delta}_\beta\right) = (\rho,\eta,h)\,\mathbb{R}^a{}_{\alpha\beta}\dot{\tilde{\partial}}_a;\\ (\rho,\eta,h)\,\mathbb{R}\left(\tilde{\delta}_\alpha,\dot{\tilde{\partial}}_b\right) = 0 = (\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_b,\tilde{\delta}_\alpha\right);\\ (\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_a,\dot{\tilde{\partial}}_b\right) = 0;\end{array} \tag{5.3.7}$$

will be called the (ρ,η,h)-*curvature associated to* (ρ,η)-*connection* $(\rho,\eta)\,\Gamma$.

In particular, if $h = Id_M$, then we obtain the (ρ,η)-*curvature associated to* (ρ,η)-*connection* $(\rho,\eta)\,\Gamma$.

Moreover, if $(\rho,\eta) = (Id_{TM}, Id_M)$, then we obtain the *curvature associated to connection* Γ.

Remark 5.3.2 If $(\rho,\eta,h)\,\mathbb{R}$ is the (ρ,η,h)-curvature associated to (ρ,η)-connection $(\rho,\eta)\,\Gamma$, then

$$(\rho,\eta,h)\,\mathbb{R}\,(X,Y) = -\,(\rho,\eta,h)\,\mathbb{R}\,(Y,X)\,, \tag{5.3.8}$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$.

Definition 5.3.4 The tensor field

$$(\rho,\eta,h)\,\mathbb{R}^{a}{}_{\alpha\beta}\frac{\partial}{\partial\tilde{y}^{a}}\otimes d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta} \tag{5.3.9}$$

will be called the *(ρ,η,h)-curvature tensor field associated to the (ρ,η)-connection $(\rho,\eta)\,\Gamma$.*

Using equality (5.1.5), we obtain

Remark 5.3.3 The horizontal interior differential system $(H\,(\rho,\eta)\,TE,(\rho,\eta)\,\tau_{E},E)$ is involutive if and only if the (ρ,η,h)-curvature tensor field associated to the (ρ,η)-connection $(\rho,\eta)\,\Gamma$ is null.

Theorem 5.3.2 *If $\mathcal{F}$ is the almost complex structure presented in Example 5.2.4.1, then $(\rho,\eta,h)\,\mathbb{T}=0$ and $(\rho,\eta,h)\,\mathbb{R}=0$ if and only if $N_{\mathcal{F}}=0$.*

Proof. After some calculations, we obtain the relations:

$$\begin{aligned}
N_{\mathcal{F}}\left(\tilde{\delta}_{b},\tilde{\delta}_{c}\right)&=(\rho,\eta,h)\,\mathbb{T}^{a}{}_{bc}\tilde{\delta}_{a}-(\rho,\eta,h)\,\mathbb{R}^{a}{}_{bc}\dot{\tilde{\partial}}_{a},\\
N_{\mathcal{F}}\left(\tilde{\delta}_{b},\dot{\tilde{\partial}}_{c}\right)&=(\rho,\eta,h)\,\mathbb{R}^{a}{}_{bc}\tilde{\delta}_{a}-(\rho,\eta,h)\,\mathbb{T}^{a}{}_{bc}\dot{\tilde{\partial}}_{a},\\
N_{\mathcal{F}}\left(\dot{\tilde{\partial}}_{b},\dot{\tilde{\partial}}_{c}\right)&=-\,(\rho,\eta,h)\,\mathbb{T}^{a}{}_{bc}\tilde{\delta}_{a}-(\rho,\eta,h)\,\mathbb{R}^{a}{}_{bc}\dot{\tilde{\partial}}_{a}.
\end{aligned}$$

Obviously, $(\rho,\eta,h)\,\mathbb{T}=0$ and $(\rho,\eta,h)\,\mathbb{R}=0$ imply $N_{\mathcal{F}}=0$.

Conservely, if $N_{\mathcal{F}}=0$, then we have the equalities:

$$\begin{aligned}
(\rho,\eta,h)\,\mathbb{T}^{a}{}_{bc}\tilde{\delta}_{a}-(\rho,\eta,h)\,\mathbb{R}^{a}{}_{bc}\dot{\tilde{\partial}}_{a}&=0,\\
(\rho,\eta,h)\,\mathbb{R}^{a}{}_{bc}\tilde{\delta}_{a}-(\rho,\eta,h)\,\mathbb{T}^{a}{}_{bc}\dot{\tilde{\partial}}_{a}&=0,\\
-\,(\rho,\eta,h)\,\mathbb{T}^{a}{}_{bc}\tilde{\delta}_{a}-(\rho,\eta,h)\,\mathbb{R}^{a}{}_{bc}\dot{\tilde{\partial}}_{a}&=0.
\end{aligned}$$

q.e.d.

5.4 Tensor d-fields. Distinguished linear (ρ,η)-connections

We consider the following diagram:

$$\begin{tikzcd}
E \arrow[d, "\pi"'] & \left(F,[,]_{F,h},(\rho,\eta)\right) \arrow[d, "\nu"] \\
M \arrow[r, "h"] & N
\end{tikzcd}$$

where $(E,\pi,M)\in|\mathbf{B}^{\mathbf{v}}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let

$$\left(\mathcal{T}^{\,p,r}_{\,q,s}\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_{E},E\right),+,\cdot\right)$$

be the $\mathcal{F}(E)$-module of tensor fields by $\binom{p,r}{q,s}$-type from the generalized tangent bundle

$$\left(H\,(\rho,\eta)\,TE\oplus V\,(\rho,\eta)\,TE,(\rho,\eta)\,\tau_{E},E\right).$$

An arbitrarily tensor field T is written as

$$T = T^{\alpha_1 \ldots \alpha_p a_1 \ldots a_r}_{\beta_1 \ldots \beta_q b_1 \ldots b_s} \tilde{\delta}_{\alpha_1} \otimes \ldots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \ldots \otimes d\tilde{z}^{\beta_q} \otimes$$
$$\dot{\tilde{\partial}}_{a_1} \otimes \ldots \otimes \dot{\tilde{\partial}}_{a_r} \otimes \delta\tilde{y}^{b_1} \otimes \ldots \otimes \delta\tilde{y}^{b_s}.$$

Let

$$(\mathcal{T} \ \left((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E \right), +, \cdot, \otimes)$$

be the tensor fields algebra of generalized tangent bundle $((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E)$.

If $T_1 \in \mathcal{T}^{p_1, r_1}_{q_1, s_1}((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$ and $T_2 \in \mathcal{T}^{p_2, r_2}_{q_2, s_2}((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$, then the components of product tensor field $T_1 \otimes T_2$ are the products of local components of T_1 and T_2.

Therefore, we obtain $T_1 \otimes T_2 \in \mathcal{T}^{p_1+p_2, r_1+r_2}_{q_1+q_2, s_1+s_2}\left((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E\right).$

Let $\mathcal{DT}((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$ be the family of tensor fields

$$T \in \mathcal{T}((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$$

for which there exists

$$T_1 \in \mathcal{T}^{p,0}_{q,0}((\rho,\eta)TE, (\rho,\eta)\tau_E, E) \text{ and } T_2 \in \mathcal{T}^{0,r}_{0,s}((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$$

such that $T = T_1 + T_2$.

The $\mathcal{F}(E)$-module $(\mathcal{DT}\left((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E\right), +, \cdot)$ will be called the *module of distinguished tensor fields* or the *module of tensor d-fields.*

Remark 5.4.1 The elements of

$$\Gamma\left((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E\right)$$

respectively

$$\Gamma(((\rho,\eta)TE)^*, ((\rho,\eta)\tau_E)^*, E)$$

are tensor d-fields.

Definition 5.4.1 Let (E, π, M) be a vector bundle endowed with a (ρ,η)-connection $(\rho,\eta)\,\Gamma$ and let

$$(X, T) \xrightarrow{(\rho,\eta)D} (\rho,\eta)\, D_X T \tag{5.4.1}$$

be a covariant (ρ,η)-derivative for the tensor algebra of generalized tangent bundle

$$((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E)$$

which preserves the horizontal and vertical IDS by parallelism.

The real local functions

$$\left((\rho,\eta)\, H^{\alpha}_{\beta\gamma}, (\rho,\eta)\, H^{a}_{b\gamma}, (\rho,\eta)\, V^{\alpha}_{\beta c}, (\rho,\eta)\, V^{a}_{bc}\right)$$

defined by the following equalities:

$$\begin{array}{ll} (\rho,\eta)\, D_{\tilde{\delta}_\gamma} \tilde{\delta}_\beta = (\rho,\eta)\, H^{\alpha}_{\beta\gamma} \tilde{\delta}_\alpha, & (\rho,\eta)\, D_{\tilde{\delta}_\gamma} \dot{\tilde{\partial}}_b = (\rho,\eta)\, H^{a}_{b\gamma} \dot{\tilde{\partial}}_a \\ (\rho,\eta)\, D_{\dot{\tilde{\partial}}_c} \tilde{\delta}_\beta = (\rho,\eta)\, V^{\alpha}_{\beta c} \tilde{\delta}_\alpha, & (\rho,\eta)\, D_{\dot{\tilde{\partial}}_c} \dot{\tilde{\partial}}_b = (\rho,\eta)\, V^{a}_{bc} \dot{\tilde{\partial}}_a \end{array} \tag{5.4.2}$$

are the components of a linear (ρ,η)-connection

$$\left((\rho,\eta)\,H,(\rho,\eta)\,V\right)$$

for the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ which will be called the *distinguished linear (ρ,η)-connection.*

If $h=Id_M$, then the distinguished linear (Id_{TM},Id_M)-connection will be called the *distinguished linear connection.*

The components of a distinguished linear connection (H,V) will be denoted

$$\left(H_{jk}^i,H_{bk}^a,V_{jc}^i,V_{bc}^a\right).$$

Theorem 5.4.1 *If $((\rho,\eta)H,(\rho,\eta)V)$ is a distinguished linear (ρ,η)- connection for the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$, then its components satisfy the change relations:*

$$\begin{aligned}(\rho,\eta)\,H_{\beta'\gamma'}^{\alpha'}&=\Lambda_{\alpha}^{\alpha'}\circ h\circ\pi\cdot\Big[\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_{\gamma}\right)\left(\Lambda_{\beta'}^{\alpha}\circ h\circ\pi\right)+\\&\quad+(\rho,\eta)\,H_{\beta\gamma}^{\alpha}\cdot\Lambda_{\beta'}^{\beta}\circ h\circ\pi\Big]\cdot\Lambda_{\gamma'}^{\gamma}\circ h\circ\pi,\\(\rho,\eta)\,H_{b'\gamma'}^{a'}&=M_a^{a'}\circ\pi\cdot\Big[\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_{\gamma}\right)\left(M_{b'}^{a}\circ\pi\right)+\\&\quad+(\rho,\eta)\,H_{b\gamma}^{a}\cdot M_{b'}^{b}\circ\pi\Big]\cdot\Lambda_{\gamma'}^{\gamma}\circ h\circ\pi,\\(\rho,\eta)\,V_{\beta'c'}^{\alpha'}&=\Lambda_{\alpha}^{\alpha'}\circ h\circ\pi\cdot(\rho,\eta)\,V_{\beta c}^{\alpha}\cdot\Lambda_{\beta'}^{\beta}\circ h\circ\pi\cdot M_{c'}^{c}\circ\pi,\\(\rho,\eta)\,V_{b'c'}^{a'}&=M_a^{a'}\circ\pi\cdot(\rho,\eta)\,V_{bc}^{a}\cdot M_{b'}^{b}\circ\pi\cdot M_{c'}^{c}\circ\pi.\end{aligned}\tag{5.4.3}$$

The components of a distinguished linear connection (H,V) verify the change relations:

$$\begin{aligned}H_{j'k'}^{i'}&=\tfrac{\partial x^{i'}}{\partial x^i}\circ\pi\cdot\left[\frac{\delta}{\delta x^k}\left(\frac{\partial x^i}{\partial x^{j'}}\circ\pi\right)+H_{jk}^i\cdot\frac{\partial x^j}{\partial x^{j'}}\circ\pi\right]\cdot\frac{\partial x^k}{\partial x^{k'}}\circ\pi,\\H_{b'k'}^{a'}&=M_a^{a'}\circ\pi\cdot\left[\frac{\delta}{\delta x^k}\left(M_{b'}^a\circ\pi\right)+H_{bk}^a\cdot M_{b'}^b\circ\pi\right]\cdot\frac{\partial x^k}{\partial x^{k'}}\circ\pi,\\V_{j'c'}^{i'}&=\frac{\partial x^{i'}}{\partial x^i}\circ\pi\cdot V_{jc}^i\frac{\partial x^j}{\partial x^{j'}}\circ\pi\cdot M_{c'}^c\circ\pi,\\V_{b'c'}^{a'}&=M_a^{a'}\circ\pi\cdot V_{bc}^a\cdot M_{b'}^b\circ\pi\cdot M_{c'}^c\circ\pi.\end{aligned}\tag{5.4.3'}$$

Example 5.4.1 If (E,π,M) is a vector bundle endowed with the (ρ,η)-connection $(\rho,\eta)\,\Gamma$, then the local real functions

$$\left(\frac{\partial\left(\rho,\eta\right)\Gamma_{\gamma}^{a}}{\partial y^b},\frac{\partial\left(\rho,\eta\right)\Gamma_{\gamma}^{a}}{\partial y^b},0,0\right)$$

are the components of a distinguished linear (ρ,η)-connection for $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$, which will by called the *Berwald linear (ρ,η)-connection.*

The Berwald linear (Id_{TM},Id_M)-connection will be called the *Berwald linear connection.*

Theorem 5.4.2 *If the generalized tangent bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *is endowed with a distinguished linear* (ρ,η)*-connection* $((\rho,\eta)H,(\rho,\eta)V)$*, then, for any*

$$X=\tilde{Z}^{\alpha}\tilde{\delta}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a}\in\Gamma((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$$

and for any

$$T\in\mathcal{T}\,{}^{pr}_{qs}((\rho,\eta)TE,(\rho,\eta)\tau_E,E),$$

we obtain the formula:

$$\begin{aligned}&(\rho,\eta)\,D_X\Big(T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\tilde{\delta}_{\alpha_1}\otimes\ldots\otimes\tilde{\delta}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes\\ &\qquad\otimes d\tilde{z}^{\beta_q}\otimes\dot{\tilde{\partial}}_{a_1}\otimes\ldots\otimes\dot{\tilde{\partial}}_{a_r}\otimes\delta\tilde{y}^{b_1}\otimes\ldots\otimes\delta\tilde{y}^{b_s}\Big)=\\ &\qquad=\tilde{Z}^{\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s|\gamma}\tilde{\delta}_{\alpha_1}\otimes\ldots\otimes\tilde{\delta}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes d\tilde{z}^{\beta_q}\otimes\dot{\tilde{\partial}}_{a_1}\otimes\ldots\otimes\\ &\qquad\otimes\dot{\tilde{\partial}}_{a_r}\otimes\delta\tilde{y}^{b_1}\otimes\ldots\otimes\delta\tilde{y}^{b_s}+Y^{c}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\mid_c\tilde{\delta}_{\alpha_1}\otimes\ldots\otimes\\ &\qquad\otimes\tilde{\delta}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes d\tilde{z}^{\beta_q}\otimes\dot{\tilde{\partial}}_{a_1}\otimes\ldots\otimes\dot{\tilde{\partial}}_{a_r}\otimes\delta\tilde{y}^{b_1}\otimes\ldots\otimes\delta\tilde{y}^{b_s},\end{aligned}$$

where

$$\begin{aligned}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s|\gamma}&=\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\gamma}\right)T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\\ &+(\rho,\eta)\,H^{\alpha_1}_{\alpha\gamma}T^{\alpha\alpha_2\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}+\ldots+(\rho,\eta)\,H^{\alpha_p}_{\alpha\gamma}T^{\alpha_1\ldots\alpha_{p-1}\alpha a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\\ &-(\rho,\eta)\,H^{\beta}_{\beta_1\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta\beta_2\ldots\beta_q b_1\ldots b_s}-\ldots-(\rho,\eta)\,H^{\beta}_{\beta_q\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_{q-1}\beta b_1\ldots b_s}\\ &+(\rho,\eta)\,H^{a_1}_{a\gamma}T^{\alpha_1\ldots\alpha_p a a_2\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}+\ldots+(\rho,\eta)\,H^{a_r}_{a\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_{r-1}a}_{\beta_1\ldots\beta_q b_1\ldots b_s}\\ &-(\rho,\eta)\,H^{b}_{b_1\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b b_2\ldots b_s}-\ldots-(\rho,\eta)\,H^{b}_{b_s\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_{s-1}b}\end{aligned}$$

and

$$\begin{aligned}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\mid_c&=\Gamma(\tilde{\rho},Id_E)\left(\dot{\tilde{\partial}}_{c}\right)T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}+\\ &+(\rho,\eta)\,V^{\alpha_1}_{\alpha c}T^{\alpha\alpha_2\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}+\ldots+(\rho,\eta)\,V^{\alpha_p}_{\alpha c}T^{\alpha_1\ldots\alpha_{p-1}\alpha a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}-\\ &-(\rho,\eta)\,V^{\beta}_{\beta_1 c}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta\beta_2\ldots\beta_q b_1\ldots b_s}-\ldots-(\rho,\eta)\,V^{\beta}_{\beta_q c}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_{q-1}\beta b_1\ldots b_s}+\\ &+(\rho,\eta)\,V^{a_1}_{ac}T^{\alpha_1\ldots\alpha_p a a_2\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}+\ldots+(\rho,\eta)\,V^{a_r}_{ac}T^{\alpha_1\ldots\alpha_p a_1\ldots a_{r-1}a}_{\beta_1\ldots\beta_q b_1\ldots b_s}-\\ &-(\rho,\eta)\,V^{b}_{b_1 c}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b b_2\ldots b_s}-\ldots-(\rho,\eta)\,V^{b}_{b_s c}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_{s-1}b}.\end{aligned}$$

Definition 5.4.2 We assume that $(E,\pi,M)=(F,\nu,N)$.

If $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M) and

$$\left((\rho,\eta)\,H^{a}_{bc},(\rho,\eta)\,\tilde{H}^{a}_{bc},(\rho,\eta)\,V^{a}_{bc},(\rho,\eta)\,\tilde{V}^{a}_{bc}\right)$$

are the components of a distinguished linear (ρ,η)-connection for the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ such that

$$(\rho,\eta)\,H^{a}_{bc}=(\rho,\eta)\,\tilde{H}^{a}_{bc}\text{ and }(\rho,\eta)\,V^{a}_{bc}=(\rho,\eta)\,\tilde{V}^{a}_{bc},$$

then we will say that *the generalized tangent bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *is endowed with a normal distinguished linear* (ρ,η)*-connection on components* $((\rho,\eta)H^a_{bc},(\rho,\eta)V^a_{bc})$.

The components of a normal distinguished linear (Id_{TM},Id_M)-connection (H,V) will be denoted $\left(H^i_{jk},V^i_{jk}\right)$.

5.5 The lift of accelerations for a differentiable curve

We consider the following diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & (F,[,]_{F,h},(\rho,\eta)) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd} \tag{5.5.1}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)\in|\mathbf{GLA}|$.

Let $(\rho,\eta)\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M).

We admit that $((\rho,\eta)H,(\rho,\eta)V)$ is a distinguished linear (ρ,η)-connection for the vector bundle $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$.

Let $g\in\mathbf{Man}(E,F)$ be such that (g,h) is a $\mathbf{B^v}$-morphism of (E,π,M) source and (F,ν,N) target.

Let

$$\begin{array}{ccc} I & \xrightarrow{\dot{c}} & E_{|\operatorname{Im}(\eta\circ h\circ c)} \\ t & \longmapsto & y^a(t)\,s_a(\eta\circ h\circ c(t)) \end{array} \tag{5.5.2}$$

be the (g,h)-lift of differentiable curve $I\ \xrightarrow{c}\ M$.

Definition 5.5.1 The differentiable curve

$$\begin{array}{ccc} I & \xrightarrow{\ddot{c}} & (\rho,\eta)TE_{|\operatorname{Im}\dot{c}} \\ t & \longmapsto & \left(g^\alpha_a\circ h\circ c(t)\cdot y^a(t)\right)\dfrac{\partial}{\partial\tilde{z}^\alpha}(\dot{c}(t))+\dfrac{dy^a(t)}{dt}\dfrac{\partial}{\partial\tilde{y}^a}(\dot{c}(t)) \end{array} \tag{5.5.3}$$

will be called the *differentiable* (g,h)*-lift of accelerations of the differentiable curve* c.

The section

$$\begin{array}{ccc} \operatorname{Im}(\dot{c}) & \xrightarrow{u(c,\dot{c},\ddot{c})} & (\rho,\eta)TE_{|\operatorname{Im}(\dot{c})} \\ \dot{c}(t) & \longmapsto & \left(g^\alpha_b\circ h\circ c(t)\cdot y^b(t)\right)\dfrac{\partial}{\partial\tilde{z}^\alpha}(\dot{c}(t))+\dfrac{dy^a(t)}{dt}\dfrac{\partial}{\partial\tilde{y}^a}(\dot{c}(t)) \end{array} \tag{5.5.4}$$

will be called the *canonical section associated to the triple* $(c,\dot{c},\ddot{c})$.

Remark 5.5.1 For any $t\in I$, we obtain:

$$\begin{array}{l} u(c,\dot{c},\ddot{c})(\dot{c}(t))=\left(g^\alpha_b\circ h\circ c(t)\,y^b(t)\right)\dfrac{\delta}{\delta\tilde{z}^\alpha}(\dot{c}(t))+\dfrac{dy^a(t)}{dt}\dfrac{\partial}{\partial\tilde{y}^a}(\dot{c}(t)) \\ \qquad+(\rho,\eta)\Gamma^a_\alpha\circ u(c,\dot{c})\circ\eta\circ h\circ c(t)\cdot\left(g^\alpha_b\circ h\circ c(t)\,y^b(t)\right)\dfrac{\partial}{\partial\tilde{y}^a}(\dot{c}(t)). \end{array} \tag{5.5.5}$$

We observe easily that $u(c,\dot{c},\ddot{c})(\dot{c}(t)) \in H(\rho,\eta)TE_{|\operatorname{Im}(\dot{c})}$ if and only if the components functions $\left(y^a,\ a\in\overline{1,n}\right)$ are solutions for the differentiable equations

$$\frac{du^a}{dt} + (\rho,\eta)\Gamma^a_\alpha \circ u(c,\dot{c})\circ\eta\circ h\circ c\cdot\left(g^\alpha_b\circ h\circ c\right)\cdot u^b = 0,\ a\in\overline{1,r}. \tag{5.5.6}$$

Remark 5.5.2 In particular, if $(\rho,\eta,h) = (Id_{TM}, Id_M, Id_M)$ and (g, Id_M) is locally invertible, then, using the (g, Id_M)-lift

$$\begin{array}{rcl} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \tilde{g}^i_j \dfrac{dc^j(t)}{dt}\dfrac{\partial}{\partial x^i}(c(t)), \end{array} \tag{5.5.7}$$

the differentiable (g, Id_M)-lift of accelerations for the differentiable curve c is

$$\begin{array}{rcl} I & \xrightarrow{\ddot{c}} & (Id_{TM}, Id_M)\,TTM_{|\operatorname{Im}(\dot{c})} \\ t & \longmapsto & \dfrac{dc^i(t)}{dt}\dfrac{\partial}{\partial \tilde{z}^i}(\dot{c}(t)) + \tilde{g}^i_j(c(t))\dfrac{dc^j(t)}{dt}\dfrac{\partial}{\partial \tilde{y}^i}(\dot{c}(t)). \end{array} \tag{5.5.8}$$

Definition 5.5.2 If the component functions

$$\left(\left(g^\alpha_a\circ h\circ c\right)y^a,\ a\in\overline{1,r}\right)$$

are solutions for the differentiable system of equations

$$\frac{dz^\alpha}{dt} + (\rho,\eta)H^\alpha_{\beta\gamma}\circ u(c,\dot{c})\circ\eta\circ h\circ c\cdot z^\beta\cdot z^\gamma = 0,\ \alpha\in\overline{1,p}, \tag{5.5.9}$$

then the differentiable curve $\dot{c}$ will be called *horizontal parallel with respect to the distinguished linear (ρ,η)-connection $((\rho,\eta)H,(\rho,\eta)V)$.*

If the component functions $\left(y^a,\ a\in\overline{1,n}\right)$ are solutions for the differentiable system of equations

$$\frac{du^a}{dt} + (\rho,\eta)V^a_{bc}\circ u(c,\dot{c})\circ\eta\circ h\circ c\cdot u^b\cdot u^c = 0,\ a\in\overline{1,r}, \tag{5.5.10}$$

then the differentiable curve $\dot{c}$ will be called *vertical parallel with respect to the distinguished linear (ρ,η)-connection $((\rho,\eta)H,(\rho,\eta)V)$.*

Remark 5.5.3 In particular, if $(\rho,\eta,h) = (Id_{TM}, Id_M, Id_M)$ and (g, Id_M) is locally invertible, then the (g, Id_M)-lift of tangent vectors (5.5.7) is horizontal parallel with respect to the distinguished linear connection (H,V) if the component functions $\left(\dfrac{dc^i}{dt},\ i\in\overline{1,m}\right)$ are solutions for the differentiable system of equations

$$\frac{dz^i}{dt} + H^i_{jk}\circ u(c,\dot{c})\circ c\cdot z^j\cdot z^k = 0,\ i\in\overline{1,m}. \tag{5.5.12}$$

Moreover, the (g, Id_M)-lift of tangent vectors (5.5.7) is vertical parallel with respect to the distinguished linear connection (H,V) if the component functions

$$\left(\tilde{g}^i_j\circ c\cdot\frac{dc^j(t)}{dt},\ i\in\overline{1,m}\right)$$

are solutions for the differentiable system of equations

$$\frac{du^i}{dt} + V^i_{jk}\circ u(c,\dot{c})\circ c\cdot u^j\cdot u^k = 0,\ i\in\overline{1,m}. \tag{5.5.13}$$

5.6 The (ρ,η,h)-torsion and the (ρ,η,h)-curvature of a distinguished linear (ρ,η)-connection

We consider the following diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & (F,[,]_{F,h},(\rho,\eta)) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $((F,\nu,M),[,]_{F.h},(\rho,\eta))\in|\mathbf{GLA}|$. Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M) and let $((\rho,\eta)\,H,(\rho,\eta)\,V)$ be a distinguished linear (ρ,η)-connection for the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$.

Definition 5.6.1 The application

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)^2 & \xrightarrow{(\rho,\eta,h)\mathbb{T}} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\\ (X,Y) & \longmapsto & (\rho,\eta)\,\mathbb{T}\,(X,Y)\end{array}$$

defined by

$$(\rho,\eta,h)\,\mathbb{T}\,(X,Y)=(\rho,\eta)\,D_XY-(\rho,\eta)\,D_YX-[X,Y]_{(\rho,\eta)TE}\,, \tag{5.6.1}$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$, will be called the *$(\rho,\eta,h)$-torsion associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$.*

The applications

$$\mathcal{H}\,(\rho,\eta,h)\,\mathbb{T}\,(\mathcal{H}\,(\cdot)\,,\mathcal{H}\,(\cdot))\,,\ \mathcal{V}\,(\rho,\eta,h)\,\mathbb{T}\,(\mathcal{H}\,(\cdot)\,,\mathcal{H}\,(\cdot))\,,...,\mathcal{V}\,(\rho,\eta,h)\,\mathbb{T}\,(\mathcal{V}\,(\cdot)\,,\mathcal{V}\,(\cdot))$$

are called $\mathcal{H}\,(\mathcal{HH})\,,\ \mathcal{V}\,(\mathcal{HH})\,,...,\mathcal{V}\,(\mathcal{VV})$ *(ρ,η,h)-torsions associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$.*

Proposition 5.6.1 *The (ρ,η,h)-torsion $(\rho,\eta,h)\,\mathbb{T}$ associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$, is $\mathbb{R}$-bilinear and antisymmetric in the lower indices.*

Using the notations:

$$\begin{aligned}&\mathcal{H}\,(\rho,\eta,h)\,\mathbb{T}\left(\tilde{\delta}_\gamma,\tilde{\delta}_\beta\right)=(\rho,\eta,h)\,\mathbb{T}^\alpha{}_{\beta\gamma}\tilde{\delta}_\alpha,\\ &\mathcal{V}\,(\rho,\eta,h)\,\mathbb{T}\left(\tilde{\delta}_\gamma,\tilde{\delta}_\beta\right)=(\rho,\eta,h)\,\mathbb{T}^a{}_{\beta\gamma}\dot{\tilde{\partial}}_a,\\ &\mathcal{H}\,(\rho,\eta,h)\,\mathbb{T}\left(\dot{\tilde{\partial}}_c,\tilde{\delta}_\beta\right)=(\rho,\eta,h)\,\mathbb{P}^\alpha{}_{\beta c}\tilde{\delta}_\alpha,\\ &\mathcal{V}\,(\rho,\eta,h)\,\mathbb{T}\left(\dot{\tilde{\partial}}_c,\tilde{\delta}_\beta\right)=(\rho,\eta,h)\,\mathbb{P}^a{}_{\beta c}\dot{\tilde{\partial}}_a,\\ &\mathcal{V}\,(\rho,\eta,h)\,\mathbb{T}\left(\dot{\tilde{\partial}}_c,\dot{\tilde{\partial}}_b\right)=(\rho,\eta,h)\,\mathbb{S}^a{}_{bc}\dot{\tilde{\partial}}_a.\end{aligned} \tag{5.6.2}$$

we can easily prove the following

Theorem 5.6.1 *The (ρ,η,h)-torsion $(\rho,\eta,h)\,\mathbb{T}$ associated to the distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$, is characterized by the tensor fields with local components:*

$$
\begin{aligned}
(\rho,\eta,h)\,\mathbb{T}^{\alpha}{}_{\beta\gamma} &= (\rho,\eta)\,H^{\alpha}_{\beta\gamma}-(\rho,\eta)\,H^{\alpha}_{\gamma\beta}-L^{\alpha}_{\beta\gamma}\circ h\circ\pi,\\
(\rho,\eta,h)\,\mathbb{T}^{a}{}_{\beta\gamma} &= (\rho,\eta,h)\,\mathbb{R}^{a}{}_{\beta\gamma},\\
(\rho,\eta,h)\,\mathbb{P}^{\alpha}{}_{\beta c} &= (\rho,\eta)\,V^{\alpha}_{\beta c},\\
(\rho,\eta,h)\,\mathbb{P}^{a}{}_{\beta c} &= \frac{\partial}{\partial y^{c}}\left((\rho,\eta)\,\Gamma^{a}_{\beta}\right)-(\rho,\eta)\,H^{a}_{c\beta},\\
(\rho,\eta,h)\,\mathbb{S}^{a}{}_{bc} &= (\rho,\eta)\,V^{a}_{bc}-(\rho,\eta)\,V^{a}_{cb}.
\end{aligned}
\tag{5.6.3}
$$

In particular, when $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, we regain the local components of torsion associated to distinguished linear connection (H,V):

$$
\begin{aligned}
\mathbb{T}^{i}{}_{jk} &= H^{i}_{jk}-H^{i}_{kj}, & \mathbb{T}^{a}{}_{jk} &= \mathbb{R}^{a}{}_{jk},\\
\mathbb{P}^{i}{}_{jc} &= V^{i}_{jc}, & \mathbb{P}^{a}{}_{jc} &= \frac{\partial\Gamma^{a}_{j}}{\partial y^{c}}-H^{a}_{cj},\\
\mathbb{S}^{a}{}_{bc} &= V^{a}_{bc}-V^{a}_{cb}.
\end{aligned}
\tag{5.6.3'}
$$

Definition 5.6.2 The application

$$
\begin{array}{ccc}
\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\right)^3 & \xrightarrow{(\rho,\eta,h)\mathbb{R}} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\\
\left((Y,Z),X\right) & \longmapsto & (\rho,\eta,h)\,\mathbb{R}\,(Y,Z)\,,X
\end{array}
$$

defined by:

$$
\begin{aligned}
(\rho,\eta,h)\,\mathbb{R}\,(Y,Z)\,X &= (\rho,\eta)\,D_Y\left((\rho,\eta)\,D_Z X\right)\\
&\quad-(\rho,\eta)\,D_Z\left((\rho,\eta)\,D_Y X\right)-(\rho,\eta)\,D_{[Y,Z]_{(\rho,\eta)TE}}X,
\end{aligned}
\tag{18.4}
$$

for any $X,Y,Z\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$, will be called the *$(\rho,\eta,h)$-curvature associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$.*

Proposition 5.6.2 *The (ρ,η,h)-curvature $(\rho,\eta,h)\,\mathbb{R}$ associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$, is $\mathbb{R}$-linear in each argument and antisymmetric in the first two arguments.*

Using the notations:

$$
\begin{aligned}
(\rho,\eta,h)\,\mathbb{R}\left(\tilde{\delta}_{\varepsilon},\tilde{\delta}_{\gamma}\right)\tilde{\delta}_{\beta} &= (\rho,\eta,h)\,\mathbb{R}^{\alpha}{}_{\beta\ \gamma\varepsilon}\tilde{\delta}_{\alpha},\\
(\rho,\eta,h)\,\mathbb{R}\left(\tilde{\delta}_{\varepsilon},\tilde{\delta}_{\gamma}\right)\dot{\tilde{\partial}}_{b} &= (\rho,\eta,h)\,\mathbb{R}^{a}{}_{b\ \gamma\varepsilon}\dot{\tilde{\partial}}_{a},\\
(\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_{c},\tilde{\delta}_{\gamma}\right)\tilde{\delta}_{\varepsilon} &= (\rho,\eta,h)\,\mathbb{P}^{\alpha}{}_{\varepsilon\ \gamma c}\tilde{\delta}_{\alpha},\\
(\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_{c},\tilde{\delta}_{\gamma}\right)\dot{\tilde{\partial}}_{b} &= (\rho,\eta,h)\,\mathbb{P}^{a}{}_{b\ \gamma c}\dot{\tilde{\partial}}_{a},\\
(\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_{c},\dot{\tilde{\partial}}_{b}\right)\tilde{\delta}_{\beta} &= (\rho,\eta,h)\,\mathbb{S}^{\alpha}{}_{\beta\ bc}\tilde{\delta}_{\alpha},\\
(\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_{d},\dot{\tilde{\partial}}_{c}\right)\dot{\tilde{\partial}}_{b} &= (\rho,\eta,h)\,\mathbb{S}^{\alpha}{}_{b\ cd}\dot{\tilde{\partial}}_{a}.
\end{aligned}
\tag{5.6.5}
$$

we can easily prove the following

Theorem 5.6.2 *The (ρ,η,h)-curvature $(\rho,\eta,h)\mathbb{R}$ associated to distinguished linear (ρ,η)-connection $((\rho,\eta)H,(\rho,\eta)V)$, is characterized by the tensor fields with local components:*

$$(5.6.6)\quad \begin{cases} (\rho,\eta,h)\mathbb{R}^{\alpha}{}_{\beta\ \gamma\varepsilon} & =\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_\varepsilon\right)(\rho,\eta)H^{\alpha}_{\beta\gamma}-\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_\gamma\right)(\rho,\eta)H^{\alpha}_{\beta\varepsilon} \\ & +(\rho,\eta)H^{\alpha}_{\theta\varepsilon}(\rho,\eta)H^{\theta}_{\beta\gamma}-(\rho,\eta)H^{\alpha}_{\theta\gamma}(\rho,\eta)H^{\theta}_{\beta\varepsilon} \\ & -(\rho,\eta,h)\mathbb{R}^{c}{}_{\gamma\varepsilon}(\rho,\eta)H^{\alpha}_{\beta c}-L^{\theta}_{\gamma\varepsilon}\circ h\circ\pi(\rho,\eta)H^{\alpha}_{\beta\theta}, \\ (\rho,\eta,h)\mathbb{R}^{a}{}_{b\ \gamma\varepsilon} & =\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_\varepsilon\right)(\rho,\eta)H^{a}_{b\gamma}-\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_\gamma\right)(\rho,\eta)H^{a}_{b\varepsilon} \\ & +(\rho,\eta)H^{a}_{d\varepsilon}(\rho,\eta)H^{d}_{b\gamma}-(\rho,\eta)H^{a}_{d\gamma}(\rho,\eta)H^{d}_{b\varepsilon} \\ & -(\rho,\eta,h)\mathbb{R}^{c}{}_{\varepsilon\gamma}(\rho,\eta)V^{a}_{bc}-L^{\theta}_{\gamma\varepsilon}\circ h\circ\pi(\rho,\eta)V^{a}_{b\theta}, \end{cases}$$

$$(5.6.7)\quad \begin{cases} (\rho,\eta,h)\mathbb{P}^{\alpha}{}_{\varepsilon\ \gamma c} & =\Gamma(\tilde{\rho},Id_E)(\dot{\tilde{\partial}}_c)(\rho,\eta)H^{\alpha}_{\varepsilon\gamma}-\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_\gamma\right)(\rho,\eta)V^{\alpha}_{\varepsilon c} \\ & =(\rho,\eta)V^{\alpha}_{\theta c}(\rho,\eta)H^{\theta}_{\varepsilon\gamma}-(\rho,\eta)H^{\alpha}_{\theta\gamma}(\rho,\eta)V^{\theta}_{\varepsilon c} \\ & +\frac{\partial}{\partial y^c}\left((\rho,\eta)\Gamma^{d}_{\gamma}\right)(\rho,\eta)V^{\alpha}_{\varepsilon d}, \\ (\rho,\eta,h)\mathbb{P}^{a}{}_{b\ \gamma c} & =\Gamma(\tilde{\rho},Id_E)\left(\dot{\tilde{\partial}}_c\right)(\rho,\eta)H^{a}_{b\gamma}- \\ & -\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_\gamma\right)(\rho,\eta)V^{a}_{bc}+(\rho,\eta)V^{a}_{dc}(\rho,\eta)H^{d}_{b\gamma}- \\ & -(\rho,\eta)H^{a}_{d\gamma}(\rho,\eta)V^{d}_{bc}+\frac{\partial}{\partial y^c}(\left(\rho,\eta)\Gamma^{d}_{\gamma}\right)(\rho,\eta)V^{a}_{bd}, \end{cases}$$

$$(5.6.8)\quad \begin{cases} (\rho,\eta,h)\,\mathbb{S}^{\alpha}{}_{\beta\ bc} & =\Gamma(\tilde{\rho},Id_E)\left(\dot{\tilde{\partial}}_c\right)(\rho,\eta)\,V^{\alpha}_{\beta b} \\ & -\Gamma(\tilde{\rho},Id_E)\left(\dot{\tilde{\partial}}_b\right)(\rho,\eta)\,V^{\alpha}_{\beta c}+(\rho,\eta)\,V^{\alpha}_{\theta c}(\rho,\eta)\,V^{\theta}_{\beta b} \\ & -(\rho,\eta)\,V^{\alpha}_{\theta b}(\rho,\eta)\,V^{\theta}_{\beta c}, \\ (\rho,\eta,h)\,\mathbb{S}^{a}{}_{b\ cd} & =\Gamma(\tilde{\rho},Id_E)\left(\dot{\tilde{\partial}}_d\right)(\rho,\eta)\,V^{a}_{bc} \\ & -\Gamma(\tilde{\rho},Id_E)\left(\dot{\tilde{\partial}}_c\right)(\rho,\eta)\,V^{a}_{bd}+(\rho,\eta)\,V^{a}_{ed}(\rho,\eta)\,V^{e}_{bc} \\ & -(\rho,\eta)\,V^{a}_{ec}(\rho,\eta)\,V^{e}_{bd}. \end{cases}$$

In particular, when $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, we see the local components of the curvature associated to distinguished linear connection (H,V) in the followings:

$$(5.6.6')\quad \begin{array}{l} \mathbb{R}^{i}{}_{j\ kl}=\delta_l\left(H^{i}_{jk}\right)-\delta_k\left(H^{i}_{jl}\right)+H^{i}_{hl}H^{h}_{jk}-H^{i}_{hk}H^{h}_{jl}-\mathbb{R}^{c}{}_{kl}H^{i}_{jc}, \\ \mathbb{R}^{a}{}_{b\ kl}=\delta_l\left(H^{a}_{bk}\right)-\delta_k\left(H^{a}_{bl}\right)+H^{a}_{dl}H^{d}_{bk}-H^{a}_{dk}H^{d}_{bl}-\mathbb{R}^{c}{}_{lk}V^{a}_{bc}, \end{array}$$

$$
(5.6.7') \quad \begin{aligned}
\mathbb{P}^{i}{}_{l\ kc} &= \frac{\partial}{\partial y^c}\left(H^i_{lk}\right) - \delta_k\left(V^i_{lc}\right) + V^i_{hc}H^h_{lk} - H^i_{hk}V^h_{lc} + \frac{\partial}{\partial y^c}\left(\Gamma^d_k\right)V^i_{ld},\\
\mathbb{P}^{a}{}_{b\ kc} &= \frac{\partial}{\partial y^c}\left(H^a_{bk}\right) - \delta_k\left(V^a_{bc}\right) + V^a_{dc}H^d_{bk} - H^a_{dk}V^d_{bc} + \frac{\partial}{\partial y^c}\left(\Gamma^d_k\right)V^a_{bd},
\end{aligned}
$$

$$
(5.6.8') \quad \begin{aligned}
\mathbb{S}^{i}{}_{j\ bc} &= \frac{\partial}{\partial y^c}\left(V^i_{jb}\right) - \frac{\partial}{\partial y^b}\left(V^i_{jc}\right) + V^i_{hc}V^h_{jb} - V^i_{hb}V^h_{jc},\\
\mathbb{S}^{a}{}_{b\ cd} &= \frac{\partial}{\partial y^d}\left(V^a_{bc}\right) - \frac{\partial}{\partial y^c}\left(V^a_{bd}\right) + V^a_{ed}V^e_{bc} - V^a_{ec}V^e_{bd}.
\end{aligned}
$$

Definition 5.6.3 The tensor field

$$
(5.6.9) \quad \begin{aligned}
&\mathbf{Ric}\left(\left(\rho,\eta\right)H,\left(\rho,\eta\right)V\right) =\\
&\quad = \left(\rho,\eta,h\right)\mathbb{R}_{\ \alpha\ \beta}d\tilde{z}^\alpha\otimes d\tilde{z}^\beta + \left(\rho,\eta,h\right)\mathbb{P}_{\ \alpha\ b}d\tilde{z}^\alpha\otimes\delta\tilde{y}^b\\
&\quad + \left(\rho,\eta,h\right)\mathbb{P}_{\ a\ \beta}\delta\tilde{y}^a\otimes d\tilde{z}^\beta + \left(\rho,\eta,h\right)\mathbb{S}_{\ a\ b}\delta\tilde{y}^a\otimes\delta\tilde{y}^b,
\end{aligned}
$$

$$
(5.6.10) \quad \begin{array}{ll}
\left(\rho,\eta,h\right)\mathbb{R}_{\ \alpha\ \beta} = \left(\rho,\eta,h\right)\mathbb{R}^{\gamma}{}_{\alpha\ \beta\gamma} & \left(\rho,\eta,h\right)\mathbb{P}_{\ \alpha\ b} = \left(\rho,\eta,h\right)\mathbb{P}^{\beta}{}_{\alpha\ \beta b}\\
\left(\rho,\eta,h\right)\mathbb{P}_{\ a\ \beta} = \left(\rho,\eta,h\right)\mathbb{P}^{c}{}_{a\ \beta c} & \left(\rho,\eta,h\right)\mathbb{S}_{\ a\ b} = \left(\rho,\eta,h\right)\mathbb{S}^{c}{}_{a\ cb}
\end{array},
$$

will be called *the Ricci tensor field associated to distinguished linear* (ρ,η)*-connection* $\left(\left(\rho,\eta\right)H,\left(\rho,\eta\right)V\right)$.

This tensor field will be used to write the Einstein equations in Subsection 5.10.

5.7 Formulas of Ricci type. Identities of Cartan and Bianchi type

We consider the following diagram:

$$
\begin{array}{ccc}
E & & \left(F,\left[,\right]_{F,h},\left(\rho,\eta\right)\right)\\
\pi\downarrow & & \downarrow\nu\\
M & \xrightarrow{h} & N
\end{array}
$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left(\left(F,\nu,M\right),\left[,\right]_{F.h},\left(\rho,\eta\right)\right)\in|\mathbf{GLA}|$. Let $\left(\rho,\eta\right)\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M) and let $\left(\left(\rho,\eta\right)H,\left(\rho,\eta\right)V\right)$ be a distinguished linear (ρ,η)-connection for the generalized tangent bundle $\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$.

Theorem 5.7.1 *Using the definition of* (ρ,η,h)*-curvature associated to the distinguished linear* (ρ,η)*-connection* $\left(\left(\rho,\eta\right)H,\left(\rho,\eta\right)V\right)$*, it results the following formulas:*

$$
(\mathcal{B}_1) \quad \left\{\begin{aligned}
&\left(\rho,\eta\right)D_{\mathcal{H}X}\left(\rho,\eta\right)D_{\mathcal{H}Y}\mathcal{H}Z - \left(\rho,\eta\right)D_{\mathcal{H}Y}\left(\rho,\eta\right)D_{\mathcal{H}X}\mathcal{H}Z\\
&= \left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{H}X,\mathcal{H}Y\right)\mathcal{H}Z + \left(\rho,\eta\right)D_{\mathcal{H}[\mathcal{H}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{H}Z\\
&+ \left(\rho,\eta\right)D_{\mathcal{V}[\mathcal{H}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{H}Z,\\
&\left(\rho,\eta\right)D_{\mathcal{V}X}\left(\rho,\eta\right)D_{\mathcal{H}Y}\mathcal{H}Z - \left(\rho,\eta\right)D_{\mathcal{H}Y}\left(\rho,\eta\right)D_{\mathcal{V}X}\mathcal{H}Z\\
&= \left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{V}X,\mathcal{H}Y\right)\mathcal{H}Z + \left(\rho,\eta\right)D_{\mathcal{H}[\mathcal{V}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{H}Z\\
&+ \left(\rho,\eta\right)D_{\mathcal{V}[\mathcal{V}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{H}Z,\\
&\left(\rho,\eta\right)D_{\mathcal{V}X}\left(\rho,\eta\right)D_{\mathcal{V}Y}\mathcal{H}Z - \left(\rho,\eta\right)D_{\mathcal{V}Y}\left(\rho,\eta\right)D_{\mathcal{V}X}\mathcal{H}Z\\
&= \left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{V}X,\mathcal{V}Y\right)\mathcal{H}Z + \left(\rho,\eta\right)D_{\mathcal{V}[\mathcal{V}X,\mathcal{V}Y]_{(\rho,\eta)TE}}\mathcal{H}Z,
\end{aligned}\right.
$$

and

$$(\mathcal{B}_2)\quad \left\{\begin{array}{l}(\rho,\eta)\,D_{\mathcal{H}X}\,(\rho,\eta)\,D_{\mathcal{H}Y}\mathcal{V}Z-(\rho,\eta)\,D_{\mathcal{H}Y}\,(\rho,\eta)\,D_{\mathcal{H}X}\mathcal{V}Z\\ =(\rho,\eta,h)\,\mathbb{R}\,(\mathcal{H}X,\mathcal{H}Y)\,\mathcal{V}Z+(\rho,\eta)\,D_{\mathcal{H}[\mathcal{H}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{V}Z\\ +(\rho,\eta)\,D_{\mathcal{V}[\mathcal{H}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{V}Z,\\ (\rho,\eta)\,D_{\mathcal{V}X}\,(\rho,\eta)\,D_{\mathcal{H}Y}\mathcal{V}Z-(\rho,\eta)\,D_{\mathcal{H}Y}\,(\rho,\eta)\,D_{\mathcal{V}X}\mathcal{V}Z\\ =(\rho,\eta,h)\,\mathbb{R}\,(\mathcal{V}X,\mathcal{H}Y)\,\mathcal{V}Z+(\rho,\eta)\,D_{h[\mathcal{V}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{V}Z\\ +(\rho,\eta)\,D_{\mathcal{V}[\mathcal{V}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{V}Z,\\ (\rho,\eta)\,D_{\mathcal{V}X}\,(\rho,\eta)\,D_{\mathcal{V}Y}\mathcal{V}Z-(\rho,\eta)\,D_{\mathcal{V}Y}\,(\rho,\eta)\,D_{\mathcal{V}X}\mathcal{V}Z\\ =(\rho,\eta,h)\,\mathbb{R}\,(\mathcal{V}X,\mathcal{V}Y)\,\mathcal{V}Z+(\rho,\eta)\,D_{\mathcal{V}[\mathcal{V}X,\mathcal{V}Y]_{(\rho,\eta)TE}}\mathcal{V}Z.\end{array}\right.$$

Using the previous theorem, the horizontal and vertical sections of adapted base and an arbitrary section

$$Z^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y^a\frac{\partial}{\partial\tilde{y}^a}\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right),$$

we propose the following

Theorem 5.7.2 *We obtain the following formulas of Ricci type:*

$$(\mathcal{R}_1)\quad \left\{\begin{array}{ll}\tilde{Z}^\alpha{}_{|\gamma|\beta}-\tilde{Z}^\alpha{}_{|\beta|\gamma} & =(\rho,\eta,h)\,\mathbb{R}^\alpha_{\theta\ \gamma\beta}\tilde{Z}^\theta+\left(L^\theta_{\beta\gamma}\circ h\circ\pi\right)\tilde{Z}^\alpha{}_{|\theta}\\ & +(\rho,\eta,h)\,\mathbb{T}^a{}_{\beta\gamma}\tilde{Z}^\alpha|_a+(\rho,\eta,h)\,\mathbb{T}^\theta{}_{\beta\gamma}\tilde{Z}^\alpha{}_{|\theta},\\ \tilde{Z}^\alpha{}_{|\gamma}|_b-\tilde{Z}^\alpha|_{b|\gamma} & =(\rho,\eta,h)\,\mathbb{P}^\alpha_{\theta\ \gamma b}\tilde{Z}^\theta-(\rho,\eta,h)\,\mathbb{P}^a{}_{\gamma b}\tilde{Z}^\alpha|_a\\ & -(\rho,\eta)\,\mathbb{H}^a_{b\gamma}\tilde{Z}^\alpha|_a,\\ \tilde{Z}^\alpha|_c|_b-\tilde{Z}^\alpha|_b|_c & =(\rho,\eta,h)\,\mathbb{S}^\alpha_{\theta\ cb}\tilde{Z}^\theta+(\rho,\eta,h)\,\mathbb{S}^a{}_{bc}\tilde{Z}^\alpha|_a,\end{array}\right.$$

and

$$(\mathcal{R}_2)\quad \left\{\begin{array}{ll}Y^a{}_{|\gamma|\beta}-Y^a{}_{|\beta|\gamma} & =(\rho,\eta,h)\,\mathbb{R}^a{}_{c\ \gamma\beta}Y^c+\left(L^\theta_{\beta\gamma}\circ h\circ\pi\right)Y^c{}_{|\theta}\\ & +(\rho,\eta)\,\mathbb{T}^b{}_{\beta\gamma}Y^a|_b+(\rho,\eta,h)\,\mathbb{T}^\theta{}_{\beta\gamma}Y^a{}_{|\theta},\\ Y^a{}_{|\gamma}|_b-Y^a|_{b|\gamma} & =(\rho,\eta,h)\,\mathbb{P}^a{}_{\theta\ \gamma b}Y^\theta-(\rho,\eta,h)\,\mathbb{P}^c{}_{\gamma b}Y^a|_c\\ & -(\rho,\eta)\,\mathbb{H}^c_{b\gamma}Y^a|_c,\\ Y^a|_c|_b-Y^a|_b|_c & =(\rho,\eta,h)\,\mathbb{S}^a{}_{d\ cb}Y^d+(\rho,\eta,h)\,\mathbb{S}^d{}_{bc}Y^a|_d.\end{array}\right.$$

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,id_M)$ *and the Lie bracket* $[,]_{TM}$ *is the usual Lie bracket, then the formulas of Ricci type* $(\mathcal{R}_1)$ *and* $(\mathcal{R}_2)$ *become:*

$$(\mathcal{R}_1)'\quad \left\{\begin{array}{ll}\tilde{Z}^i{}_{|k|j}-\tilde{Z}^i{}_{|j|k} & =\mathbb{R}^i{}_{h\ kj}\tilde{Z}^h+\mathbb{T}^a{}_{jk}\tilde{Z}^i|_a+\mathbb{T}^h{}_{jk}\tilde{Z}^i{}_{|h},\\ \tilde{Z}^i{}_{|k|b}-\tilde{Z}^i{}_{|b|k} & =\mathbb{P}^i{}_{h\ kb}\tilde{Z}^h-\mathbb{P}^a{}_{kb}\tilde{Z}^i|_a-\mathbb{H}^a_{bk}\tilde{Z}^i|_a,\\ \tilde{Z}^i{}_{|c|b}-\tilde{Z}^i{}_{|b|c} & =\mathbb{S}^i{}_{h\ cb}\tilde{Z}^h+\mathbb{S}^a{}_{bc}\tilde{Z}^i|_a,\end{array}\right.$$

and

$$(\mathcal{R}_2)'\quad \left\{\begin{array}{ll}Y^a_{|k|j}-Y^a_{|j|k} & =\mathbb{R}^a{}_{c\ kj}Y^c+\mathbb{T}^b{}_{jk}Y^a|_b+\mathbb{T}^h{}_{jk}Y^a{}_{|h},\\ Y^a_{|k|b}-Y^a_{|b|k} & =\mathbb{P}^a{}_{h\ kb}Y^h-\mathbb{P}^c{}_{kb}Y^a|_c-\mathbb{H}^c_{bk}Y^a|_c,\\ Y^a_{|c|b}-Y^a_{|b|c} & =\mathbb{S}^a{}_{d\ cb}Y^d+\mathbb{S}^d{}_{bc}Y^a|_d.\end{array}\right.$$

Using the 1-forms associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$

$$(5.7.1)\qquad \begin{aligned}(\rho,\eta)\,\omega_\beta^\alpha &= (\rho,\eta)\,H^\alpha_{\beta\gamma}d\tilde{z}^\gamma + (\rho,\eta)\,V^\alpha_{\beta c}\delta\tilde{y}^c,\\ (\rho,\eta)\,\omega_b^a &= (\rho,\eta)\,H^a_{b\gamma}d\tilde{z}^\gamma + (\rho,\eta)\,V^a_{bc}\delta\tilde{y}^c,\end{aligned}$$

the torsion 2-forms

$$(5.7.2)\qquad \begin{cases}(\rho,\eta,h)\,\mathbb{T}^\alpha & = \frac{1}{2}(\rho,\eta,h)\,\mathbb{T}^\alpha{}_{\beta\gamma}d\tilde{z}^\beta\wedge d\tilde{z}^\gamma + (\rho,\eta,h)\,\mathbb{P}^\alpha{}_{\beta c}d\tilde{z}^\beta\wedge\delta\tilde{y}^c,\\ (\rho,\eta,h)\,\mathbb{T}^a & = \frac{1}{2}(\rho,\eta,h)\,\mathbb{T}^a{}_{\beta\gamma}d\tilde{z}^\beta\wedge d\tilde{z}^\gamma + (\rho,\eta,h)\,\mathbb{P}^a{}_{\beta c}d\tilde{z}^\beta\wedge\delta\tilde{y}^c\\ & +\frac{1}{2}(\rho,\eta,h)\,\mathbb{S}^a{}_{bc}\delta\tilde{y}^b\wedge\delta\tilde{y}^c\end{cases}$$

and the curvature 2-forms

$$(5.7.3)\qquad \begin{cases}(\rho,\eta,h)\,\mathbb{R}^\alpha{}_\beta & = \frac{1}{2}(\rho,\eta,h)\,\mathbb{R}^\alpha{}_{\beta\ \gamma\theta}d\tilde{z}^\gamma\wedge d\tilde{z}^\theta + (\rho,\eta,h)\,\mathbb{P}^\alpha{}_{\beta\ \gamma c}d\tilde{z}^\gamma\wedge\delta\tilde{y}^c\\ & +\frac{1}{2}(\rho,\eta,h)\,\mathbb{S}^\alpha{}_{\beta\ cd}\delta\tilde{y}^c\wedge\delta\tilde{y}^d,\\ (\rho,\eta,h)\,\mathbb{R}^a{}_b & = \frac{1}{2}(\rho,\eta,h)\,\mathbb{R}^a{}_{b\ \gamma\theta}d\tilde{z}^\gamma\wedge d\tilde{z}^\theta + (\rho,\eta,h)\,\mathbb{P}^a{}_{b\ \gamma c}d\tilde{z}^\gamma\wedge\delta\tilde{y}^c\\ & +\frac{1}{2}(\rho,\eta,h)\,\mathbb{S}^a{}_{b\ cd}\delta\tilde{y}^c\wedge\delta\tilde{y}^d,\end{cases}$$

we obtain the following

Theorem 5.7.3 *We obtain the following identities of Cartan type:*

$$(\mathcal{C}_1)\qquad \begin{aligned}(\rho,\eta,h)\,\mathbb{T}^\alpha &= d^{(\rho,\eta)TE}\left(d\tilde{z}^\alpha\right) + (\rho,\eta)\,\omega_\beta^\alpha\wedge d\tilde{z}^\beta,\\ (\rho,\eta,h)\,\mathbb{T}^a &= d^{(\rho,\eta)TE}\left(\delta\tilde{y}^a\right) + (\rho,\eta)\,\omega_b^a\wedge\delta\tilde{y}^b,\end{aligned}$$

$$(\mathcal{C}_2)\qquad \begin{aligned}(\rho,\eta,h)\,\mathbb{R}^\alpha{}_\beta &= d^{(\rho,\eta)TE}\left((\rho,\eta)\,\omega_\beta^\alpha\right) + (\rho,\eta)\,\omega_\gamma^\alpha\wedge(\rho,\eta)\,\omega_\beta^\gamma,\\ (\rho,\eta,h)\,\mathbb{R}^a{}_b &= d^{(\rho,\eta)TE}\left((\rho,\eta)\,\omega_b^a\right) + (\rho,\eta)\,\omega_c^a\wedge(\rho,\eta)\,\omega_b^c.\end{aligned}$$

In particular, if $(\rho,\eta,h) = (Id_{TM}, Id_M, Id_M)$ and the Lie bracket $[,]_{TM}$ is the usual Lie bracket, then the identities of Cartan type $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ become:

$$(\mathcal{C}_1)'\qquad \begin{aligned}\mathbb{T}^i &= d^{(Id_{TE},Id_E)TE}\left(d\tilde{z}^i\right) + \omega_j^i\wedge d\tilde{z}^j,\\ \mathbb{T}^a &= d^{(Id_{TE},Id_E)TE}\left(\delta\tilde{y}^a\right) + \omega_b^a\wedge\delta\tilde{y}^b,\end{aligned}$$

and

$$(\mathcal{C}_2)'\qquad \begin{aligned}\mathbb{R}^i{}_j &= d^{(Id_{TE},Id_E)TE}\left(\omega_j^i\right) + \omega_k^i\wedge\omega_j^k,\\ \mathbb{R}^a{}_b &= d^{(Id_{TE},Id_E)TE}\left(\omega_b^a\right) + \omega_c^a\wedge\omega_b^c.\end{aligned}$$

Remark 5.7.1 For any $X,Y,Z\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$, the following identities

$$(5.7.4)\qquad \begin{aligned}\mathcal{V}\,(\rho,\eta,h)\,\mathbb{R}\,(X,Y)\,\mathcal{H}Z &= 0,\\ \mathcal{H}\,(\rho,\eta,h)\,\mathbb{R}\,(X,Y)\,\mathcal{V}Z &= 0,\end{aligned}$$

$$
\begin{aligned}
\mathcal{V}D_X\left(\left(\rho,\eta,h\right)\mathbb{R}\left(Y,Z\right)\mathcal{H}U\right) &= 0,\\
\mathcal{H}D_X\left(\left(\rho,\eta,h\right)\mathbb{R}\left(Y,Z\right)\mathcal{V}U\right) &= 0,
\end{aligned}
\tag{5.7.5}
$$

and

$$
\left(\rho,\eta,h\right)\mathbb{R}\left(X,Y\right)Z = \mathcal{H}\left(\rho,\eta,h\right)\mathbb{R}\left(X,Y\right)\mathcal{H}Z + \mathcal{V}\left(\rho,\eta,h\right)\mathbb{R}\left(X,Y\right)\mathcal{V}Z.
\tag{5.7.6}
$$

hold. Using the formulas of Bianchi type from Theorem 4.2.3 and Remark 5.7.1, we obtain the following

Theorem 5.7.4 *The identities of Bianchi type:*

$$
(\mathcal{B}_1)\quad
\begin{cases}
\displaystyle\sum_{cyclic(X,Y,Z)} \{\mathcal{H}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\mathbb{T}\left(Y,Z\right)\right) - \mathcal{H}\left(\rho,\eta,h\right)\mathbb{R}\left(X,Y\right)Z\\
\qquad +\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)\\
\qquad +\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)\} = 0,\\
\displaystyle\sum_{cyclic(X,Y,Z)} \{\mathcal{V}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\mathbb{T}\left(Y,Z\right)\right) - \mathcal{V}\left(\rho,\eta,h\right)\mathbb{R}\left(X,Y\right)Z\\
\qquad +\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)\\
\qquad +\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)\} = 0.
\end{cases}
$$

and

$$
(\mathcal{B}_2)\quad
\begin{cases}
\displaystyle\sum_{cyclic(X,Y,Z,U)} \{\mathcal{H}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\mathbb{R}\left(Y,Z\right)U\right)\\
\qquad -\mathcal{H}\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)U\\
\qquad -\mathcal{H}\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)U\} = 0,\\
\displaystyle\sum_{cyclic(X,Y,Z,U)} \{\mathcal{V}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\mathbb{R}\left(Y,Z\right)U\right)\\
\qquad -\mathcal{V}\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)U\\
\qquad -\mathcal{V}\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)U\} = 0,
\end{cases}
$$

hold good for any $X,Y,Z\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$.

Corollary 5.7.1 *Using the following sections* $\left(\delta_\theta,\delta_\gamma,\delta_\beta\right)$, *the identities* $(\mathcal{B}_1)$ *become:*

$$
(\mathcal{B}_1)'\quad
\begin{cases}
\displaystyle\sum_{cyclic(\beta,\gamma,\theta)} \Big\{\left(\rho,\eta,h\right)\mathbb{T}^{\alpha}{}_{\beta\gamma_{|\theta}} - \left(\rho,\eta,h\right)\mathbb{R}^{\alpha}_{\beta\ \gamma\theta}\\
\qquad + \left(\rho,\eta,h\right)\mathbb{T}^{\lambda}{}_{\gamma\theta}\left(\rho,\eta,h\right)\mathbb{T}^{\alpha}{}_{\beta\gamma} + \left(\rho,\eta,h\right)\mathbb{T}^{a}{}_{\gamma\theta}\left(\rho,\eta,h\right)\mathbb{T}^{\alpha}{}_{\beta a}\Big\} = 0,\\
\displaystyle\sum_{cyclic(\beta,\gamma,\theta)} \Big\{\left(\rho,\eta,h\right)\mathbb{T}^{a}{}_{\beta\gamma_{|\theta}} + \left(\rho,\eta,h\right)\mathbb{T}^{\alpha}{}_{\gamma\theta}\left(\rho,\eta,h\right)\mathbb{P}^{a}{}_{\beta\alpha}\\
\qquad + \left(\rho,\eta,h\right)\mathbb{P}^{b}{}_{\gamma\theta}\left(\rho,\eta,h\right)\mathbb{P}^{a}{}_{b\beta}\Big\} = 0,
\end{cases}
$$

and using the following sections $(\delta_\lambda, \delta_\theta, \delta_\gamma, \delta_\beta)$*, the identities* $(\mathcal{B}_2)$ *become:*

$$(\mathcal{B}_2)' \quad \left\{ \begin{array}{l} \displaystyle\sum_{cyclic(\beta,\gamma,\theta,\lambda)} \Big\{ (\rho,\eta,h)\,\mathbb{R}^{\alpha}{}_{\beta\ \gamma\theta_{|\lambda}} - (\rho,\eta,h)\,\mathbb{T}^{\mu}{}_{\theta\lambda}\,(\rho,\eta,h)\,\mathbb{R}^{\alpha}{}_{\beta\ \gamma\mu} \\ \qquad\qquad - (\rho,\eta,h)\,\mathbb{T}^{a}{}_{\theta\lambda}\,(\rho,\eta,h)\,\mathbb{P}^{\alpha}{}_{\beta\ \gamma a} \Big\} = 0, \\ \displaystyle\sum_{cyclic(\beta,\gamma,\theta,\lambda)} \Big\{ (\rho,\eta,h)\,\mathbb{R}^{a}{}_{b\ \gamma\theta_{|\lambda}} - (\rho,\eta,h)\,\mathbb{T}^{\mu}{}_{\theta\lambda}\,(\rho,\eta,h)\,\mathbb{R}^{a}{}_{\beta\ \gamma\mu} \\ \qquad\qquad - (\rho,\eta,h)\,\mathbb{T}^{a}{}_{\theta\lambda}\,(\rho,\eta,h)\,\mathbb{P}^{\alpha}{}_{\beta\ \gamma a} \Big\} = 0. \end{array} \right.$$

Using another base of sections, we shall obtain new identities of Bianchi type necessary in the applications.

5.8 The (ρ,η)-(pseudo)metrizability

We consider the following diagram:

$$\begin{array}{ccc} E & & (F,[,]_{F,h},(\rho,\eta)) \\ \pi\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E,\pi,M) \in |\mathbf{B^v}|$ and $\Big((F,\nu,M)\,, [,]_{F,h}\,, (\rho,\eta)\Big) \in |\mathbf{GLA}|$. Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M) and let $((\rho,\eta)\,H, (\rho,\eta)\,V)$ be a distinguished linear (ρ,η)-connection for the generalized tangent bundle $((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E)$.

Definition 5.8.1 A tensor d-field

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab}\delta\tilde{y}^a \otimes \delta\tilde{y}^b \in \mathcal{DT}^{00}_{22}\left((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E\right)$$

will be called *pseudometrical structure* if its components are symmetric and the matrices $\|g_{\alpha\beta}\,(u_x)\|$and $\|g_{ab}\,(u_x)\|$ are nondegenerate, for any point $u_x \in E$.

Moreover, if the matrices $\|g_{\alpha\beta}\,(u_x)\|$ and $\|g_{ab}\,(u_x)\|$ has constant signature, then the tensor d-field G will be called *metrical structure.*

Let

$$G = g_{\alpha\beta} d\tilde{z}^\alpha \otimes d\tilde{z}^\beta + g_{ab}\delta\tilde{y}^a \otimes \delta\tilde{y}^b$$

be a (pseudo)metrical structure. If $\alpha,\beta \in \overline{1,p}$ and $a,b \in \overline{1,r}$, then for any vector local $(m+r)$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, we consider the real functions

$$\pi^{-1}(U) \xrightarrow{\tilde{g}^{\beta\alpha}} \mathbb{R}$$

and

$$\pi^{-1}(U) \xrightarrow{\tilde{g}^{ba}} \mathbb{R}$$

such that

$$\left\|\tilde{g}^{\beta\alpha}\,(u_x)\right\| = \|g_{\alpha\beta}\,(u_x)\|^{-1}\,,\ \forall u_x \in \pi^{-1}(U) \setminus \{0_x\}$$

and

$$\left\|\tilde{g}^{ba}\left(u_x\right)\right\| = \left\|g_{ab}\left(u_x\right)\right\|^{-1}, \ \forall u_x \in \pi^{-1}\left(U\right) \setminus \{0_x\}$$

respectively.

Definition 5.8.2 We will say that the *(pseudo)metrical structure*

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab}\delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b}$$

is Riemannian (pseudo)metrical structure if around each point $x \in M$ it exists a local vector $m+r$-chart (U, s_U) and a local m-chart (U, ξ_U) such that $g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x, y)$ and $g_{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x, y)$ depends only on x, for any $u_x \in \pi^{-1}(U)$.

If only the condition is verified:

"$g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x, y)$ depends only on x, for any $u_x \in \pi^{-1}(U)$" respectively "$g_{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x, y)$ depends only on x, for any $u_x \in \pi^{-1}(U)$", then we will say that the *(pseudo)metrical structure G is a Riemannian $\mathcal{H}$-(pseudo)metrical structure* respectively a *Riemannian $\mathcal{V}$-(pseudo)metrical structure.*

Definition 5.8.3 We will say that the *(pseudo)metrical structure*

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab}\delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b}$$

is a locally Minkowski structure if around each point $x \in M$ there exists a local vector $m+r$-chart (U, s_U) and a local m-chart (U, ξ_U) such that $g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x, y)$ and $g_{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x, y)$ depends only on y, *for any* $u_x \in \pi^{-1}(U)$.

If only the condition is verified:

"$g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x, y)$ depends only on y, for any $u_x \in \pi^{-1}(U)$" respectively "$g_{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x, y)$ depends only on y, for any $u_x \in \pi^{-1}(U)$", then we will say that *the (pseudo)metrical structure G is a (pseudo)metrical structure $\mathcal{H}$-locally Minkowski* and *$\mathcal{V}$-locally Minkowski, respectively.*

Definition 5.8.4 The generalized tangent bundle $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ will be called (ρ, η)-*(pseudo)metrizable* if there exists a (pseudo)metrical structure

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab}\delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b}$$

and a distinguished linear (ρ, η)-connection

$$\left(\left(\rho, \eta\right) H, \left(\rho, \eta\right) V\right)$$

such that

$$(5.8.1) \qquad \left(\rho, \eta\right) D_{\tilde{X}} G = 0, \ \forall \tilde{X} \in \Gamma\left(\left(\rho, \eta\right) TE, \left(\rho, \eta\right) \tau_E, E\right).$$

Condition (5.8.1) is equivalent with the following equalities:

$$(5.8.2) \qquad g_{\alpha\beta|\gamma} = 0, \ g_{ab|\gamma} = 0, \ g_{\alpha\beta} \left.\right|_c = 0, \ g_{ab} \left.\right|_c = 0.$$

If $g_{\alpha\beta|\gamma}$=0 and $g_{ab|\gamma}$=0, then we will say that *the vector bundle $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ is $\mathcal{H}$-(ρ, η)-(pseudo)metrizable.*

If $g_{\alpha\beta}|_c$=0 and $g_{ab}|_c$=0, then we will say that *the vector bundle $((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$ is $\mathcal{V}$-(ρ, η)-(pseudo)metrizable.*

Theorem 5.8.1 *If* $\left(\left(\rho,\eta\right)\overset{0}{H},\left(\rho,\eta\right)\overset{0}{V}\right)$ *is a distinguished linear* (ρ,η)*-connection for the generalized tangent bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *and* $G=g_{\alpha\beta}d\tilde{z}^\alpha\otimes d\tilde{z}^\beta+g_{ab}\delta\tilde{y}^a\otimes\delta\tilde{y}^b$ *is a (pseudo)metrical structure, then the following real local functions:*

$$
(5.8.3)\qquad
\begin{aligned}
\left(\rho,\eta\right)H_{\beta\gamma}^{\alpha} &= \frac{1}{2}\tilde{g}^{\alpha\varepsilon}\Big(\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\gamma\right)g_{\varepsilon\beta}\\
&\quad+\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\beta\right)g_{\varepsilon\gamma}-\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\varepsilon\right)g_{\beta\gamma}\\
&\quad+g_{\theta\varepsilon}L_{\gamma\beta}^{\theta}\circ h\circ\pi-g_{\beta\theta}L_{\gamma\varepsilon}^{\theta}\circ h\circ\pi-g_{\theta\gamma}L_{\beta\varepsilon}^{\theta}\circ h\circ\pi\Big),\\
\left(\rho,\eta\right)H_{b\gamma}^{a} &= \left(\rho,\eta\right)\overset{0}{H}{}_{b\gamma}^{a}+\frac{1}{2}\tilde{g}^{ac}g_{bc\overset{0}{|}\gamma},\\
\left(\rho,\eta\right)V_{\beta c}^{\alpha} &= \left(\rho,\eta\right)\overset{0}{V}{}_{\beta c}^{\alpha}+\frac{1}{2}\tilde{g}^{\alpha\varepsilon}g_{\beta\varepsilon\overset{0}{|}c},\\
\left(\rho,\eta\right)V_{bc}^{a} &= \frac{1}{2}\tilde{g}^{ae}\Big(\Gamma\left(\tilde{\rho},Id_E\right)\left(\dot{\tilde{\partial}}_c\right)g_{eb}\\
&\quad+\Gamma\left(\tilde{\rho},Id_E\right)\left(\dot{\tilde{\partial}}_b\right)g_{ec}-\Gamma\left(\tilde{\rho},Id_E\right)\left(\dot{\tilde{\partial}}_e\right)g_{bc}\Big)
\end{aligned}
$$

are components of a distinguished linear (ρ,η)*-connection such that the generalized tangent bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *becomes* (ρ,η)*-(pseudo)metrizable.*

Corollary 5.8.1 *If the distinguished linear* (ρ,η)*-connection* $\left(\left(\rho,\eta\right)\overset{0}{H},\left(\rho,\eta\right)\overset{0}{V}\right)$ *coincides with the Berwald linear* (ρ,η)*-connection, then the local real functions:*

$$
(5.8.4)\qquad
\begin{aligned}
\left(\rho,\eta\right)\overset{c}{H}{}_{\beta\gamma}^{\alpha} &= \frac{1}{2}\tilde{g}^{\alpha\varepsilon}\Big(\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\gamma\right)g_{\varepsilon\beta}+\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\beta\right)g_{\varepsilon\gamma}\\
&\quad-\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_\varepsilon\right)g_{\beta\gamma}+g_{\theta\varepsilon}L_{\gamma\beta}^{\theta}\circ h\circ\pi\\
&\quad-g_{\beta\theta}L_{\gamma\varepsilon}^{\theta}\circ h\circ\pi-g_{\theta\gamma}L_{\beta\varepsilon}^{\theta}\circ h\circ\pi\Big),\\
\left(\rho,\eta\right)\overset{c}{H}{}_{b\gamma}^{a} &= \frac{\partial\left(\rho,\eta\right)\Gamma_\gamma^a}{\partial y^b}+\frac{1}{2}\tilde{g}^{ac}g_{bc\overset{0}{|}\gamma},\\
\left(\rho,\eta\right)\overset{c}{V}{}_{\beta c}^{\alpha} &= \frac{1}{2}\tilde{g}^{\alpha\varepsilon}\frac{\partial g_{\beta\varepsilon}}{\partial y^c},\\
\left(\rho,\eta\right)\overset{c}{V}{}_{bc}^{a} &= \frac{1}{2}\tilde{g}^{ae}\left(\frac{\partial g_{e\beta}}{\partial y^c}+\frac{\partial g_{ec}}{\partial y^b}-\frac{\partial g_{bc}}{\partial y^e}\right)
\end{aligned}
$$

are the components of a distinguished linear (ρ,η)*-connection such that the generalized tangent bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *becomes* (ρ,η)*-(pseudo)metrizable.*

Moreover, if the (pseudo)metrical structure G *is* $\mathcal{H}$*- and* $\mathcal{V}$*-Riemannian, then the local real functions:*

$$
\begin{aligned}
&(\rho,\eta)\overset{c}{H}{}^{\alpha}_{\beta\gamma}=\frac{1}{2}\tilde{g}^{\alpha\varepsilon}\left(\rho^k_\gamma\circ h\circ\pi\frac{\partial g_{\varepsilon\beta}}{\partial x^k}+\rho^j_\beta\circ h\circ\pi\frac{\partial g_{\varepsilon\gamma}}{\partial x^j}-\rho^e_\varepsilon\circ h\circ\pi\frac{\partial g_{\beta\gamma}}{\partial x^e}+\right.\\
&\qquad\left.+g_{\theta\varepsilon}L^\theta_{\gamma\beta}\circ h\circ\pi-g_{\beta\theta}L^\theta_{\gamma\varepsilon}\circ h\circ\pi-g_{\theta\gamma}L^\theta_{\beta\varepsilon}\circ h\circ\pi\right),\\
&(\rho,\eta)\overset{c}{H}{}^{a}_{b\gamma}=\frac{\partial\left(\rho,\eta\right)\Gamma^a_\gamma}{\partial y^b}+\frac{1}{2}\tilde{g}^{ac}\left(\rho^i_\gamma\circ h\circ\pi\frac{\partial g_{bc}}{\partial x^i}-\frac{\partial\rho\Gamma^e_\gamma}{\partial y^b}g_{ec}-\frac{\partial\rho\Gamma^e_\gamma}{\partial y^c}g_{eb}\right),\\
&(\rho,\eta)\overset{c}{V}{}^{\alpha}_{\beta c}=0,\ \ (\rho,\eta)\overset{c}{V}{}^{a}_{bc}=0
\end{aligned}
\tag{5.8.5}
$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ becomes (ρ,η)-(pseudo)metrizable.

Theorem 5.8.2 *Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M). Let*

$$\left((\rho,\eta)\overset{0}{H},(\rho,\eta)\overset{0}{V}\right)$$

be a distinguished linear (ρ,η)-connection for $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ and let

$$G=g_{\alpha\beta}d\tilde{z}^\alpha\otimes d\tilde{z}^\beta+g_{ab}\delta\tilde{y}^a\otimes\delta\tilde{y}^b$$

be a (pseudo)metrical structure.

Let

$$
\begin{aligned}
O^{\alpha\varepsilon}_{\beta\gamma}&=\frac{1}{2}\left(\delta^\alpha_\beta\delta^\varepsilon_\gamma-g_{\beta\gamma}\tilde{g}^{\alpha\varepsilon}\right),\quad O^{*\alpha\varepsilon}_{\beta\gamma}=\frac{1}{2}\left(\delta^\alpha_\beta\delta^\varepsilon_\gamma+g_{\beta\gamma}\tilde{g}^{\alpha\varepsilon}\right),\\
O^{ae}_{bc}&=\frac{1}{2}\left(\delta^a_b\delta^e_c-g_{bc}\tilde{g}^{ae}\right),\qquad O^{*ae}_{bc}=\frac{1}{2}\left(\delta^a_b\delta^e_c+g_{bc}\tilde{g}^{ae}\right),
\end{aligned}
\tag{5.8.6}
$$

be the Obata operators.

If the real local functions $X^\alpha_{\beta\gamma},X^\alpha_{\beta c},Y^a_{b\gamma},Y^a_{bc}$ are components of tensor fields, then the local real functions are given in the following:

$$
\begin{aligned}
(\rho,\eta)\,H^\alpha_{\beta\gamma}&=(\rho,\eta)\,\overset{c}{H}{}^\alpha_{\beta\gamma}+O^{\alpha\varepsilon}_{\gamma\eta}X^\eta_{\varepsilon\beta},\\
(\rho,\eta)\,H^a_{b\gamma}&=(\rho,\eta)\,\overset{c}{H}{}^a_{b\gamma}+O^{ae}_{bd}Y^d_{e\gamma},\\
(\rho,\eta)\,V^\alpha_{\beta c}&=(\rho,\eta)\,\overset{c}{V}{}^\alpha_{\beta c}+O^{*\alpha\varepsilon}_{\beta\eta}X^\eta_{\varepsilon c},\\
(\rho,\eta)\,V^a_{bc}&=(\rho,\eta)\,\overset{c}{V}{}^a_{bc}+O^{*ae}_{bd}Y^d_{ec},
\end{aligned}
\tag{5.8.7}
$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ becomes (ρ,η)-(pseudo)metrizable.

Theorem 5.8.3 *Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M).*

If

$$\left((\rho,\eta)\overset{0}{H},(\rho,\eta)\overset{0}{V}\right)$$

is a distinguished linear (ρ,η)-connection for the generalized tangent bundle

$$((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$$

and

$$G=g_{\alpha\beta}d\tilde{z}^\alpha\otimes d\tilde{z}^\beta+g_{ab}\delta\tilde{y}^a\otimes\delta\tilde{y}^b$$

is a (pseudo)metrical structure, then the real local functions:

$$(5.8.8)\qquad \begin{aligned} (\rho,\eta)\,H^{\alpha}_{\beta\gamma} &= (\rho,\eta)\,\overset{0}{H}{}^{\alpha}_{\beta\gamma} + \frac{1}{2}\tilde{g}^{\alpha\varepsilon} g_{\underset{\varepsilon\beta|\gamma}{0}}, \\ (\rho,\eta)\,H^{a}_{b\gamma} &= (\rho,\eta)\,\overset{0}{H}{}^{a}_{b\gamma} + \frac{1}{2}\tilde{g}^{ae} g_{\underset{eb|\gamma}{0}}, \\ (\rho,\eta)\,V^{\alpha}_{\beta c} &= (\rho,\eta)\,\overset{0}{V}{}^{\alpha}_{\beta c} + \frac{1}{2}\tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta} \overset{0}{|}_{c}, \\ (\rho,\eta)\,V^{a}_{bc} &= (\rho,\eta)\,\overset{0}{V}{}^{a}_{bc} + \frac{1}{2}\tilde{g}^{ae} g_{eb} \overset{0}{|}_{c} \end{aligned}$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ becomes (ρ,η)-(pseudo)metrizable.

5.9 Generalized Lagrange (ρ,η)-spaces, Lagrange (ρ,η)-spaces and Finsler (ρ,η)-spaces

We consider the following diagram:

$$\begin{array}{ccc} E & & (F,[,]_{F,h},(\rho,\eta)) \\ \pi\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array}$$

such that $(E,\pi,M)=(F,\nu,N)$ and the generalized tangent bundle

$$((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$$

is (ρ,η)-(pseudo)metrizable.

Definition 5.9.1 A smooth *Lagrange fundamental function* on the vector bundle (E,π,M) is a mapping

$$E \overset{L}{\longrightarrow} \mathbb{R}$$

which satisfies the following conditions:

1. $L\circ u\in C^{\infty}(M)$, for any $u\in\Gamma(E,\pi,M)\setminus\{0\}$;
2. $L\circ 0\in C^{0}(M)$, where 0 means the null section of (E,π,M).

Let L be a Lagrangian defined on the total space of the vector bundle (E,π,M).

If (U,s_U) is a local vector $(m+r)$-chart for (E,π,M), then we obtain the following real functions defined on $\pi^{-1}(U)$:

$$(5.9.1)\qquad \begin{array}{ll} L_i \overset{put}{=} \dfrac{\partial L}{\partial x^i} \overset{put}{=} \dfrac{\partial}{\partial x^i}(L) & L_{ib} \overset{put}{=} \dfrac{\partial^2 L}{\partial x^i\partial y^b} \overset{put}{=} \dfrac{\partial}{\partial x^i}\left(\dfrac{\partial}{\partial y^b}(L)\right) \\ L_a \overset{put}{=} \dfrac{\partial L}{\partial y^a} \overset{put}{=} \dfrac{\partial}{\partial y^a}(L) & L_{ab} \overset{put}{=} \dfrac{\partial^2 L}{\partial y^a\partial y^b} \overset{put}{=} \dfrac{\partial}{\partial y^a}\left(\dfrac{\partial}{\partial y^b}(L)\right). \end{array}$$

Definition 5.9.2 If for any local vector $m+r$-chart (U,s_U) of (E,π,M), we have:

$$(5.9.2)\qquad rank\,\|L_{ab}(u_x)\| = r,$$

for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$, then we will say that *the Lagrangian L is regular.*

Proposition 5.9.1 *If the Lagrangian L is regular, then for any local vector $m+r$-chart (U, s_U) of (E, π, M), we obtain the real functions $\tilde{L}^{ab}$ locally defined by*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{L^{ab}} & \mathbb{R} \\ u_x & \longmapsto & \tilde{L}^{ab}(u_x) \end{array}, \tag{5.9.3}$$

where $\left\|\tilde{L}^{ab}(u_x)\right\| = \|L_{ab}(u_x)\|^{-1}$, for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 5.9.3 A smooth *Finsler fundamental function* on the vector bundle (E, π, M) is a mapping

$$E \xrightarrow{F} \mathbb{R}_+$$

which satisfies the following conditions:

1. $F \circ u \in C^\infty(M)$, for any $u \in \Gamma(E, \pi, M) \setminus \{0\}$;
2. $F \circ 0 \in C^0(M)$, where 0 means the null section of (E, π, M);
3. F is positively 1-homogenous on the fibres of vector bundle (E, π, M);
4. For any local vector $m + r$-chart (U, s_U) of (E, π, M), the hessian:

$$\left\|F^2_{ab}(u_x)\right\| \tag{5.9.4}$$

is positively define for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 5.9.4 If the (pseudo)metrical structure G is determined by a (pseudo)metrical structure

$$g \in \mathcal{T}^{\,0}_{\,2}(V(\rho, \eta) TE, (\rho, \eta), \tau_E, E),$$

then the (ρ, η)-(pseudo)metrizable vector bundle

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

will be called the *generalized Lagrange (ρ, η)-space.*

In particular, if the (pseudo)metrical structure g is determined with the help of a Lagrange fundamental function or Finsler fundamental function, then the (ρ, η)-(pseudo)metrizable vector bundle

$$((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

will be called the *Lagrange (ρ, η)-space* or the *Finsler (ρ, η)-space, respectively.*

The generalized Lagrange (Id_{TM}, Id_M)-space, the Lagrange (Id_{TM}, Id_M)-space, and the Finsler (Id_{TM}, Id_M)-space will be called the *generalized Lagrange space, Lagrange space*, *Finsler space.*

Definition 5.9.5 The normal distinguished linear (ρ, η)-connections of a Lagrange or Finsler (ρ, η)-space will be called *Lagrange* or *Finsler linear (ρ, η)-connections.*

The Lagrange and Finsler linear (Id_{TM}, Id_M)-connections will be called *Lagrange* and *Finsler linear connections*, respectively.

Theorem 5.9.1 *If the (pseudo)metrical structure G is determined by a (pseudo)metrical structure*

$$g\in \mathcal{T}\,_{2}^{0}\left(V\left(\rho ,\eta \right) TE,\left(\rho ,\eta \right) ,\tau _{E},E\right) ,$$

then, the real local functions:

$$\begin{array}{ll} & \left(\rho ,\eta \right) H_{bc}^{a}=\dfrac{1}{2}\tilde{g}^{ae}\left(\Gamma \left(\tilde{\rho},Id_{E}\right) \left(\delta _{b}\right) g_{ec}+\Gamma \left(\tilde{\rho},Id_{E}\right) \left(\delta _{c}\right) g_{be}-\Gamma \left(\tilde{\rho},Id_{E}\right) \left(\delta _{e}\right) g_{bc}\right. \\ & \qquad \left. -g_{cd}L_{be}^{d}\circ \left(h\circ \pi \right) +g_{bd}L_{ec}^{d}\circ \left(h\circ \pi \right) -g_{ed}L_{bc}^{d}\circ \left(h\circ \pi \right) \right) , \\ (5.9.5) & \left(\rho ,\eta \right) V_{bc}^{a}=\dfrac{1}{2}\tilde{g}^{ae}\left(\Gamma \left(\tilde{\rho},Id_{E}\right) \left(\dot{\tilde{\partial}}_{c}\right) g_{eb}\right. \\ & \qquad \left. +\Gamma \left(\tilde{\rho},Id_{E}\right) \left(\dot{\tilde{\partial}}_{b}\right) g_{ec}-\Gamma \left(\tilde{\rho},Id_{E}\right) \left(\dot{\tilde{\partial}}_{e}\right) g_{bc}\right) \end{array}$$

are the components of a normal distinguished linear $\left(\rho ,\eta \right)$-connection with $\left(\rho ,\eta \right)$-$\mathcal{H}\left(\mathcal{HH}\right)$ and $\left(\rho ,\eta \right)$-$\mathcal{V}\left(\mathcal{VV}\right)$ torsions free such that the generalized tangent bundle $\left(\left(\rho ,\eta \right) TE,\left(\rho ,\eta \right) \tau _{E},E\right)$ derives generalized Lagrange $\left(\rho ,\eta \right)$-space.

This normal distinguished linear (ρ ,η)-connection will be called *generalized linear (ρ ,η)-connection of Levi-Civita type.*

If the *(pseudo)metrical structure* g is determined with the help of a Lagrange and Finsler fundamental function, then the Lagrange and Finsler linear (ρ ,η)-connections will be called *canonical Lagrange* and *Finsler linear (ρ ,η)-connection, respectively.*

The canonical Lagrange and Finsler linear (Id_{TM},Id_{M})-connection will be called the *canonical Lagrange* and *Finsler linear connection* respectively.

Theorem 5.9.2 *Let $\left(\left(\rho ,\eta \right) H,\left(\rho ,\eta \right) V\right)$ be the normal distinguished linear (ρ ,η)-connection presented in the previous theorem.*

If

$$\mathbb{T}_{bc}^{a}\tilde{\delta}_{a}\otimes d\tilde{z}^{b}\otimes d\tilde{z}^{c}\in \mathcal{T}_{20}^{10}\left(\left(\rho ,\eta \right) TE,\left(\rho ,\eta \right) \tau _{E},E\right)$$

and

$$\mathbb{S}_{bc}^{a}\dot{\tilde{\partial}}_{a}\otimes \delta \tilde{y}^{b}\otimes \delta \tilde{y}^{c}\in \mathcal{T}_{02}^{01}\left(\left(\rho ,\eta \right) TE,\left(\rho ,\eta \right) \tau _{E},E\right)$$

such that they satisfy the conditions:

$$\mathbb{T}_{bc}^{a}=-\mathbb{T}_{cb}^{a},\ \mathbb{S}_{bc}^{a}=-\mathbb{S}_{cb}^{a},\ \forall b,c\in \overline{1,n},$$

then the following real local functions:

$$\begin{array}{ll} & \left(\rho ,\eta \right) \tilde{H}_{bc}^{a}=\left(\rho ,\eta \right) H_{bc}^{a}+\dfrac{1}{2}\tilde{g}^{ae}\left(g_{ed}\mathbb{T}_{bc}^{d}-g_{bd}\mathbb{T}_{ec}^{d}+g_{cd}\mathbb{T}_{be}^{d}\right) , \\ (5.9.6) & \\ & \left(\rho ,\eta \right) \tilde{V}_{bc}^{a}=\left(\rho ,\eta \right) V_{bc}^{a}+\dfrac{1}{2}\tilde{g}^{ae}\left(g_{ed}\mathbb{S}_{bc}^{d}-g_{bd}\mathbb{S}_{ec}^{d}+g_{cd}\mathbb{S}_{be}^{d}\right) \end{array}$$

are the components of a normal distinguished linear (ρ ,η)-connection with (ρ ,η)-$\mathcal{H}\left(\mathcal{HH}\right)$ and (ρ ,η)-$\mathcal{V}\left(\mathcal{VV}\right)$ torsions a priori given such that the generalized tangent bundle $\left(\left(\rho ,\eta \right) TE,\left(\rho ,\eta \right) \tau _{E},E\right)$ derives generalized Lagrange (ρ ,η)-space.

Moreover, we obtain:

$$\begin{array}{ll} & \mathbb{T}_{bc}^{a}=\left(\rho ,\eta \right) H_{bc}^{a}-\left(\rho ,\eta \right) H_{cb}^{a}-L_{bc}^{a}\circ h\circ \pi , \\ (5.9.87 & \\ & \mathbb{S}_{bc}^{a}=\left(\rho ,\eta \right) V_{bc}^{a}-\left(\rho ,\eta \right) V_{cb}^{a}. \end{array}$$

5.10 Einstein equations

We shall consider a metric structure

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab} \delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b}$$

and a distinguished linear (ρ, η)-connection $((\rho, \eta) H, (\rho, \eta) V)$ compatible with the structure metric G having $\mathcal{H}(\mathcal{HH})$ and $\mathcal{V}(\mathcal{VV})$-torsions prescribed.

Definition 5.10.1 If $(\rho, \eta, h) \mathbb{R}_{\ \alpha\ \beta}$ and $(\rho, \eta, h) \mathbb{S}_{\ a\ b}$ are the components of tensor Ricci associated to distinguished linear (ρ, η)-connection

$$((\rho, \eta) H, (\rho, \eta) V),$$

then the scalar

$$(\rho, \eta, h) \mathbb{R} = (\rho, \eta, h) \mathbb{R}_{\ \alpha\ \beta} \tilde{g}^{\alpha\beta} + (\rho, \eta, h) \mathbb{S}_{\ a\ b} \tilde{g}^{ab} \tag{5.10.1}$$

will be called the *scalar of curvature of distinguished linear (ρ, η)-connection* $((\rho, \eta) H, (\rho, \eta) V)$.

Definition 5.10.2 The tensor field

$$\begin{aligned} (\rho, \eta, h) \mathbb{T} &= (\rho, \eta, h) \mathbb{T}_{\ \alpha\ \beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + (\rho, \eta, h) \mathbb{T}_{\ \alpha\ b} d\tilde{z}^{\alpha} \otimes \delta\tilde{y}^{b} \\ &\quad + (\rho, \eta, h) \mathbb{T}_{\ a\ \beta} \delta\tilde{y}^{a} \otimes d\tilde{z}^{\alpha} + (\rho, \eta, h) \mathbb{T}_{\ a\ b} \delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b} \end{aligned} \tag{5.10.2}$$

such that its components verify the following conditions:

$$\begin{aligned} \varkappa (\rho, \eta, h) \mathbb{T}_{\ \alpha\ \beta} &= (\rho, \eta, h) \mathbb{R}_{\ \alpha\ \beta} - \frac{1}{2} (\rho, \eta, h) \mathbb{R} \cdot g_{\alpha\beta}, \\ -\varkappa (\rho, \eta, h) \mathbb{T}_{\ \alpha\ b} &= (\rho, \eta, h) \mathbb{P}_{\ \alpha\ b}, \\ \varkappa (\rho, \eta, h) \mathbb{T}_{\ a\ \beta} &= (\rho, \eta, h) \mathbb{P}_{\ a\ \beta}, \\ \varkappa (\rho, \eta, h) \mathbb{T}_{\ a\ b} &= (\rho, \eta, h) \mathbb{S}_{\ a\ b} - \frac{1}{2} (\rho, \eta, h) \mathbb{R} \cdot g_{ab}, \end{aligned} \tag{5.10.3}$$

where $\varkappa$ is a constant, will be called *the energy-momentum tensor field associated to distinguished linear (ρ, η)-connection $((\rho, \eta) H, (\rho, \eta) V)$ and metrical structure G.*

The equations (5.10.3) will be called *the Einstein equations associated to distinguished linear (ρ, η)-connection $((\rho, \eta) H, (\rho, \eta) V)$ and metrical structure G.*

Formally, the Einstein equations will be written

$$\mathbf{Ric}((\rho, \eta) H, (\rho, \eta) V) - \frac{1}{2} (\rho, \eta, h) \mathbb{R} \cdot G = \varkappa \cdot (\rho, \eta, h) \mathbb{T}. \tag{5.10.3'}$$

5.11 Mechanical systems

Using the diagram:

$$\begin{array}{ccc} E & & \left(E, [,]_{E,h}, (\rho, \eta)\right) \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{h} & M \end{array} \tag{5.11.1}$$

where $\left(\left(E,\pi,M\right),[,]_{E,h},\left(\rho,\eta\right)\right)$ is a generalized Lie algebroid, we build the generalized tangent bundle

$$(((\rho,\eta)TE,(\rho,\eta)\tau_E,E),[,]_{(\rho,\eta)TE},(\tilde{\rho},Id_E)).$$

Definition 5.11.1 A triple

$$\left(\left(E,\pi,M\right),F_e,\left(\rho,\eta\right)\Gamma\right), \tag{5.11.2}$$

where

$$F_e=F^a\frac{\partial}{\partial\tilde{y}^a}\in\Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right) \tag{5.11.3}$$

is an external force and $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection, will be called the *mechanical* (ρ,η)*-system.*

A mechanical (ρ,η)-system

$$\left(\left(E,\pi,M\right),F_e,\left(\rho,\eta\right)\Gamma\right)$$

endowed with a (pseudo)metrical structure G determined with the help of a (pseudo)metrical structure

$$g=g_{ab}d\tilde{y}^a\otimes d\tilde{y}^b\in\mathcal{T}\,^{00}_{02}\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$$

will be denoted

$$\left(\left(E,\pi,M\right),F_e,\left(\rho,\eta\right)\Gamma,G\right). \tag{5.11.4}$$

and will be called *generalized Lagrange mechanical* (ρ,η)*-system.*

Any mechanical (Id_{TM},Id_M)-system and any generalized Lagrange mechanical (Id_{TM},Id_M)-system will be called *mechanical system* and *generalized Lagrange mechanical system,* respectively.

Definition 5.11.2 If L (respectively F) is a smooth Lagrange (respectively Finsler function), then we put the triples

$$\left(\left(E,\pi,M\right),F_e,L\right)\quad(\text{ respectively }(E,F_e,F))$$

where $F_e=F^a\dfrac{\partial}{\partial\tilde{y}^a}\in\Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$ is an external force. These are called *Lagrange mechanical* (ρ,η)*-system* (*Finsler mechanical* (ρ,η)*-system, respectively*).

Any Lagrange mechanical (Id_{TM},Id_M)-system and any Finsler mechanical (Id_{TM},Id_M)-system will be called *Lagrange mechanical system* and *Finsler mechanical system*, respectively.

5.11.1 (ρ,η)-semisprays and (ρ,η)-sprays for mechanical (ρ,η)-systems

Let $\left(\left(E,\pi,M\right),F_e,(\rho,\eta)\Gamma\right)$ be an arbitrary mechanical (ρ,η)-system.

Definition 5.11.1.1 The vertical section

$$\mathbb{C}=y^a\dot{\tilde{\partial}}_a, \tag{5.11.1.1}$$

will be called the *Liouville section.*

Definition 5.11.1.2 The section $S\in\Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right)$ will be called (ρ,η)-*semispray* if there exists an almost tangent structure e such that

$$e\left(S\right)=\mathbb{C}. \tag{5.11.1.2}$$

Let $g\in\mathbf{Man}\left(E,E\right)$ be such that (g,h) is a locally invertible $\mathbf{B^v}$-morphism of (E,π,M) source and (E,π,M) target.

Theorem 5.11.1.1 *The section*

$$S=\left(g_b^a\circ h\circ\pi\right)y^b\frac{\partial}{\partial\tilde{z}^a}-2\left(G^a-\frac{1}{4}F^a\right)\frac{\partial}{\partial\tilde{y}^a} \tag{5.11.1.3}$$

is a (ρ,η)-semispray such that the real local functions $G^a,\ a\in\overline{1,n}$, satisfy the following conditions

$$\begin{array}{rl}(\rho,\eta)\,\Gamma_c^a & =\tilde{g}_c^e\circ h\circ\pi\dfrac{\partial\left(G^a-\frac{1}{4}F^a\right)}{\partial y^e}\\ & -\dfrac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi,\ a,b\in\overline{1,r}.\end{array} \tag{5.11.1.4}$$

In addition, we remark that the local real functions

$$\begin{array}{rl}(\rho,\eta)\,\mathring{\Gamma}_c^a & \overset{put}{=}\tilde{g}_c^e\circ h\circ\pi\frac{\partial G^a}{\partial y^e}\\ & -\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi,\ a,b\in\overline{1,r}\end{array} \tag{5.11.1.5}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\,\mathring{\Gamma}$ for the vector bundle (E,π,M).

The (ρ,η)-semispray S will be called *the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $\left((E,\pi,M),F_e,(\rho,\eta)\,\Gamma\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).*

Proof. We consider the **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right) & \xrightarrow{\mathbb{P}} & \Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right)\\ X & \longmapsto & \mathcal{J}_{(g,h)}\left[S,X\right]_{(\rho,\eta)TE}-\left[S,\mathcal{J}_{(g,h)}X\right]_{(\rho,\eta)TE}.\end{array}$$

Let $X=\tilde{Z}^a\tilde{\partial}_a+Y^a\dot{\tilde{\partial}}_a$ be an arbitrary section. Since

$$\begin{array}{rl}\left[S,X\right]_{(\rho,\eta)TE} & =\left[\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_a,\tilde{Z}^b\tilde{\partial}_b\right]_{(\rho,\eta)TE}+\left[\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_a,Y^b\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE}\\ & -\left[2\left(G^a-\dfrac{1}{4}F^a\right)\dot{\tilde{\partial}}_a,\tilde{Z}^b\tilde{\partial}_b\right]_{(\rho,\eta)TE}-\left[2\left(G^a-\dfrac{1}{4}F^a\right)\dot{\tilde{\partial}}_a,Y^b\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE}\end{array}$$

and

$$\begin{array}{rl}\left[\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_a,\tilde{Z}^b\tilde{\partial}_b\right]_{(\rho,\eta)TE} & =\left(g_e^a\circ h\circ\pi\cdot y^e\right)\rho_a^i\circ h\circ\pi\dfrac{\partial\tilde{Z}^c}{\partial x^i}\tilde{\partial}_c\\ & -\tilde{Z}^b\rho_b^j\circ h\circ\pi\dfrac{\partial\left(g_e^c\circ h\circ\pi\cdot y^e\right)}{\partial x^j}\tilde{\partial}_c\\ & +\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{Z}^bL_{ab}^c\tilde{\partial}_c,\end{array}$$

$$\left[\left(g_e^a \circ h \circ \pi \cdot y^e\right) \tilde{\partial}_a, Y^b \dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE} = \left(g_e^a \circ h \circ \pi \cdot y^e\right) \rho_a^i \circ h \circ \pi \frac{\partial Y^c}{\partial x^i} \dot{\tilde{\partial}}_c - Y^b \frac{\partial \left(g_e^c \circ h \circ \pi \cdot y^e\right)}{\partial y^b} \tilde{\partial}_c,$$

$$\left[2\left(G^a - \frac{1}{4}F^a\right) \dot{\tilde{\partial}}_a, \tilde{Z}^b \tilde{\partial}_b\right]_{(\rho,\eta)TE} = 2\left(G^a - \frac{1}{4}F^a\right) \frac{\partial \tilde{Z}^c}{\partial y^a} \tilde{\partial}_c - 2\tilde{Z}^b \rho_b^j \circ h \circ \pi \frac{\partial \left(G^c - \frac{1}{4}F^c\right)}{\partial x^j} \dot{\tilde{\partial}}_c,$$

$$\left[2\left(G^a - \tfrac{1}{4}F^a\right) \dot{\tilde{\partial}}_a, Y^b \dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE} = 2\left(G^a - \tfrac{1}{4}F^a\right) \tfrac{\partial Y^c}{\partial y^a} \dot{\tilde{\partial}}_c - 2Y^b \frac{\partial \left(G^c - \frac{1}{4}F^c\right)}{\partial y^b} \dot{\tilde{\partial}}_c,$$

it results that

$$(P_1) \qquad \begin{aligned} \mathcal{J}_{(g,h)} [S, X]_{(\rho,\eta)TE} &= \left(g_e^a \circ h \circ \pi \cdot y^e\right) \rho_a^i \circ h \circ \pi \frac{\partial \tilde{Z}^c}{\partial x^i} \cdot \tilde{g}_c^d \circ h \circ \pi \dot{\tilde{\partial}}_d \\ &- \tilde{Z}^b \rho_b^j \circ h \circ \pi \frac{\partial \left(g_e^c \circ h \circ \pi \cdot y^e\right)}{\partial x^j} \cdot \tilde{g}_c^d \circ h \circ \pi \dot{\tilde{\partial}}_d \\ &+ \left(g_e^a \circ h \circ \pi \cdot y^e\right) \tilde{Z}^b L_{ab}^c \cdot \tilde{g}_c^d \circ h \circ \pi \dot{\tilde{\partial}}_d \\ &- Y^b \frac{\partial \left(g_e^c \circ h \circ \pi \cdot y^e\right)}{\partial y^b} \cdot \tilde{g}_c^d \circ h \circ \pi \dot{\tilde{\partial}}_d \\ &- 2\left(G^a - \frac{1}{4}F^a\right) \frac{\partial \tilde{Z}^c}{\partial y^a} \cdot \tilde{g}_c^d \circ h \circ \pi \dot{\tilde{\partial}}_d. \end{aligned}$$

Since

$$\begin{aligned} \left[S, \mathcal{J}_{(g,h)} X\right]_{(\rho,\eta)TE} &= \left[\left(g_e^a \circ h \circ \pi \cdot y^e\right) \tilde{\partial}_a, \tilde{g}_b^c \circ h \circ \pi \tilde{Z}^b \dot{\tilde{\partial}}_c\right]_{(\rho,\eta)TE} \\ &- \left[2\left(G^a - \frac{1}{4}F^a\right) \dot{\tilde{\partial}}_a, \tilde{g}_b^c \circ h \circ \pi \tilde{Z}^b \dot{\tilde{\partial}}_c\right]_{(\rho,\eta)TE} \end{aligned}$$

and

$$\left[\left(g_e^a \circ h \circ \pi \cdot y^e\right) \tilde{\partial}_a, \tilde{g}_b^c \circ h \circ \pi \tilde{Z}^b \dot{\tilde{\partial}}_c\right]_{(\rho,\eta)TE} = \left(g_e^a \circ h \circ \pi \cdot y^e\right) \rho_a^i \circ h \circ \pi \frac{\partial \tilde{g}_b^d \circ h \circ \pi \tilde{Z}^b}{\partial x^i} \dot{\tilde{\partial}}_d - \tilde{g}_b^c \circ h \circ \pi \cdot \tilde{Z}^b \frac{\partial \left(g_e^d \circ h \circ \pi \cdot y^e\right)}{\partial y^c} \tilde{\partial}_d,$$

$$\left[2\left(G^a - \frac{1}{4}F^a\right) \dot{\tilde{\partial}}_a, \tilde{g}_b^c \circ h \circ \pi \tilde{Z}^b \dot{\tilde{\partial}}_c\right]_{(\rho,\eta)TE} = 2\left(G^a - \frac{1}{4}F^a\right) \frac{\partial \left(\tilde{g}_b^d \circ h \circ \pi \cdot \tilde{Z}^b\right)}{\partial y^a} \dot{\tilde{\partial}}_d - 2\tilde{g}_b^c \circ h \circ \pi \cdot \tilde{Z}^b \frac{\partial \left(G^d - \frac{1}{4}F^d\right)}{\partial y^c} \dot{\tilde{\partial}}_d$$

it results that

$$(P_2)\qquad \begin{aligned}\left[S,\mathcal{J}_{(g,h)}X\right]_{(\rho,\eta)TE} &= \left(g_e^a\circ h\circ\pi\cdot y^e\right)\rho_a^i\circ h\circ\pi\frac{\partial\tilde{g}_b^d\circ h\circ\pi\tilde{Z}^b}{\partial x^i}\dot{\tilde{\partial}}_d\\ &-\tilde{g}_b^c\circ h\circ\pi\cdot\tilde{Z}^b\frac{\partial\left(g_e^d\circ h\circ\pi\cdot y^e\right)}{\partial y^c}\tilde{\partial}_d\\ &-2\left(G^a-\frac{1}{4}F^a\right)\frac{\partial\left(\tilde{g}_b^d\circ h\circ\pi\cdot\tilde{Z}^b\right)}{\partial y^a}\dot{\tilde{\partial}}_d\\ &+2\tilde{g}_b^c\circ h\circ\pi\cdot\tilde{Z}^b\frac{\partial\left(G^d-\frac{1}{4}F^d\right)}{\partial y^c}\dot{\tilde{\partial}}_d.\end{aligned}$$

We remark that

$$\begin{aligned}\left(g_e^a\circ h\circ\pi\cdot y^e\right)\rho_a^i\circ h\circ\pi\frac{\partial\tilde{g}_b^d\circ h\circ\pi\tilde{Z}^b}{\partial x^i} &= g_e^a\circ h\circ\pi\cdot y^e\rho_a^i\circ h\circ\pi\frac{\partial\tilde{Z}^c}{\partial x^i}\cdot\tilde{g}_c^d\circ h\circ\pi\\ &-\tilde{Z}^b\rho_b^j\circ h\circ\pi\frac{\partial\left(g_e^c\circ h\circ\pi\cdot y^e\right)}{\partial x^j}\cdot\tilde{g}_c^d\circ h\circ\pi,\end{aligned}$$

$$Y^b = Y^b\frac{\partial\left(g_e^c\circ h\circ\pi\cdot y^e\right)}{\partial y^b}\cdot\tilde{g}_c^d\circ h\circ\pi$$

and

$$\tilde{Z}^d = \tilde{g}_b^c\circ h\circ\pi\cdot\tilde{Z}^b\frac{\partial\left(g_e^d\circ h\circ\pi\cdot y^e\right)}{\partial y^c}.$$

Using equalities (P_1) and (P_2), we obtain:

$$\mathbb{P}\left(\tilde{Z}^a\tilde{\partial}_a+Y^a\dot{\tilde{\partial}}_a\right)=\tilde{Z}^a\tilde{\partial}_a+$$
$$+\left(-Y^a-2\tilde{g}_b^c\circ h\circ\pi\frac{\partial\left(G^a-\frac{1}{4}F^a\right)}{\partial y^c}\tilde{Z}^b+\left(g_e^d\circ h\circ\pi\cdot y^e\right)\tilde{Z}^bL_{db}^c\circ h\circ\pi\cdot\tilde{g}_c^a\circ h\circ\pi\right)\dot{\tilde{\partial}}_a.$$

After some calculations, it results that $\mathbb{P}$ is an almost product structure.

Using the equality

$$\mathbb{P}=Id-2\left(\rho,\eta\right)\Gamma,$$

we obtain that

$$\left(\rho,\eta\right)\Gamma\left(\tilde{Z}^a\tilde{\partial}_a+Y^a\dot{\partial}_a\right)=$$
$$=\left(Y^a+\tilde{g}_b^c\circ h\circ\pi\frac{\partial\left(G^a-\frac{1}{4}F^a\right)}{\partial y^c}\tilde{Z}^b-\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)\tilde{Z}^bL_{db}^c\circ h\circ\pi\cdot\tilde{g}_c^a\circ h\circ\pi\right)\dot{\tilde{\partial}}_a$$

Since

$$\left(\rho,\eta\right)\Gamma\left(\tilde{Z}^a\tilde{\partial}_a+Y^a\dot{\tilde{\partial}}_a\right)=\left(Y^a+\left(\rho,\eta\right)\Gamma_b^a\tilde{Z}^b\right)\dot{\tilde{\partial}}_a$$

it results that relations (5.11.1.4) are satisfied. In addition, since

$$\left(\rho,\eta\right)\mathring{\Gamma}_c^a=\left(\rho,\eta\right)\Gamma_c^a+\frac{1}{4}\tilde{g}_c^d\circ h\circ\pi\frac{\partial F^a}{\partial y^d}$$

and

$$
\begin{aligned}
(\rho,\eta)\,\mathring{\Gamma}_{\acute{c}}^{\acute{a}} &= (\rho,\eta)\,\Gamma_{\acute{c}}^{\acute{a}} + \frac{1}{2}\tilde{g}_{\acute{c}}^{\acute{b}}\circ h\circ\pi\frac{\partial F^{\acute{a}}}{\partial y^{\acute{b}}}\\
&= M_{a}^{\acute{a}}\circ\pi\left(\rho_{c}^{i}\circ h\circ\pi\cdot\frac{\partial M_{\acute{b}}^{a}}{\partial x^{i}}y^{\acute{b}} + (\rho,\eta)\,\Gamma_{c}^{a}\right)M_{\acute{c}}^{c}\circ h\circ\pi\\
&+M_{a}^{\acute{a}}\circ\pi\left(\frac{1}{4}\tilde{g}_{c}^{b}\circ h\circ\pi\cdot\frac{\partial F^{a}}{\partial y^{b}}\right)M_{\acute{c}}^{c}\circ h\circ\pi\\
&= M_{a}^{\acute{a}}\circ\pi\left(\rho_{c}^{i}\circ h\circ\pi\cdot\frac{\partial M_{\acute{b}}^{a}}{\partial x^{i}}y^{\acute{b}} + \left((\rho,\eta)\,\Gamma_{c}^{a} + \frac{1}{4}\tilde{g}_{c}^{b}\circ h\circ\pi\cdot\frac{\partial F^{a}}{\partial y^{b}}\right)\right)M_{\acute{c}}^{c}\circ h\circ\pi\\
&= M_{a}^{\acute{a}}\circ\pi\left(\rho_{c}^{i}\circ h\circ\pi\cdot\frac{\partial M_{\acute{b}}^{a}}{\partial x^{i}}y^{\acute{b}} + (\rho,\eta)\,\mathring{\Gamma}_{c}^{a}\right)M_{\acute{c}}^{c}\circ h\circ\pi
\end{aligned}
$$

it results the conclusion of the theorem. *q.e.d.*

Remarks

1. If $(\rho,\eta) = (Id_{TM}, Id_M)$, $(g,h) = (Id_E, Id_M)$, and $F_e \neq 0$, then we obtain the canonical semispray associated to connection Γ which is not the same canonical semispray presented by I. Bucataru and R. Miron in [7].
2. If $(\rho,\eta) = (Id_{TM}, Id_M)$, $(g,h) = (Id_E, Id_M)$, and $F_e = 0$, then we obtain the canonical semispray associated to connection Γ which is not the classical canonical semispray associated to connection Γ.

Using *Theorem 5.11.1.1*, we obtain the following:

Theorem 5.11.1.2 *The following properties hold good:*

1° *Since* $\mathring{\tilde{\delta}}_c = \tilde{\partial}_c - (\rho,\eta)\,\mathring{\Gamma}_c^a\dot{\tilde{\partial}}_a,\ c\in\overline{1,r}$, *it results that*

$$
\mathring{\tilde{\delta}}_c = \tilde{\delta}_c - \frac{1}{4}\tilde{g}_c^b\circ h\circ\pi\cdot\frac{\partial F^a}{\partial y^b}\dot{\tilde{\partial}}_a,\ c\in\overline{1,r}. \tag{5.11.1.6}
$$

2° *Since* $\mathring{\delta}\tilde{y}^a = (\rho,\eta)\,\mathring{\Gamma}_c^a d\tilde{z}^c + d\tilde{y}^a$, *it results that*

$$
\mathring{\delta}\tilde{y}^a = \delta\tilde{y}^a + \frac{1}{4}\tilde{g}_c^b\circ h\circ\pi\frac{\partial F^a}{\partial y^b}d\tilde{z}^c,\ a\in\overline{1,r}. \tag{5.11.1.7}
$$

Theorem 5.11.1.3 *The real local functions*

$$
\left(\frac{\partial\,(\rho,\eta)\,\Gamma_c^a}{\partial y^b}, \frac{\partial\,(\rho,\eta)\,\Gamma_c^a}{\partial y^b}, 0,\ 0\right),\ a,b,c\in\overline{1,r}, \tag{5.11.1.8}
$$

and

$$
\left(\frac{\partial\,(\rho,\eta)\,\mathring{\Gamma}_c^a}{\partial y^b}, \frac{\partial\,(\rho,\eta)\,\mathring{\Gamma}_c^a}{\partial y^b}, 0,\ 0\right),\ a,b,c\in\overline{1,r}, \tag{5.11.1.8$'$}
$$

respectively, are the coefficients to a normal Berwald linear (ρ,η)-connection for the generalized tangent bundle $((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E)$.

Theorem 5.11.1.4 *The tensor of integrability of the (ρ,η)-connection $(\rho,\eta)\overset{\circ}{\Gamma}$ is as follows:*

$$\begin{array}{c}(\rho,\eta,h)\,\overset{\circ}{\mathbb{R}}{}^{a}_{cd}=(\rho,\eta,h)\,\mathbb{R}^{a}_{cd}+\dfrac{1}{4}\left(\tilde{g}^{e}_{d}\circ h\circ\pi\dfrac{\partial F^{a}}{\partial y^{e}}_{|c}-\tilde{g}^{e}_{c}\circ h\circ\pi\dfrac{\partial F^{a}}{\partial y^{e}}_{|d}\right)\\ +\dfrac{1}{16}\left(\tilde{g}^{e}_{d}\circ h\circ\pi\dfrac{\partial F^{b}}{\partial y^{e}}\tilde{g}^{f}_{c}\circ h\circ\pi\dfrac{\partial^{2}F^{a}}{\partial y^{b}\partial y^{f}}-\tilde{g}^{f}_{c}\circ h\circ\pi\dfrac{\partial F^{b}}{\partial y^{f}}\tilde{g}^{e}_{d}\circ h\circ\pi\dfrac{\partial^{2}F^{a}}{\partial y^{b}\partial y^{e}}\right)\\ +\dfrac{1}{4}\left(L^{f}_{cd}\circ h\circ\pi\right)\left(\tilde{g}^{e}_{f}\circ h\circ\pi\right)\dfrac{\partial F^{a}}{\partial y^{e}},\end{array}\tag{5.11.1.9}$$

where $_{|c}$ is the h-covariant derivation with respect to the normal Berwald linear (ρ,η)-connection (5.11.1.8).

Proof. Since

$$\begin{aligned}(\rho,\eta,h)\,\overset{\circ}{\mathbb{R}}{}^{a}_{cd}=\;&\Gamma(\tilde{\rho},Id_{E})\left(\overset{\circ}{\tilde{\delta}}_{c}\right)\left((\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{d}\right)-\Gamma(\tilde{\rho},Id_{E})\left(\overset{\circ}{\tilde{\delta}}_{d}\right)\left((\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{c}\right)\\ &+L^{e}_{cd}\circ h\circ(h\circ\pi)\,(\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{e},\end{aligned}$$

and

$$\begin{aligned}\Gamma(\tilde{\rho},Id_{E})\left(\overset{\circ}{\tilde{\delta}}_{c}\right)\left((\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{d}\right)\;&=\Gamma(\tilde{\rho},Id_{E})\left(\tilde{\delta}_{c}\right)\left((\rho,\eta)\,\Gamma^{a}_{d}\right)\\ &+\frac{1}{4}\Gamma(\tilde{\rho},Id_{E})\left(\tilde{\delta}_{c}\right)\left(\tilde{g}^{e}_{d}\circ h\circ\pi\frac{\partial F^{a}}{\partial y^{e}}\right)\\ &-\frac{1}{4}\tilde{g}^{e}_{c}\circ h\circ\pi\frac{\partial F^{f}}{\partial y^{e}}\frac{\partial}{\partial y^{f}}\left((\rho,\eta)\,\Gamma^{a}_{d}\right)\\ &-\frac{1}{16}\tilde{g}^{e}_{c}\circ h\circ\pi\frac{\partial F^{f}}{\partial y^{e}}\frac{\partial}{\partial y^{f}}\left(\tilde{g}^{e}_{d}\circ h\circ\pi\frac{\partial F^{a}}{\partial y^{e}}\right),\end{aligned}$$

$$\begin{aligned}\Gamma(\tilde{\rho},Id_{E})\left(\overset{\circ}{\tilde{\delta}}_{d}\right)\left((\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{c}\right)\;&=\Gamma(\tilde{\rho},Id_{E})\left(\tilde{\delta}_{d}\right)\left((\rho,\eta)\,\Gamma^{a}_{c}\right)\\ &+\frac{1}{4}\Gamma(\tilde{\rho},Id_{E})\left(\tilde{\delta}_{d}\right)\left(\tilde{g}^{e}_{c}\circ h\circ\pi\frac{\partial F^{a}}{\partial y^{e}}\right)\\ &-\frac{1}{4}\tilde{g}^{e}_{d}\circ h\circ\pi\frac{\partial F^{f}}{\partial y^{e}}\frac{\partial}{\partial y^{f}}\left((\rho,\eta)\,\Gamma^{a}_{c}\right)\\ &-\frac{1}{16}\tilde{g}^{e}_{d}\circ h\circ\pi\frac{\partial F^{f}}{\partial y^{e}}\frac{\partial}{\partial y^{f}}\left(\tilde{g}^{e}_{c}\circ h\circ\pi\frac{\partial F^{a}}{\partial y^{e}}\right),\\ L^{e}_{cd}\circ h\circ\pi\cdot(\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{e}\;&=L^{e}_{cd}\circ h\circ\pi\cdot(\rho,\eta)\,\Gamma^{a}_{e}\\ &+L^{e}_{cd}\circ h\circ\pi\cdot\left(\tilde{g}^{f}_{e}\circ h\circ\pi\frac{\partial F^{a}}{\partial y^{f}}\right)\end{aligned}$$

it results the conclusion of the theorem. *q.e.d.*

Theorem 5.11.1.5 *Let*

$$\mathbb{T}^{a}_{bc}\delta_{a}\otimes d\tilde{z}^{b}\otimes d\tilde{z}^{c}\in\mathcal{T}^{10}_{20}\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_{E},E\right)$$

and

$$\mathbb{S}^a_{bc}\dot{\tilde{\partial}}_a\otimes\delta\tilde{y}^b\otimes\delta\tilde{y}^c\in\mathcal{T}^{01}_{02}\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$$

such that they verify the following conditions:

$$\mathbb{T}^a_{bc}=-\mathbb{T}^a_{cb},\ \mathbb{S}^a_{bc}=-\mathbb{S}^a_{cb},\ \forall b,c\in\overline{1,r}.$$

If $\left(\left(\rho,\eta\right)\widetilde{H},\left(\rho,\eta\right)\widetilde{V}\right)$ *is the distinguished linear* (ρ,η)*-connection presented in the Theorem 5.9.2, then the local real functions:*

$$\begin{aligned}\left(\rho,\eta\right)\mathring{H}^a_{bc}&=\left(\rho,\eta\right)\widetilde{H}^a_{bc}+\frac{1}{8}\tilde{g}^{ae}\left(-\tilde{g}^f_c\circ h\circ\pi\frac{\partial F^d}{\partial y^f}\frac{\partial g_{be}}{\partial y^d}\right.\\&\qquad\left.+\tilde{g}^f_e\circ h\circ\pi\frac{\partial F^d}{\partial y^f}\frac{\partial g_{bc}}{\partial y^d}-\tilde{g}^f_b\circ h\circ\pi\frac{\partial F^d}{\partial y^f}\frac{\partial g_{ec}}{\partial y^d}\right),\\\left(\rho,\eta\right)\mathring{V}^a_{bc}&=\left(\rho,\eta\right)\widetilde{V}^a_{bc}\end{aligned}\tag{5.11.1.10}$$

are the components of a normal distinguished linear (ρ,η)*-connection with* (ρ,η)*-*$\mathcal{H}\left(\mathcal{HH}\right)$ *and* (ρ,η)*-*$\mathcal{V}\left(\mathcal{VV}\right)$ *torsions a priori given such that the generalized tangent bundle* $\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$ *derives generalized Lagrange* (ρ,η)*-space.*

In addition, we have:

$$\begin{aligned}\left(\rho,\eta,h\right)\mathring{\mathbb{T}}^a_{bc}&=\mathbb{T}^a_{bc}\\\left(\rho,\eta,h\right)\mathring{\mathbb{S}}^a_{bc}&=\mathbb{S}^a_{bc}.\end{aligned}\tag{5.11.1.11}$$

Proposition 5.11.1.1 *If S is the canonical (ρ,η)-semispray associated to the mechanical (ρ,η)-system $\left(\left(E,\pi,M\right),F_e,\left(\rho,\eta\right)\Gamma\right)$ and from $\mathbf{B^v}$-morphism (g,h), then*

$$2G^{\acute{a}}=2G^aM^{\acute{a}}_a\circ h\circ\pi-\left(g^a_b\circ h\circ\pi\right)y^b\left(\rho^i_a\circ h\circ\pi\right)\frac{\partial y^{\acute{a}}}{\partial x^i}.\tag{5.11.1.12}$$

Proof. Since the Jacobian matrix of coordinates transformation is

$$\left\|\begin{array}{cc}M^{\acute{a}}_a\circ h\circ\pi & 0\\ \rho^i_a\circ\left(h\circ\pi\right)\dfrac{\partial M^{\acute{a}}_a\circ\pi}{\partial x^i}y^a & M^{\acute{a}}_a\circ\pi\end{array}\right\|=\left\|\begin{array}{cc}M^{\acute{a}}_a\circ h\circ\pi & 0\\ \rho^i_a\circ\left(h\circ\pi\right)\dfrac{\partial y^{\acute{a}}}{\partial x^i} & M^{\acute{a}}_a\circ\pi\end{array}\right\|$$

and

$$\left\|\begin{array}{cc}M^{\acute{a}}_a\circ h\circ\pi & 0\\ \rho^i_a\circ\left(h\circ\pi\right)\dfrac{\partial y^{\acute{a}}}{\partial x^i} & M^{\acute{a}}_a\circ\pi\end{array}\right\|\cdot\left(\begin{array}{c}\left(g^a_b\circ h\circ\pi\right)y^b\\-2\left(G^a-\dfrac{1}{4}F^a\right)\end{array}\right)=\left(\begin{array}{c}\left(g^{\acute{a}}_{\acute{b}}\circ h\circ\pi\right)y^{\acute{b}}\\-2\left(G^{\acute{a}}-\dfrac{1}{4}F^{\acute{a}}\right)\end{array}\right),$$

the conclusion results immediately.

In the following, we consider a differentiable curve $I\overset{c}{\to}M$ and its (g,h)-lift $I\overset{\dot{c}}{\to}E$. *q.e.d.*

Definition 5.11.1.3 The curve $\dot{c}$ is a integral curve of the (ρ,η)-semispray S of the mechanical (ρ,η)-system $\left(\left(E,\pi,M\right),F_e,\left(\rho,\eta\right)\Gamma\right)$, if it is verifies the following equality:

$$\frac{d\dot{c}\left(t\right)}{dt}=\Gamma\left(\tilde{\rho},Id_E\right)S\left(\dot{c}\left(t\right)\right).\tag{5.11.1.13}$$

Theorem 5.11.1.6 *All (g,h)-lifts solutions of the equations:*

$$\frac{dy^{a}(t)}{dt}+2G^{a}\circ u(c,\dot{c})(x(t))=\frac{1}{2}F^{a}\circ u(c,\dot{c})(x(t)),\ a\in\overline{1,r}, \tag{5.11.1.14}$$

where $x(t)=(\eta\circ h\circ c)(t)$, are integral curves of the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,(\rho,\eta)\Gamma)$ and from locally invertible $\mathbf{B}^{\mathbf{v}}$-morphism (g,h).

Proof. Since the equality

$$\frac{d\dot{c}(t)}{dt}=\Gamma(\tilde{\rho},Id_E)S(\dot{c}(t))$$

is equivalent to

$$\begin{array}{l}\frac{d}{dt}((\eta\circ h\circ c)^{i}(t),y^{a}(t))\\ \qquad=\left(\rho_{a}^{i}\circ\eta\circ h\circ c(t)g_{b}^{a}\circ h\circ c(t)y^{b}(t),-2\left(G^{a}-\frac{1}{4}F^{a}\right)((\eta\circ h\circ c)^{i}(t),y^{a}(t))\right),\end{array}$$

it results

$$\begin{array}{l}\frac{dy^{a}(t)}{dt}+2G^{a}\left(x^{i}(t),y^{a}(t)\right)=\frac{1}{2}F^{a}\left(x^{i}(t),y^{a}(t)\right),\quad a\in\overline{1,n},\\ \frac{dx^{i}(t)}{dt}=\rho_{a}^{i}\circ\eta\circ h\circ c(t)\,g_{b}^{a}\circ h\circ c(t)\,y^{b}(t),\end{array}$$

where $x^{i}(t)=(\eta\circ h\circ c)^{i}(t)$. *q.e.d.*

Definition 5.11.1.4 If S is a (ρ,η)-semispray, then the vector field

$$[\mathbb{C},S]_{(\rho,\eta)TE}-S \tag{5.11.1.15}$$

will be called the *derivation of (ρ,η)-semispray S.*

The (ρ,η)-semispray S will be called *(ρ,η)-spray* if the following conditions are verified:

1. $S\circ 0\in C^{1}$, where 0 is the null section;
2. Its derivation is the null vector field.

The (ρ,η)-semispray S will be called *quadratic (ρ,η)-spray* if there are verified the following conditions:

1. $S\circ 0\in C^{2}$, where 0 is the null section;
2. Its derivation is the null vector field.

In particular, if $(\rho,\eta)=(id_{TM},Id_M)$ and $(g,h)=(Id_E,Id_M)$, then we obtain the *spray* and the *quadratic spray* which is similar with the classical spray and quadratic spray.

Theorem 5.11.1.7 *If S is the canonical (ρ,η)-spray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,(\rho,\eta)\Gamma)$ and from locally invertible $\mathbf{B}^{\mathbf{v}}$-morphism (g,h), then*

$$\begin{array}{l}2\left(G^{a}-\frac{1}{4}F^{a}\right)=(\rho,\eta)\Gamma_{c}^{a}\left(g_{f}^{c}\circ h\circ\pi\cdot y^{f}\right)\\ \qquad+\frac{1}{2}\left(g_{e}^{d}\circ h\circ\pi\cdot y^{e}\right)L_{dc}^{b}\circ h\circ\pi\tilde{g}_{b}^{a}\circ h\circ\pi\left(g_{f}^{c}\circ h\circ\pi\cdot y^{f}\right),\ a\in\overline{1,r}.\end{array} \tag{5.11.1.16}$$

Then, we obtain the spray

$$(5.11.1.17)\quad \begin{aligned} S &= \left(g_b^a\circ h\circ \pi\right) y^b\frac{\partial}{\partial \tilde{z}^a}+\left(\rho,\eta\right)\Gamma_c^a\left(g_f^c\circ h\circ\pi\cdot y^f\right)\frac{\partial}{\partial\tilde{y}^a}\\ &+\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi\left(g_f^c\circ h\circ\pi\cdot y^f\right)\frac{\partial}{\partial\tilde{y}^a}.\end{aligned}$$

This (ρ,η)-spray will be called the canonical (ρ,η)-spray associated to mechanical system $((E,\pi,M),F_e,(\rho,\eta)\Gamma)$ and from locally invertible $\mathbf{B}^{\mathbf{v}}$-morphism (g,h).

In particular, if $(\rho,\eta)=(id_{TM},Id_M)$ and $(g,h)=(Id_E,Id_M)$, then we get the canonical spray associated to connection Γ which is similar with the classical canonical spray associated to connection Γ.

Proof. Since

$$[\mathbb{C},S]_{(\rho,\eta)TE}=\left[y^a\dot{\tilde{\partial}}_a,\left(g_e^b\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_b\right]_{(\rho,\eta)TE}-2\left[y^a\dot{\tilde{\partial}}_a,\left(G^b-\frac{1}{4}F^b\right)\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE},$$

$$\begin{aligned}\left[y^a\dot{\tilde{\partial}}_a,\left(g_e^b\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_b\right]_{(\rho,\eta)TE} &= y^a\frac{\partial\left(g_e^b\circ h\circ\pi\cdot y^e\right)}{\partial y^a}\tilde{\partial}_b-\left(g_e^b\circ h\circ\pi\cdot y^e\right)\rho_\beta^j\circ h\circ\pi\frac{\partial y^a}{\partial x^i}\\ &= y^a g_e^b\circ h\circ\pi\cdot\delta_a^e\tilde{\partial}_b-0=\left(g_e^b\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_b\end{aligned}$$

and

$$\begin{aligned}\left[y^a\dot{\tilde{\partial}}_a,\left(G^b-\tfrac{1}{4}F^b\right)\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE} &= y^a\frac{\partial\left(G^b-\frac{1}{4}F^b\right)}{\partial y^a}\dot{\tilde{\partial}}_b-\left(G^b-\frac{1}{4}F^b\right)\delta_b^a\dot{\tilde{\partial}}_a\\ &= y^a\frac{\partial\left(G^b-\frac{1}{4}F^b\right)}{\partial y^a}\dot{\tilde{\partial}}_b-\left(G^b-\frac{1}{4}F^b\right)\dot{\tilde{\partial}}_b\end{aligned}$$

it results that

$$(S_1)\qquad [\mathbb{C},S]_{(\rho,\eta)TE}-S=2\left(-y^f\frac{\partial\left(G^a-\frac{1}{4}F^a\right)}{y^f}+2\left(G^a-\frac{1}{4}F^a\right)\right)\dot{\tilde{\partial}}_a$$

Using equality (5.11.1.4), it results that

$$(S_2)\qquad \begin{aligned}\frac{\partial\left(G^a-\frac{1}{4}F^a\right)}{y^f} &= \left(\rho,\eta\right)\Gamma_c^a\cdot g_f^c\circ h\circ\pi\\ &+\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi\cdot g_f^c\circ h\circ\pi.\end{aligned}$$

Using equalities (S_1) and (S_2), it results the conclusion of the theorem. *q.e.d.*

Remark 5.11.1.2. If $(\rho,\eta)=(id_{TM},Id_M)$ and $(g,h)=(Id_E,Id_M)$, then we get the canonical spray associated to connection Γ.

Theorem 5.11.1.8 *All (g,h)-lifts solutions of the following system of equations:*

$$(5.11.1.17)\quad \begin{aligned}&\frac{dy^a}{dt}+\left(\rho,\eta\right)\Gamma_c^a\left(g_f^c\circ h\circ\pi\cdot y^f\right)\\ &+\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi\left(g_f^c\circ h\circ\pi\cdot y^f\right)=0,\end{aligned}$$

are the integral curves of canonical (ρ,η)-spray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,\rho\Gamma)$ and from locally invertible $\mathbf{B}^{\mathbf{v}}$-morphism (g,h).

5.11.2 The Lagrangian formalism for Lagrange mechanical (ρ, η)-systems

Let $((E, \pi, M), F_e, L)$ be an arbitrarily Lagrange mechanical (ρ, η)-system.

The *natural dual* (ρ, η)*-base* $(d\tilde{z}^\alpha, d\tilde{y}^a)$ of natural (ρ, η)-base $\left(\frac{\partial}{\partial \tilde{z}^\alpha}, \frac{\partial}{\partial \tilde{y}^a}\right)$ is determined by the equations

$$\begin{cases} \left\langle d\tilde{z}^\alpha, \frac{\partial}{\partial \tilde{z}^\beta} \right\rangle = \delta^\alpha_\beta, & \left\langle d\tilde{z}^\alpha, \frac{\partial}{\partial \tilde{y}^a} \right\rangle = 0, \\ \left\langle d\tilde{y}^a, \frac{\partial}{\partial \tilde{z}^\beta} \right\rangle = 0, & \left\langle d\tilde{y}^a, \frac{\partial}{\partial \tilde{y}^b} \right\rangle = \delta^a_b. \end{cases}$$

It is very important to remark that the 1-forms $d\tilde{z}^\alpha,\ \alpha \in \overline{1,p}$ and $d\tilde{y}^a,\ a \in \overline{1,n}$ are not the differentials of coordinates functions as in the classical case, but we will use the same notations. In this case

$$(d\tilde{z}^\alpha) \neq d^{(\rho,\eta)TE}(\tilde{z}^\alpha) = 0,$$

where $d^{(\rho,\eta)TE}$ is the exterior differentiation operator associated to exterior differential $\mathcal{F}(E)$-algebra

$$(\Lambda((\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot, \wedge).$$

Let L be a regular Lagrangian and let (g, h) be a locally invertible $\mathbf{B^v}$-morphism of (E, π, M) source and (E, π, M) target.

Definition 5.11.2.1 The 1-form

(5.11.2.1) $$\theta_L = (\tilde{g}^e_a \circ h \circ \pi \cdot L_e)\, d\tilde{z}^a$$

will be called the 1*-form of Poincaré-Cartan type associated to the Lagrangian* L *and to the locally invertible* $\mathbf{B^v}$*-morphism* (g, h).

We obtain easily:

(5.11.2.2) $$\theta_L\left(\frac{\partial}{\partial \tilde{z}^b}\right) = \tilde{g}^e_b \circ h \circ \pi \cdot L_e, \quad \theta_L\left(\frac{\partial}{\partial \tilde{y}^b}\right) = 0.$$

Definition 5.11.2.2 The 2-form

$$\omega_L = d^{(\rho,\eta)TE}\theta_L$$

will be called the 2*-form of Poincaré-Cartan type associated to the Lagrangian* L *and to the locally invertible* $\mathbf{B^v}$*-morphism* (g, h).

By the definition of $d^{(\rho,\eta)TE}$, we obtain:

(5.11.2.3) $$\begin{aligned} \omega_L(U, V) &= \Gamma(\tilde{\rho}, Id_E)(U)(\theta_L(V)) \\ &\quad - \Gamma(\tilde{\rho}, Id_E)(V)(\theta_L(U)) - \theta_L\left([U, V]_{(\rho,\eta)TE}\right), \end{aligned}$$

for any $U, V \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$.

It follows:

$$
(5.11.2.4)\qquad \begin{cases}
\omega_L\left(\dfrac{\partial}{\partial \tilde{z}^a}, \dfrac{\partial}{\partial \tilde{z}^b}\right) &= \left(\rho_a^i \circ h \circ \pi\right) \cdot L_{ib} \\
&\quad - \left(\rho_b^i \circ h \circ \pi\right) \cdot L_{ia} - L_{ab}^c \circ h \circ \pi \cdot \tilde{g}_c^e \circ h \circ \pi \cdot L_e; \\
\omega_L\left(\dfrac{\partial}{\partial \tilde{z}^a}, \dfrac{\partial}{\partial \tilde{y}^b}\right) &= -\tilde{g}_a^e \circ h \circ \pi \cdot L_{eb}; \\
\omega_L\left(\dfrac{\partial}{\partial \tilde{y}^a}, \dfrac{\partial}{\partial \tilde{y}^b}\right) &= 0.
\end{cases}
$$

Definition 5.11.2.3 The real function

$$
(5.11.2.5)\qquad \mathcal{E}_L = y^a \cdot L_a - L
$$

will be called the *energy of regular Lagrangian L.*

Theorem 5.11.2.1 *The equation*

$$
(5.11.2.6)\qquad i_S\left(\omega_L\right) = -d^{(\rho,\eta)TE}\left(\mathcal{E}_L\right),\ S \in \Gamma\left(\left(\rho,\eta\right)TE, \left(\rho,\eta\right)\tau_E, E\right),
$$

has an unique solution $S_L\left(g,h\right)$ of the type:

$$
(5.11.2.7)\qquad \left(g_e^a \circ h \circ \pi\right) y^e \frac{\partial}{\partial \tilde{z}^a} - 2\left(G^a - \frac{1}{4}F^a\right)\frac{\partial}{\partial \tilde{y}^a},
$$

where

$$
(5.11.2.8)\qquad 2G^a = g_e^a \circ h \circ \pi \cdot \tilde{L}^{eb} \cdot E_b\left(L,g,h\right) + \frac{1}{2}F^a
$$

and

$$
(5.11.2.9)\qquad \begin{aligned} E_b(L,g,h) &= \rho_b^i \circ h \circ \pi \cdot L_i - g_e^a \circ h \circ \pi \cdot y^e \cdot \rho_a^i \circ h \circ \pi \cdot \frac{\partial\left(\tilde{g}_b^e \circ h \circ \pi \cdot L_e\right)}{\partial x^i} \\ &\quad + g_e^a \circ h \circ \pi \cdot y^e \cdot L_{ab}^d \circ h \circ \pi \cdot \left(\tilde{g}_d^e \circ h \circ \pi \cdot L_e\right). \end{aligned}
$$

$S_L\left(g,h\right)$ will be called *the canonical (ρ,η)-semispray associated to Lagrange mechanical (ρ,η)-system $\left(\left(E,\pi,M\right), F_e, L\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).*

Proof. We obtain

$$
\begin{aligned} i_S\left(\omega_L\right) = -d^{(\rho,\eta)TE}\left(\mathcal{E}_L\right) \Longleftrightarrow \omega_L\left(S,X\right) &= -\Gamma\left(\tilde{\rho}, Id_E\right)\left(X\right)\left(\mathcal{E}_L\right), \\ &\forall X \in \Gamma\left(\left(\rho,\eta\right)TE, \left(\rho,\eta\right)\tau_E, E\right) \end{aligned}
$$

Particularly, we obtain:

$$
\omega_L\left(S, \frac{\partial}{\partial \tilde{z}^b}\right) = -\Gamma\left(\tilde{\rho}, Id_E\right)\left(\frac{\partial}{\partial \tilde{z}^b}\right)\left(\mathcal{E}_L\right).
$$

If we expand this equality by using (5.11.2.2) and (5.11.2.4), we obtain

$$
\begin{aligned} & g_e^a \circ h \circ \pi \cdot y^e \cdot \left[\rho_a^i \circ h \circ \pi \cdot \frac{\partial\left(\tilde{g}_b^e \circ h \circ \pi \cdot L_e\right)}{\partial x^i} - \rho_b^i \circ h \circ \pi \cdot \frac{\partial\left(\tilde{g}_a^e \circ h \circ \pi \cdot L_e\right)}{\partial x^i}\right. \\ & \left. - L_{ab}^d \circ h \circ \pi \cdot \left(\tilde{g}_d^e \circ h \circ \pi \cdot L_e\right)\right] + 2\left(G^a - \frac{1}{4}F^a\right)\left(\tilde{g}_a^e \circ h \circ \pi\right) \cdot L_{eb} \\ & = -\rho_b^i \circ h \circ \pi \cdot \left(g_e^a \circ h \circ \pi \cdot y^e\right) \cdot \frac{\partial\left(\tilde{g}_a^e \circ h \circ \pi \cdot L_e\right)}{\partial x^i} + \rho_b^i \circ h \circ \pi \cdot L_i. \end{aligned}
$$

After some calculations, we obtain

$$2\left(G^{a}-\frac{1}{4}F^{a}\right)=g_{e}^{a}\circ h\circ\pi\cdot\tilde{L}^{eb}\cdot E_{b}\left(L,g,h\right),$$

where

$$\begin{array}{ll}E_{b}\left(L,g,h\right) & =\rho_{b}^{i}\circ h\circ\pi\cdot L_{i}-g_{e}^{a}\circ h\circ\pi\cdot y^{e}\cdot\rho_{a}^{i}\circ h\circ\pi\cdot\dfrac{\partial\left(\tilde{g}_{b}^{e}\circ h\circ\pi\cdot L_{e}\right)}{\partial x^{i}}+\\ & +\,g_{e}^{a}\circ h\circ\pi\cdot y^{e}\cdot L_{ab}^{d}\circ h\circ\pi\cdot\left(\tilde{g}_{d}^{e}\circ h\circ\pi\cdot L_{e}\right).\end{array}$$

q.e.d.

Remarks

1. If $F_{e}=0$ and $\eta=Id_{M}$, then $S_{L}\left(Id_{E},Id_{M}\right)\overset{put}{=}S_{L}$ is the canonical ρ-semispray associated to regular Lagrangian L which is similar with the semispray presented in [27] by M. de Leon, J. Marrero and E. Martinez.

2. If $F_{e}\neq 0$ and $\left(\rho,\eta\right)=\left(Id_{TM},Id_{M}\right)$, then $S_{L}\left(Id_{E},Id_{M}\right)\overset{put}{=}S_{L}$ will be called *the canonical semispray* which is not the same canonical semispray presented by I. Bucataru and R. Miron in [7].

3. If $F_{e}=0$ and $\left(\rho,\eta\right)=\left(Id_{TM},Id_{M}\right)$, then $S_{L}\left(Id_{M},Id_{E}\right)\overset{put}{=}S_{L}$ will be called *the canonical semispray* which is not the same canonical semispray presented by R. Miron and M. Anastasiei in [41].

Theorem 5.11.2.2 *The real local functions*

$$\begin{array}{ll}\left(\rho,\eta\right)\Gamma_{c}^{a} & =\dfrac{1}{2}\tilde{g}_{c}^{e}\circ h\circ\pi\dfrac{\partial\left(g_{e}^{a}\circ h\circ\pi\cdot L^{eb}\cdot E_{b}\left(L,g,h\right)\right)}{\partial y^{e}}\\ & -\dfrac{1}{2}\left(g_{e}^{d}\circ h\circ\pi\cdot y^{e}\right)L_{dc}^{b}\circ h\circ\pi\cdot\tilde{g}_{b}^{a}\circ h\circ\pi,\ a,c\in\overline{1,r}.\end{array}\tag{5.11.2.10}$$

are the components of a $\left(\rho,\eta\right)$-connection $\left(\rho,\eta\right)\Gamma$ for the vector bundle $\left(E,\pi,M\right)$ which will be called the $\left(\rho,\eta\right)$-connection associated to Lagrange mechanical $\left(\rho,\eta\right)$-system $\left(\left(E,\pi,M\right),F_{e},L\right)$ and from $\mathbf{B}^{\mathbf{v}}$-morphism $\left(g,h\right)$.

Corollary 5.11.2.1 *The real local functions*

$$\begin{array}{ll}(\rho,\eta)\mathring{\Gamma}^{a}{}_{c} & =\left(\tilde{g}_{c}^{b}\circ h\circ\pi\right)\dfrac{\partial G^{a}}{\partial y^{b}}\\ & -\dfrac{1}{2}\left(g_{e}^{d}\circ h\circ\pi\cdot y^{e}\right)L_{dc}^{b}\circ h\circ\pi\cdot\tilde{g}_{b}^{a}\circ h\circ\pi,\ a,c\in\overline{1,r}\end{array}\tag{5.11.2.11}$$

are the components of a $\left(\rho,\eta\right)$-connection $\left(\rho,\eta\right)\mathring{\Gamma}$ for the vector bundle $\left(E,\pi,M\right)$.

In addition, we have

$$\left(\rho,\eta\right)\mathring{\Gamma}_{c}^{a}=\left(\rho,\eta\right)\Gamma_{c}^{a}+\frac{1}{4}(\tilde{g}_{c}^{b}\circ h\circ\pi)\cdot\frac{\partial F^{a}}{\partial y^{b}},\ \forall a,c\in\overline{1,r}.\tag{5.11.2.12}$$

Theorem 5.11.2.3 *The parallel (g,h)-lifts with respect to (ρ,η)-connection $(\rho,\eta)\Gamma$ are the integral curves of the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,L)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).*

Definition 5.11.2.4 The equations

$$\frac{dy^a(t)}{dt}+\left(g_e^a\circ h\circ\pi\cdot\tilde{L}^{eb}\cdot E_b(L,g,h)\right)\circ u(c,\dot{c})(x(t))=0, \tag{5.11.2.13}$$

where $x(t)=\eta\circ h\circ c(t)$, will be called the *equations of Euler-Lagrange type associated to Lagrange mechanical (ρ,η)-system $((E,\pi,M),F_e,L)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).*

The equations

$$\frac{dy^a(t)}{dt}+\left(\tilde{L}^{ab}\cdot E_b(L,Id_E,Id_M)\right)\circ u(c,\dot{c})(x(t))=0, \tag{5.11.2.13'}$$

where $x(t)=h\circ\eta\circ c(t)$, will be called the *equations of Euler-Lagrange type associated to Lagrange mechanical (ρ,η)-system $((E,\pi,M),F_e,L)$.*

Remark 5.11.2.1 The integral curves of the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,L)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h) are the (g,h)-lifts solutions for the equations of Euler-Lagrange type (5.11.2.13).

It is known that, in classical sense, a geodesic with respect to a Finsler metric

$$TM\ \stackrel{F}{\longrightarrow}\ \mathbb{R}_+$$

is a curve c on the manifold M such that the components of its tangent lift

$$\frac{dc^i}{dt}\cdot\frac{\partial}{\partial x^i}$$

are solutions for the Euler-Lagrange equations

$$\tfrac{d}{dt}\left(\tfrac{\partial F^2}{\partial y^i}\right)-\tfrac{\partial F^2}{\partial x^i}=0,\ i\in\overline{1,m}\ . \tag{5.11.2.14}$$

If

$$\left((TM,\tau_M,M),\widetilde{[,]}_{TM,h},(\rho,\eta)\right)$$

is a generalized Lie algebroid different by the generalized Lie algebroid

$$\left((TM,\tau_M,M),[,]_{TM,Id_M},(Id_{TM},Id_M)\right),$$

then, using the classical method by work, we can not determine the geodesics on the manifold M such that the components of their lifts (different by the tangent lift) are solutions for the Euler-Lagrange equations (5.11.2.14).

Using our theory, we obtain the following

Theorem 5.11.2.4 *If F is a Finsler fundamental function, then the geodesics on the manifold M are the curves such that the components of their (g,h)-lifts are solutions for the equations of Euler-Lagrange type* (5.11.2.13).

Therefore, it is natural to propose to extend the study of the Finsler geometry from the usual Lie algebroid

$$((TM,\tau_M,M),[,]_{TM},(Id_{TM},Id_M)),$$

to an arbitrary (generalized) Lie algebroid

$$\left((E,\pi,M),[,]_{E,h},(\rho,\eta)\right).$$

6 The geometry of total space of the generalized tangent bundle for dual vector bundle

6.1 Adapted (ρ,η)-basis and adapted dual (ρ,η)-basis

In the following we consider the following diagram:

$$\begin{array}{ccc}\overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\ \downarrow{\scriptstyle \overset{*}{\pi}} & & \downarrow{\scriptstyle \nu}\\ M & \xrightarrow{h} & N\end{array}$$

where $(E,\pi,M)\in|\mathbf{B}^{\mathbf{v}}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

If we put the problem of finding a base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module

$$\left(\Gamma\left(H(\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

of the type

$$\frac{\overset{*}{\delta}}{\delta\tilde{z}^{\alpha}}=\tilde{Z}_{\alpha}^{\beta}\frac{\overset{*}{\partial}}{\partial\tilde{z}^{\beta}}+Y_{b\alpha}\frac{\partial}{\partial\tilde{p}_b},\alpha\in\overline{1,p}$$

which satisfies the following conditions:

(6.1.1)
$$\begin{array}{rcl}\Gamma\left((\rho,\eta)\overset{*}{\pi}!,Id_{\overset{*}{E}}\right)\left(\dfrac{\overset{*}{\delta}}{\delta\tilde{z}^{\alpha}}\right)&=&\overset{*}{\tilde{T}}_{\alpha}\\ \Gamma\left((\rho,\eta)\overset{*}{\Gamma},Id_{\overset{*}{E}}\right)\left(\dfrac{\overset{*}{\delta}}{\delta\tilde{z}^{\alpha}}\right)&=&0,\end{array}$$

then we obtain the sections

(6.1.2)
$$\frac{\overset{*}{\delta}}{\delta\tilde{z}^{\alpha}}=\frac{\overset{*}{\partial}}{\partial\tilde{z}^{\alpha}}+(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\frac{\partial}{\partial\tilde{p}_b}\ .$$

We observe that their law of change is a tensorial law under a change of vector fiber charts.

Definition 6.1.1 The base

$$\left(\frac{\overset{*}{\delta}}{\delta\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{p}_{a}}\right)\overset{put}{=}\left(\tilde{\overset{*}{\delta}}_{\alpha},\tilde{\overset{\cdot\, a}{\partial}}\right)$$

will be called the *adapted* (ρ,η)*-base.*

The following equality holds good

$$\Gamma\left(\tilde{\overset{*}{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\overset{*}{\delta}}_{\alpha}\right)=\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\partial}_{i}+\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\dot{\partial}^{b},\tag{6.1.3}$$

where $\left(\overset{*}{\partial}_{i},\dot{\partial}^{a}\right)$ is the natural base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module $\left(\Gamma\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$.

Moreover, if $(\rho,\eta)\overset{*}{\Gamma}$ is the (ρ,η)-connection associated to connection $\overset{*}{\Gamma}$, then we obtain

$$\Gamma\left(\tilde{\overset{*}{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\overset{*}{\delta}}_{\alpha}\right)=\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\overset{*}{\delta}_{i},\tag{6.1.4}$$

where $\left(\overset{*}{\delta}_{i},\dot{\partial}_{a}\right)$ is the adapted base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module $\left(\Gamma\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$.

Theorem 6.1.1 *The following equality holds good*

$$\left[\tilde{\overset{*}{\delta}}_{\alpha},\tilde{\overset{*}{\delta}}_{\beta}\right]_{(\rho,\eta)T\overset{*}{E}}=L_{\alpha\beta}^{\gamma}\circ\left(h\circ\overset{*}{\pi}\right)\tilde{\overset{*}{\delta}}_{\gamma}+\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}_{b\ \ \alpha\beta}\tilde{\overset{\cdot\, b}{\partial}},\tag{6.1.5}$$

where

$$\begin{aligned}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}_{b\ \ \alpha\beta}&=\Gamma\left(\tilde{\overset{*}{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\overset{*}{\delta}}_{\beta}\right)\left(\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\right)\\&+\Gamma\left(\tilde{\overset{*}{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\overset{*}{\delta}}_{\alpha}\right)\left(\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\beta}\right)-\left(L_{\alpha\beta}^{\gamma}\circ h\circ\overset{*}{\pi}\right)\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\gamma},\end{aligned}\tag{6.1.6}$$

Moreover, we have:

$$\left[\tilde{\overset{*}{\delta}}_{\alpha},\tilde{\overset{\cdot\, a}{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}}=-\Gamma\left(\tilde{\overset{*}{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\overset{\cdot\, a}{\partial}}\right)\left(\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\right)\tilde{\overset{\cdot\, b}{\partial}},\tag{6.1.7}$$

and

$$\Gamma\left(\tilde{\overset{*}{\rho}},Id_{\overset{*}{E}}\right)\left[\tilde{\overset{*}{\delta}}_{\alpha},\tilde{\overset{*}{\delta}}_{\beta}\right]_{(\rho,\eta)T\overset{*}{E}}=\left[\Gamma\left(\tilde{\overset{*}{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\overset{*}{\delta}}_{\alpha}\right),\Gamma\left(\tilde{\overset{*}{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\overset{*}{\delta}}_{\beta}\right)\right]_{T\overset{*}{E}}.\tag{6.1.8}$$

If we consider the problem of finding a base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module

$$\left(\Gamma\left(\left(V\left(\rho,\eta\right)T\overset{*}{E}\right)^{*},\left(\left(\rho,\eta\right)\tau_{\overset{*}{E}}\right)^{*},\overset{*}{E}\right),+,\cdot\right)$$

of the type

$$\delta\tilde{p}_{a}=\theta_{a\alpha}d\tilde{z}^{\alpha}+\omega_{a}^{b}d\tilde{p}_{b},\ a\in\overline{1,r}$$

which satisfies the following conditions:

$$\left\langle \delta\tilde{p}_a, \overset{.\,b}{\tilde{\partial}} \right\rangle = \delta_a^b \wedge \left\langle \delta\tilde{p}_a, \overset{*}{\tilde{\delta}}_\alpha \right\rangle = 0, \tag{6.1.9}$$

then we obtain the sections

$$\delta\tilde{p}_a = -\left(\rho,\eta\right)\overset{*}{\Gamma}_{a\alpha} d\tilde{z}^\alpha + d\tilde{p}_a, a \in \overline{1,r}. \tag{6.1.10}$$

We observe that their changing rule is tensorial under a change of vector fiber charts.

Definition 6.1.2 The base $(d\tilde{z}^\alpha, \delta\tilde{p}_a)$ will be called the *adapted dual* (ρ,η)*-base.*

6.2 Remarkable endomorphisms

Now, let us consider the following diagram:

$$\begin{tikzcd} \overset{*}{E} \arrow[d, "\overset{*}{\pi}"'] & \left(F, [,]_{F,h}, (\rho,\eta)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E,\pi,M) \in |\mathbf{B^v}|$ and $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta)\right)$ is a generalized Lie algebroid.

Definition 6.2.1 For any **Mod**-endomorphism e of

$$\left(\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E}, \left(\rho,\eta\right)\tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot\right)$$

we define the application of Nijenhuis type

$$\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E}, \left(\rho,\eta\right)\tau_{\overset{*}{E}}, \overset{*}{E}\right)^2 \xrightarrow{N_e} \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E}, \left(\rho,\eta\right)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

defined by

$$N_e\left(X,Y\right) = \left[eX, eY\right]_{(\rho,\eta)T\overset{*}{E}} + e^2\left[X,Y\right]_{(\rho,\eta)T\overset{*}{E}} - e\left[eX,Y\right]_{(\rho,\eta)T\overset{*}{E}} - e\left[X,eY\right]_{(\rho,\eta)T\overset{*}{E}},$$

for any $X, Y \in \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E}, \left(\rho,\eta\right)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$.

Remark 6.2.1 The vertical and the horizontal vector subbundles are interior differential systems for the Lie algebroid generalized tangent bundle

$$\left(\left(\left(\rho,\eta\right)T\overset{*}{E}, \left(\rho,\eta\right)\tau_{\overset{*}{E}}, \overset{*}{E}\right), [,]_{(\rho,\eta)T\overset{*}{E}}, \left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\right).$$

These interior differential systems will be called *vertical* and *horizontal interior differential systems.*

6.2.1 Projectors

Definition 6.2.1.1 Any **Mod**-endomorphism e of

$$\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

with the property

$$e^2=e \tag{6.2.1.1}$$

will be called a *projector.*

Example 6.2.1.1 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{V}}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \tilde{Z}^{\alpha}\overset{*}{\tilde{\delta}}_{\alpha}+Y_a\overset{\cdot\, a}{\tilde{\partial}} & \longmapsto & Y_a\overset{\cdot\, a}{\tilde{\partial}}\end{array}$$

is a projector which will be called the *the vertical projector.*

Remark 6.2.1.1 We have $\overset{*}{\mathcal{V}}\left(\overset{*}{\tilde{\delta}}_{\alpha}\right)=0$ and $\overset{*}{\mathcal{V}}\left(\overset{\cdot\, a}{\tilde{\partial}}\right)=\overset{\cdot\, a}{\tilde{\partial}}$. Therefore, it follows

$$\overset{*}{\mathcal{V}}\left(\overset{*}{\tilde{\partial}}_{\alpha}\right)=-\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\overset{\cdot\, b}{\tilde{\partial}}.$$

Theorem 6.2.1.1 *A (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is characterized by the existence of a* **Mod**-*endomorphism* $\overset{*}{\mathcal{V}}$ *of*

$$\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\overset{*}{\mathcal{V}}\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\right)\subset\Gamma\left(\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\right)\\ \overset{*}{\mathcal{V}}\,(X)=X\iff X\in\Gamma\left(\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\right)\end{array} \tag{6.2.1.2}$$

Example 6.2.1.2 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{H}}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \tilde{Z}^{\alpha}\overset{*}{\tilde{\delta}}_{\alpha}+Y_a\overset{\cdot\, a}{\tilde{\partial}} & \longmapsto & \tilde{Z}^{\alpha}\overset{*}{\tilde{\delta}}_{\alpha}\end{array}$$

is a projector which will be called the *horizontal projector.*

Remark 6.2.1.2 We have $\overset{*}{\mathcal{H}}\left(\overset{*}{\tilde{\delta}}_{\alpha}\right)=\overset{*}{\tilde{\delta}}_{\alpha}$ and $\overset{*}{\mathcal{H}}(\overset{\cdot\, a}{\tilde{\partial}})=0$. Therefore, we obtain $\overset{*}{\mathcal{H}}\left(\overset{*}{\tilde{\partial}}_{\alpha}\right)=\overset{*}{\tilde{\delta}}_{\alpha}$.

Theorem 6.2.1.2 *A (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is characterized by the existence of a* **Mod**-*endomorphism $\overset{*}{\mathcal{H}}$ of*

$$\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\subset\Gamma\left(H\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \overset{*}{\mathcal{H}}\,(X)=X\Longleftrightarrow X\in\Gamma\left(H\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right).\end{array}\tag{6.2.1.3}$$

Corollary 6.2.1.1 *A (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is characterized by the existence of a* **Mod**-*endomorphism $\overset{*}{\mathcal{H}}$ of*

$$\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\overset{*}{\mathcal{H}}^{2}=\overset{*}{\mathcal{H}}\\ Ker\left(\overset{*}{\mathcal{H}}\right)=\left(\Gamma\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right).\end{array}\tag{6.2.1.4}$$

Remark 6.2.1.3 For any

$$X\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

we obtain the following unique decomposition

$$X=\overset{*}{\mathcal{H}}X+\overset{*}{\mathcal{V}}X.$$

Proposition 6.2.1.1 *After some calculations we obtain*

$$N_{\overset{*}{\mathcal{V}}}\,(X,Y)=\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{H}}X,\ \overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}=N_{\overset{*}{\mathcal{H}}}\,(X,Y)\,,\tag{6.2.1.5}$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Corollary 6.2.1.2 *The horizontal interior differential system*

$$\left(H\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is involutive if and only if $N_{\overset{*}{\mathcal{V}}}=0$ or $N_{\overset{*}{\mathcal{H}}}=0$.

6.2.2 The almost product structure

Definition 6.2.2.1 Any **Mod**-endomorphism e of

$$\left(\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

with the property

$$e^{2}=Id \tag{6.2.2.1}$$

will be called the *almost product structure.*

Example 6.2.2.1 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{P}}} & \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \tilde{Z}^{\alpha}\overset{*}{\tilde{\delta}}_{\alpha}+Y_{a}\overset{\cdot\, a}{\tilde{\partial}} & \longmapsto & \tilde{Z}^{\alpha}\overset{*}{\tilde{\delta}}_{\alpha}-Y_{a}\overset{\cdot\, a}{\tilde{\partial}}\end{array}$$

is an almost product structure**.**

Remark 6.2.2.1 The previous almost product structure has the properties:

$$\begin{array}{l}\overset{*}{\mathcal{P}}=2\overset{*}{\mathcal{H}}-Id;\\ \overset{*}{\mathcal{P}}=Id-2\overset{*}{\mathcal{V}};\\ \overset{*}{\mathcal{P}}=\overset{*}{\mathcal{H}}-\overset{*}{\mathcal{V}}.\end{array} \tag{6.2.2.2}$$

Remark 6.2.2.2 We obtain that $\overset{*}{\mathcal{P}}\left(\overset{*}{\tilde{\delta}}_{\alpha}\right)=\overset{*}{\tilde{\delta}}_{\alpha}$ and $\overset{*}{\mathcal{P}}\left(\overset{\cdot\, a}{\tilde{\partial}}\right)=-\overset{\cdot\, a}{\tilde{\partial}}$. Therefore, it follows

$$\overset{*}{\mathcal{P}}\left(\overset{*}{\tilde{\partial}}_{\alpha}\right)=\overset{*}{\tilde{\delta}}_{\alpha}-\rho\overset{*}{\Gamma}_{b\alpha}\overset{\cdot\, b}{\tilde{\partial}}.$$

Theorem 6.2.2.1 *A (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is characterized by the existence of a* **Mod**-*endomorphism $\overset{*}{\mathcal{P}}$ of*

$$\left(\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

with the following property:

$$\overset{*}{\mathcal{P}}\left(X\right)=-X\Longleftrightarrow X\in\Gamma\left(V\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right). \tag{6.2.2.3}$$

Proposition 6.2.2.1 *After some calculations, we obtain*

$$N_{\overset{*}{\mathcal{P}}}\left(X,Y\right)=4\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right],$$

for any $X,Y\in\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Corollary 6.2.2.1 *The horizontal interior differential system*

$$\left(H\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is involutive if and only if $N_{\overset{*}{\mathcal{P}}}=0$.

6.2.3 The almost tangent structure

Definition 6.2.3.1 Any **Mod**-endomorphism e of

$$\left(\Gamma((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}),+,\cdot\right)$$

with the property

$$e^2=0 \tag{6.2.3.1}$$

will be called the *almost tangent structure.*

Example 6.2.3.1 If $(E,\pi,M)=(F,\nu,N)$ and $g\in\mathbf{Man}\left(\overset{*}{E},E\right)$ such that (g,h) is a $\mathbf{B^v}$-morphism locally invertible, then the **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{J}}_{(g,h)}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \tilde{Z}^a\overset{*}{\tilde{\partial}}_a+Y_b\overset{\cdot\, b}{\tilde{\partial}} & \longmapsto & \left(\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^a\overset{\cdot\, b}{\tilde{\partial}}\end{array}$$

is an almost tangent structure which will be called the *almost tangent structure associated to* $\mathbf{B^v}$*-morphism* (g,h). (See: **Definition 4.4.2.3**)

Remark 6.2.3.1 We obtain that

$$\overset{*}{\mathcal{J}}_{(g,h)}\left(\overset{*}{\tilde{\delta}}_a\right)=\overset{*}{\mathcal{J}}_{(g,h)}\left(\overset{*}{\tilde{\partial}}_a\right)=\left(\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\, b}{\tilde{\partial}}$$

and

$$\overset{*}{\mathcal{J}}_{(g,h)}\left(\overset{\cdot\, b}{\tilde{\partial}}\right)=0.$$

Remark 6.2.3.2 The previous almost tangent structure has the following properties:

$$\begin{array}{rcl}\overset{*}{\mathcal{J}}_{(g,h)}\circ\overset{*}{\mathcal{P}} & = & \overset{*}{\mathcal{J}}_{(g,h)};\\ \overset{*}{\mathcal{P}}\circ\overset{*}{\mathcal{J}}_{(g,h)} & = & -\overset{*}{\mathcal{J}}_{(g,h)};\\ \overset{*}{\mathcal{J}}_{(g,h)}\circ\overset{*}{\mathcal{H}} & = & \overset{*}{\mathcal{J}}_{(g,h)};\\ \overset{*}{\mathcal{H}}\circ\overset{*}{\mathcal{J}}_{(g,h)} & = & 0;\\ \overset{*}{\mathcal{J}}_{(g,h)}\circ\overset{*}{\mathcal{V}} & = & 0;\\ \overset{*}{\mathcal{V}}\circ\overset{*}{\mathcal{J}}_{(g,h)} & = & \overset{*}{\mathcal{J}}_{(g,h)};\\ N_{\overset{*}{\mathcal{J}}_{(g,h)}} & = & 0.\end{array} \tag{6.2.3.2}$$

6.2.4 The almost complex structure

Let us consider in the case $(E,\pi,M)=(F,\nu,N)$.

bk**Definition 6.2.4.1** Any **Mod**-endomorphism e of

$$\left(\Gamma((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}),+,\cdot\right)$$

with the property

$$e^2=-Id \tag{6.2.4.1}$$

will be called the *almost complex structure.*

Example 6.2.4.1 If (g,h) is a $\mathbf{B^v}$-morphism of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ source and (E,π,M) target locally invertible, then the **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{F}}_{(g,h)}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \tilde{Z}^a\overset{*}{\tilde{\delta}}_a+Y_a\overset{\cdot\,a}{\tilde{\partial}} & \longmapsto & \left(g^{ab}\circ h\circ\overset{*}{\pi}\right)Y_b\overset{*}{\tilde{\delta}}_a-\left(\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^a\overset{\cdot\,b}{\tilde{\partial}}\end{array}$$

is an almost complex structure.

Remark 6.2.4.1 We have

$$\overset{*}{\mathcal{F}}_{(g,h)}\left(\overset{*}{\tilde{\delta}}_a\right)=-\left(\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\,b}{\tilde{\partial}}$$

and

$$\overset{*}{\mathcal{F}}_{(g,h)}\left(\overset{\cdot\,b}{\tilde{\partial}}\right)=\left(g^{ab}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\tilde{\delta}}_a.$$

Therefore, we obtain:

$$\overset{*}{\mathcal{F}}_{(g,h)}\left(\overset{*}{\tilde{\partial}}_c\right)=-\,(\rho,\eta)\,\overset{*}{\Gamma}_{bc}\left(g^{ab}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\tilde{\delta}}_a-\left(\tilde{g}_{bc}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\,b}{\tilde{\partial}}\,.$$

Remark 6.2.4.2 The previous almost complex structure has the following properties:

$$\begin{array}{rcl}\overset{*}{\mathcal{F}}_{(g,h)}\circ\overset{*}{\mathcal{J}}_{(g,h)} & = & \overset{*}{\mathcal{H}};\\ \overset{*}{\mathcal{F}}_{(g,h)}\circ\overset{*}{\mathcal{H}} & = & -\overset{*}{\mathcal{J}}_{(g,h)};\\ \overset{*}{\mathcal{J}}_{(g,h)}\circ\overset{*}{\mathcal{F}}_{(g,h)} & = & \overset{*}{\mathcal{V}}.\end{array} \tag{6.2.4.2}$$

6.2.5 The (ρ,η)-tension endomorphism

Since

$$\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{\acute{b}\acute{\alpha}}}{\partial p_{\acute{a}}}=M_{\acute{b}}^{b}\circ\overset{*}{\pi}\left(-\rho_{\alpha}^{i}\circ h\frac{\partial M_{b}^{\acute{a}}\circ\overset{*}{\pi}}{\partial x^{i}}+\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}}{\partial p_{a}}M_{a}^{\acute{a}}\circ\overset{*}{\pi}\right)\Lambda_{\acute{\alpha}}^{\alpha}\circ h,$$

it results that

$$\left(\rho,\eta\right)\overset{*}{\Gamma}_{\acute{b}\acute{\alpha}}-p_{\acute{a}}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{\acute{b}\acute{\alpha}}}{\partial p_{\acute{a}}}=M_{\acute{b}}^{b}\circ\overset{*}{\pi}\left(\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}-p_{a}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}}{\partial p_{a}}\right)\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\overset{*}{\pi},$$

Therefore, we can introduce the following

Definition 6.2.5.1 The **Mod**-endomorphism

$$\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)\ \xrightarrow{(\rho,\eta)\overset{*}{\mathbb{H}}}\ \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

defined by

$$(6.2.5.1)\qquad\begin{array}{rcl}\left(\rho,\eta\right)\overset{*}{\mathbb{H}}\left(\overset{*}{\tilde{\delta}}_{\alpha}\right) & = & \left(\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}-p_{a}\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}}{\partial p_{a}}\right)\overset{\cdot\,b}{\tilde{\partial}},\\ \left(\rho,\eta\right)\overset{*}{\mathbb{H}}\left(\overset{\cdot\,a}{\tilde{\partial}}\right) & = & 0_{(\rho,\eta)T\overset{*}{E}}\end{array}$$

will be called the (ρ,η)*-tension of* (ρ,η)*-connection* $(\rho,\eta)\overset{*}{\Gamma}$.

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, then we obtain the *tension of connection* $\overset{*}{\Gamma}$.

Proposition 6.2.5.1 *We obtain the following equalities*

$$\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}\circ\left(\rho,\eta\right)\overset{*}{\mathbb{H}}=0=\left(\rho,\eta\right)\overset{*}{\mathbb{H}}\circ\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}.$$

6.3 The (ρ,η,h)-torsion and the (ρ,η,h)-curvature of a (ρ,η)-connection

We consider the following diagram:

$$\begin{array}{ccc}\overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\ \downarrow{\scriptstyle\overset{*}{\pi}} & & \downarrow{\scriptstyle\nu}\\ M & \xrightarrow{\ \ h\ \ } & N\end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Definition 6.3.1 If $(E,\pi,M)=(F,\nu,N)$, then the $\mathcal{F}\left(\overset{*}{E}\right)$-bilinear application

$$\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)^{2}\ \xrightarrow{(\rho,\eta,h)\overset{*}{\mathbb{T}}}\ \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

defined by

$$
\begin{array}{ll}
(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\tilde{\delta}}_b,\overset{*}{\tilde{\delta}}_c\right) & =\left(\dfrac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}}{\partial p_a}-\dfrac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{cb}}{\partial p_a}-L_{bc}^a\circ h\circ\overset{*}{\pi}\right)\overset{*}{\tilde{\delta}}_a;\\
(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\tilde{\delta}}_b,\overset{\cdot\, c}{\tilde{\partial}}\right) & =0=(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{\cdot\, b}{\tilde{\partial}},\overset{*}{\tilde{\delta}}_c\right);\\
(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{\cdot\, b}{\tilde{\partial}},\overset{\cdot\, c}{\tilde{\partial}}\right) & =0;
\end{array}
\tag{6.3.1}
$$

will be called the (ρ,η,h)-*torsion associated to* (ρ,η)-*connection* $(\rho,\eta)\,\Gamma$.

In particular, if $h=Id_M$, then we obtain the (ρ,η)-*torsion associated to* (ρ,η)-*connection* $(\rho,\eta)\overset{*}{\Gamma}$.

Moreover, if $(\rho,\eta)=(Id_{TM},Id_M)$, then we obtain the *torsion associated to connection* $\overset{*}{\Gamma}$.

Remark 6.3.1 If $(\rho,\eta,h)\overset{*}{\mathbb{T}}$ is the (ρ,η,h)-torsion associated to (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$, then

$$
(\rho,\eta,h)\overset{*}{\mathbb{T}}(X,Y)=-\,(\rho,\eta,h)\overset{*}{\mathbb{T}}(Y,X),
\tag{6.3.2}
$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Definition 6.3.2 If we consider the notation

$$
(\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{bc}^{a}\overset{put}{=}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}}{\partial p_a}-\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{cb}}{\partial p_a}-L_{bc}^a\circ h\circ\overset{*}{\pi}
\tag{6.3.3}
$$

then the tensor field

$$
(\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{bc}^{a}\overset{*}{\tilde{\delta}}_a\otimes d\tilde{z}^b\otimes d\tilde{z}^c
\tag{6.3.4}
$$

will be called the (ρ,η,h)-*torsion tensor field associated to* (ρ,η)-*connection* $(\rho,\eta)\overset{*}{\Gamma}$.

Proposition 6.3.1 *We obtain*

$$
\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}\circ(\rho,\eta)\overset{*}{\mathbb{T}}=0
$$

and

$$
\begin{array}{rcl}
(\rho,\eta)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}X,Y\right) & = & (\rho,\eta)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}X,\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}Y\right)\\
 & = & (\rho,\eta)\overset{*}{\mathbb{T}}\left(X,\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}Y\right),
\end{array}
$$

for any $X, Y \in \Gamma\left((\rho,\eta)\, T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$.

Theorem 6.3.1 *Using the* (ρ,η)*-tension tensor field*

$$(6.3.5)\qquad (\rho,\eta)\,\overset{*}{\mathbb{H}}_{ba}\overset{\cdot\, b}{\tilde{\partial}} \otimes d\tilde{z}^a = \left((\rho,\eta)\,\overset{*}{\Gamma}_{ba} - p_c\frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{ba}}{\partial p_c}\right)\overset{\cdot\, b}{\tilde{\partial}} \otimes d\tilde{z}^a,$$

and the (ρ,η,h)*-deflection of the* (ρ,η)*-connection* $(\rho,\eta)\,\overset{*}{\Gamma}$

$$(6.3.6)\qquad (\rho,\eta,h)\,\overset{*}{\mathbb{D}}_{bc} = -\,(\rho,\eta)\,\overset{*}{\Gamma}_{bc} + p_a\frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{cb}}{\partial p_a} + p_a\cdot L^a_{bc}\circ h\circ\overset{*}{\pi},$$

we obtain that $(\rho,\eta,h)\overset{*}{\mathbb{D}}_{bc}$=0 *if and only if* $(\rho,\eta)\overset{*}{\mathbb{H}}_{bc}$=0 *and* $(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^a_{bc}$=0.

Proof. If $(\rho,\eta,h)\,\overset{*}{\mathbb{D}}_{bc}$=0, then deriving with respect to p_a, we obtain:

$$-\frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{bc}}{\partial p_a} + \frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{cb}}{\partial p_a} + L^a_{bc}\circ h\circ\overset{*}{\pi} = 0 \Longleftrightarrow (\rho,\eta,h)\,\overset{*}{\mathbb{T}}{}^a_{bc} = 0.$$

The equality $(\rho,\eta,h)\,\overset{*}{\mathbb{D}}_{bc}$=0 implies:

$$(1)\qquad (\rho,\eta)\,\overset{*}{\Gamma}_{bc} = p_a\frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{cb}}{\partial p_a} + p_a L^a_{bc}\circ h\circ\overset{*}{\pi}.$$

Since

$$\begin{aligned}(\rho,\eta)\,\overset{*}{\mathbb{H}}_{bc} &= (\rho,\eta)\,\overset{*}{\Gamma}_{bc} - p_a\frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{bc}}{\partial p_a}\\ &= p_a\frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{cb}}{\partial p_a} - p_a\frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{bc}}{\partial p_a} + p_a L^a_{bc}\circ h\circ\overset{*}{\pi} = p_a\,(\rho,\eta,h)\,\overset{*}{\mathbb{T}}{}^a_{bc}\end{aligned}$$

it results the equality $(\rho,\eta)\,\overset{*}{\mathbb{H}}_{bc}$=0.

Conservely, if $(\rho,\eta,h)\,\overset{*}{\mathbb{T}}{}^a_{bc}$=0, then, multiplying with p_a, we obtain:

$$(2)\qquad p_a\frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{cb}}{\partial p_a} - p_a\frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{bc}}{\partial p_a} + p_a L^a_{bc}\circ h\circ\overset{*}{\pi} = 0.$$

The equality $(\rho,\eta)\,\overset{*}{\mathbb{H}}_{bc}$=0 is equivalent with:

$$(3)\qquad (\rho,\eta)\,\overset{*}{\Gamma}_{bc} = p_a\frac{\partial\,(\rho,\eta)\,\overset{*}{\Gamma}_{bc}}{\partial p_a}.$$

Using (2) and (3), it results the equality $(\rho,\eta,h)\,\overset{*}{\mathbb{D}}_{bc} = 0$. *q.e.d.*

Definition 6.3.3 The $\mathcal{F}\left(\overset{*}{E}\right)$-bilinear application

$$\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)^{2}\ \xrightarrow{(\rho,\eta,h)\overset{*}{\mathbb{R}}}\ \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

defined by

$$\begin{array}{c}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(\overset{*}{\tilde{\delta}}_{\alpha},\overset{*}{\tilde{\delta}}_{\beta}\right)=\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}_{b\ \ \alpha\beta}\overset{\cdot\ b}{\tilde{\partial}}\ ;\\ \left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(\overset{*}{\tilde{\delta}}_{\alpha},\overset{\cdot\ b}{\tilde{\partial}}\right)=0=\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(\overset{\cdot\ b}{\tilde{\partial}},\overset{*}{\tilde{\delta}}_{\alpha}\right);\\ \left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(\overset{\cdot\ a}{\tilde{\partial}},\overset{\cdot\ b}{\tilde{\partial}}\right)=0;\end{array}\tag{6.3.7}$$

will be called the *(ρ,η,h)-curvature associated to (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$*.

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, then we obtain the *curvature associated to connection $\overset{*}{\Gamma}$*.

Remark 6.3.2 If $(\rho,\eta,h)\overset{*}{\mathbb{R}}$ is the (ρ,η,h)-curvature associated to (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$, then

$$\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)=-\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(Y,X\right),\tag{6.3.8}$$

for any $X,Y\in\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Definition 4.3.4 The tensor field

$$\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}_{b\ \ \alpha\beta}\overset{\cdot\ b}{\tilde{\partial}}\otimes d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}\tag{4.3.9}$$

will be called the *(ρ,η,h)-curvature tensor field associated to the (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$*.

Using equality (4.1.5) we obtain

Remark 6.3.3 The horizontal interior differential system $\left(H\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$ is involutive if and only if the (ρ,η,h)-curvature tensor field associated to the (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$ is null.

6.4 Tensor d-fields. Distinguished linear (ρ,η)-connections

We consider the following diagram:

$$\begin{array}{ccc}\overset{*}{E} & & \left(F,\left[,\right]_{F,h},\left(\rho,\eta\right)\right)\\ \downarrow{\scriptstyle\overset{*}{\pi}} & & \downarrow{\scriptstyle\nu}\\ M & \xrightarrow{\ \ h\ \ } & N\end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$ is a generalized Lie algebroid.

Let

$$\left(\mathcal{T}\,{}^{p,r}_{q,s}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

be the $\mathcal{F}\left(\overset{*}{E}\right)$-module of tensor fields by $\binom{p,r}{q,s}$-type from the generalized tangent bundle

$$\left(H\left(\rho,\eta\right)T\overset{*}{E}\oplus V\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right).$$

An arbitrarily tensor field T is written by the form:

$$T=T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\overset{*}{\tilde{\delta}}_{\alpha_1}\otimes\ldots\otimes\overset{*}{\tilde{\delta}}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes d\tilde{z}^{\beta_q}\otimes\tilde{\partial}^{\cdot\,b_1}\otimes\ldots\otimes\tilde{\partial}^{\cdot\,b_s}\otimes\delta\tilde{p}_{a_1}\otimes\ldots\otimes\delta\tilde{p}_{a_r}.$$

Let

$$\left(\mathcal{T}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot,\otimes\right)$$

be the tensor algebra of generalized tangent bundle $\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

If $T_1\in\mathcal{T}^{p_1,r_1}_{q_1,s_1}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$ and $T_2\in\mathcal{T}^{p_2,r_2}_{q_2,s_2}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$, then the components of product tensor field $T_1\otimes T_2$ are the products of local components of T_1 and T_2.

Therefore, we obtain $T_1\otimes T_2\in\mathcal{T}^{p_1+p_2,r_1+r_2}_{q_1+q_2,s_1+s_2}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Let $\mathcal{DT}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$ be the family of tensor fields

$$T\in\mathcal{T}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

for which there exists

$$T_1\in\mathcal{T}^{p,0}_{q,0}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and

$$T_2\in\mathcal{T}^{0,r}_{0,s}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

such that $T=T_1+T_2$.

The $\mathcal{F}\left(\overset{*}{E}\right)$-module $\left(\mathcal{DT}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$ will be called the *module of distinguished tensor fields* or the *module of tensor d-fields.*

Remark 6.4.1 The elements of

$$\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

respectively

$$\Gamma\left(\left(\left(\rho,\eta\right)T\overset{*}{E}\right)^*,\left(\left(\rho,\eta\right)\tau_{\overset{*}{E}}\right)^*,\overset{*}{E}\right)$$

are tensor d-fields.

Definition 6.4.1 Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and let

$$(X,T)\xrightarrow{(\rho,\eta)\overset{*}{D}}(\rho,\eta)\overset{*}{D}_XT \tag{6.4.1}$$

be a covariant (ρ,η)-derivative for the tensor algebra of generalized tangent bundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

which preserves the horizontal and vertical distributions by parallelism.

If $\left(U,\overset{*}{s}_U\right)$ is a vector local $(m+r)$-chart for $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then the real local functions

$$\left((\rho,\eta)\,\overset{*}{H}{}^{\alpha}_{\beta\gamma},(\rho,\eta)\,\overset{*}{H}{}^{a}_{b\gamma},(\rho,\eta)\,\overset{*}{V}{}^{\alpha c}_{\beta},(\rho,\eta)\,\overset{*}{V}{}^{bc}_{a}\right)$$

defined on $\overset{*}{\pi}{}^{-1}(U)$ and determined by the following equalities:

$$\begin{array}{ll}(\rho,\eta)\,\overset{*}{D}_{\overset{*}{\tilde{\delta}}_\gamma}\overset{*}{\tilde{\delta}}_\beta=(\rho,\eta)\,\overset{*}{H}{}^{\alpha}_{\beta\gamma}\overset{*}{\tilde{\delta}}_\alpha, & (\rho,\eta)\,\overset{*}{D}_{\overset{*}{\tilde{\delta}}_\gamma}\tilde{\partial}^{\cdot a}=(\rho,\eta)\,\overset{*}{H}{}^{a}_{b\gamma}\tilde{\partial}^{\cdot b}\\ (\rho,\eta)\,\overset{*}{D}_{\tilde{\partial}^{\cdot c}}\overset{*}{\tilde{\delta}}_\beta=(\rho,\eta)\,\overset{*}{V}{}^{\alpha c}_{\beta}\overset{*}{\tilde{\delta}}_\alpha, & (\rho,\eta)\,\overset{*}{D}_{\tilde{\partial}^{\cdot c}}\tilde{\partial}^{\cdot b}=(\rho,\eta)\,\overset{*}{V}{}^{bc}_{a}\tilde{\partial}^{\cdot a}\end{array} \tag{6.4.2}$$

are the components of a linear (ρ,η)-connection

$$\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$$

for the generalized tangent bundle $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ which will be called the *distinguished linear (ρ,η)-connection.*

If $h=Id_M$, then the distinguished linear (Id_{TM},Id_M)-connection will be called the *distinguished linear connection.*

The components of a distinguished linear connection $\left(\overset{*}{H},\overset{*}{V}\right)$ will be denoted

$$\left(\overset{*}{H}{}^{i}_{jk},\overset{*}{H}{}^{a}_{bk},\overset{*}{V}{}^{ic}_{j},\overset{*}{V}{}^{bc}_{a}\right).$$

Theorem 6.4.1 *If* $((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V})$ *is a distinguished linear* (ρ,η)*-connection for the generalized tangent bundle* $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$*, then its components satisfy the*

change relations:

$$
(6.4.3)\quad
\begin{aligned}
(\rho,\eta)\overset{*}{H}{}^{\acute\alpha}_{\acute\beta\acute\gamma} &= \Lambda^{\acute\alpha}_{\alpha}\circ h\circ\overset{*}{\pi}\left[\Gamma\left(\overset{*}{\tilde\rho}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde\delta}_{\gamma}\right)\left(\Lambda^{\alpha}_{\acute\beta}\circ h\circ\overset{*}{\pi}\right)+\right.\\
&\left.+\,(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\gamma}\cdot\Lambda^{\beta}_{\acute\beta}\circ h\circ\overset{*}{\pi}\right]\cdot\Lambda^{\gamma}_{\acute\gamma}\circ h\circ\overset{*}{\pi},\\
(\rho,\eta)\overset{*}{H}{}^{\acute a}_{\acute b\acute\gamma} &= M^{\acute a}_{a}\circ\overset{*}{\pi}\left[\Gamma\left(\overset{*}{\tilde\rho}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde\delta}_{\gamma}\right)\left(M^{a}_{\acute b}\circ\overset{*}{\pi}\right)+\right.\\
&\left.+\,(\rho,\eta)\overset{*}{H}{}^{a}_{b\gamma}\cdot M^{b}_{\acute b}\circ\overset{*}{\pi}\right]\cdot\Lambda^{\gamma}_{\acute\gamma}\circ h\circ\overset{*}{\pi},\\
(\rho,\eta)\overset{*}{V}{}^{\acute\alpha\acute c}_{\acute\beta} &= \Lambda^{\acute\alpha}_{\alpha}\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\beta}\cdot\Lambda^{\beta}_{\acute\beta}\circ h\circ\overset{*}{\pi}\cdot M^{\acute c}_{c}\circ\overset{*}{\pi},\\
(\rho,\eta)\overset{*}{V}{}^{\acute a\acute c}_{\acute b} &= M^{\acute a}_{a}\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{V}{}^{ac}_{b}\cdot M^{b}_{\acute b}\circ\overset{*}{\pi}\cdot M^{\acute c}_{c}\circ\overset{*}{\pi}.
\end{aligned}
$$

The components of a distinguished linear connection $\left(\overset{*}{H},\overset{*}{V}\right)$ *verify the change relations:*

$$
(6.4.3)'\quad
\begin{aligned}
\overset{*}{H}{}^{\acute i}_{\acute j\acute k} &= \frac{\partial x^{\acute i}}{\partial x^{i}}\circ\overset{*}{\pi}\cdot\left[\frac{\delta}{\delta x^{k}}\left(\frac{\partial x^{i}}{\partial x^{\acute j}}\circ\overset{*}{\pi}\right)+\overset{*}{H}{}^{i}_{jk}\cdot\frac{\partial x^{j}}{\partial x^{\acute j}}\circ\overset{*}{\pi}\right]\cdot\frac{\partial x^{k}}{\partial x^{\acute k}}\circ\overset{*}{\pi},\\
\overset{*}{H}{}^{\acute a}_{\acute b\acute k} &= M^{\acute a}_{a}\circ\overset{*}{\pi}\cdot\left[\frac{\delta}{\delta x^{k}}\left(M^{a}_{\acute b}\circ\overset{*}{\pi}\right)+\overset{*}{H}{}^{a}_{bk}\cdot M^{b}_{\acute b}\circ\overset{*}{\pi}\right]\cdot\frac{\partial x^{k}}{\partial x^{\acute k}}\circ\overset{*}{\pi},\\
\overset{*}{V}{}^{\acute i\acute c}_{\acute j} &= \frac{\partial x^{\acute i}}{\partial x^{i}}\circ\overset{*}{\pi}\cdot\overset{*}{V}{}^{ic}_{j}\,\frac{\partial x^{j}}{\partial x^{\acute j}}\circ\overset{*}{\pi}\cdot M^{c}_{\acute c}\circ\overset{*}{\pi},\\
\overset{*}{V}{}^{\acute a\acute c}_{\acute b} &= M^{\acute a}_{a}\circ\overset{*}{\pi}\cdot\overset{*}{V}{}^{ac}_{b}\cdot M^{b}_{\acute b}\circ\overset{*}{\pi}M^{c}_{\acute c}\circ\overset{*}{\pi}.
\end{aligned}
$$

Example 6.4.1 If $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is endowed with the (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$, then the local real functions

$$\left(\frac{\partial\,(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}}{\partial p_{a}},\frac{\partial\,(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}}{\partial p_{a}},0,0\right)$$

are the components of a distinguished linear (ρ,η)-connection for the generalized tangent bundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

which will by called the *Berwald linear* (ρ,η)*-connection.*

The Berwald linear (Id_{TM},Id_{M})-connection will be called the *Berwald linear connection.*

Theorem 6.4.2 *If the generalized tangent bundle* $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ *is endowed with a distinguished linear* (ρ,η)*-connection* $((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V})$, *then, for any*

$$X=\tilde Z^{\gamma}\overset{*}{\tilde\delta}_{\gamma}+Y_{a}\dot{\tilde\partial}^{a}\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and for any

$$T \in \mathcal{T}\,{}^{pr}_{qs}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

we obtain the formula:

$$\begin{array}{l}(\rho,\eta)\,D_X\left(T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\overset{*}{\tilde{\delta}}_{\alpha_1}\otimes\ldots\otimes\overset{*}{\tilde{\delta}}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes\right.\\ \left.\otimes d\tilde{z}^{\beta_q}\otimes\tilde{\partial}^{\cdot\, b_1}\otimes\ldots\otimes\tilde{\partial}^{\cdot\, b_s}\otimes\delta\tilde{p}_{a_1}\otimes\ldots\otimes\delta\tilde{p}_{a_r}\right)=\\ =\tilde{Z}^{\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s|\gamma}\overset{*}{\tilde{\delta}}_{\alpha_1}\otimes\ldots\otimes\overset{*}{\tilde{\delta}}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes d\tilde{z}^{\beta_q}\otimes\tilde{\partial}^{\cdot\, b_1}\otimes\ldots\otimes\\ \otimes\tilde{\partial}^{\cdot\, b_s}\otimes\delta\tilde{p}_{a_1}\otimes\ldots\otimes\delta\tilde{p}_{a_r}+Y_cT^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\,|^c\,\overset{*}{\tilde{\delta}}_{\alpha_1}\otimes\ldots\otimes\\ \otimes\overset{*}{\tilde{\delta}}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes d\tilde{z}^{\beta_q}\otimes\tilde{\partial}^{\cdot\, b_1}\otimes\ldots\otimes\tilde{\partial}^{\cdot\, b_s}\otimes\delta\tilde{p}_{a_1}\otimes\ldots\otimes\delta\tilde{p}_{a_r},\end{array}$$

where

$$\begin{array}{l}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s|\gamma}=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\\ \quad+(\rho,\eta)\,\overset{*}{H}{}^{\alpha_1}_{\alpha\gamma}T^{\alpha\alpha_2\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}+\ldots+(\rho,\eta)\,\overset{*}{H}{}^{\alpha_p}_{\alpha\gamma}T^{\alpha_1\ldots\alpha_{p-1}\alpha a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\\ \quad-(\rho,\eta)\,\overset{*}{H}{}^{\beta}_{\beta_1\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta\beta_2\ldots\beta_q b_1\ldots b_s}-\ldots-(\rho,\eta)\,\overset{*}{H}{}^{\beta}_{\beta_q\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_{q-1}\beta b_1\ldots b_s}\\ \quad-(\rho,\eta)\,\overset{*}{H}{}^{a_1}_{a\gamma}T^{\alpha_1\ldots\alpha_p a a_2\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}-\ldots-(\rho,\eta)\,\overset{*}{H}{}^{a_r}_{a\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_{r-1}a}_{\beta_1\ldots\beta_q b_1\ldots b_s}\\ \quad+(\rho,\eta)\,\overset{*}{H}{}^{b}_{b_1\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b b_2\ldots b_s}+\ldots+(\rho,\eta)\,\overset{*}{H}{}^{b}_{b_s\gamma}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_{s-1}b}\end{array}$$

and

$$\begin{array}{l}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\,|^c=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\partial}^{\cdot\, c}\right)T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}+\\ \quad+(\rho,\eta)\,\overset{*}{V}{}^{\alpha_1 c}_{\alpha}\,T^{\alpha\alpha_2\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}+\ldots+(\rho,\eta)\,\overset{*}{V}{}^{\alpha_p c}_{\alpha}\,T^{\alpha_1\ldots\alpha_{p-1}\alpha a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\\ \quad-(\rho,\eta)\,\overset{*}{V}{}^{\beta c}_{\beta_1}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta\beta_2\ldots\beta_q b_1\ldots b_s}-\ldots-(\rho,\eta)\,\overset{*}{V}{}^{\beta c}_{\beta_q}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_{q-1}\beta b_1\ldots b_s}\\ \quad-(\rho,\eta)\,\overset{*}{V}{}^{a_1 c}_{a}\,T^{\alpha_1\ldots\alpha_p a a_2\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}-\ldots-(\rho,\eta)\,\overset{*}{V}{}^{a_r c}_{a}\,T^{\alpha_1\ldots\alpha_p a_1\ldots a_{r-1}a}_{\beta_1\ldots\beta_q b_1\ldots b_s}\\ \quad+(\rho,\eta)\,\overset{*}{V}{}^{bc}_{b_1}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b b_2\ldots b_s}\ldots+(\rho,\eta)\,\overset{*}{V}{}^{bc}_{b_s}T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_{s-1}b}.\end{array}$$

Definition 6.4.2 We assume that $(E,\pi,M)=(F,\nu,N)$.

If $(\rho,\eta)\,\overset{*}{\Gamma}$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and

$$\left((\rho,\eta)\,\overset{*}{H}{}^{a}_{bc},(\rho,\eta)\,\overset{*}{\tilde{H}}{}^{a}_{bc},(\rho,\eta)\,\overset{*}{V}{}^{ac}_{b},(\rho,\eta)\,\overset{*}{\tilde{V}}{}^{ac}_{b}\right)$$

are the components of a distinguished linear (ρ,η)-connection for the generalized tangent bundle $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ such that

$$(\rho,\eta)\,\overset{*}{H}{}^{a}_{bc}=(\rho,\eta)\,\overset{*}{\tilde{H}}{}^{a}_{bc} \text{ and } (\rho,\eta)\,\overset{*}{V}{}^{ac}_{b}=(\rho,\eta)\,\overset{*}{\tilde{V}}{}^{ac}_{b},$$

then we will say that *the generalized tangent bundle* $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ *is endowed with a normal distinguished linear* (ρ,η)*-connection on components*

$$\left((\rho,\eta)\,\overset{*}{H}{}^{a}_{bc},(\rho,\eta)\,\overset{*}{V}{}^{ac}_{b}\right).$$

The components of a normal distinguished linear (Id_{TM},Id_M)-connection $\left(\overset{*}{H},\overset{*}{V}\right)$ will be denoted $\left(\overset{*}{H}{}^{i}_{jk},\overset{*}{V}{}^{i}_{jk}\right)$.

6.5 The lift of accelerations for a differentiable curve

We consider the following diagram:

$$\begin{tikzcd}
\overset{*}{E} \arrow[d, "\overset{*}{\pi}"'] & \left(F,[,]_{F,h},(\rho,\eta)\right) \arrow[d, "\nu"] \\
M \arrow[r, "h"] & N
\end{tikzcd}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)\in|\mathbf{GLA}|$.

Let $(\rho,\eta)\,\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

We admit that $\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$ is a distinguished linear (ρ,η)-connection for the vector bundle $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Let $g\in\mathbf{Man}\left(\overset{*}{E},E\right)$ be such that (g,h) is a $\mathbf{B^v}$-morphism of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ source and (E,π,M) target.

Let

$$\begin{array}{ccc} I & \xrightarrow{\dot{c}} & \overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)} \\ t & \longmapsto & p_a(t)\,s^a(\eta\circ h\circ c(t)) \end{array} \tag{6.5.1}$$

be the (g,h)-lift of differentiable curve $I\xrightarrow{c}M$.

Definition 6.5.1 The differentiable curve

$$\begin{array}{ccc} I & \xrightarrow{\ddot{c}} & (\rho,\eta)\,T\overset{*}{E}_{|\operatorname{Im}\dot{c}} \\ t & \longmapsto & g^{\alpha a}(h\circ c(t))\,p_a(t)\,\overset{*}{\tilde{\partial}}_{\alpha}(\dot{c}(t))+\dfrac{dp_a(t)}{dt}\dot{\tilde{\partial}}{}^{a}(\dot{c}(t)) \end{array} \tag{6.5.2}$$

will be called the *differentiable (g,h)-lift of accelerations of the differentiable curve c.*

The section

$$\text{(6.5.3)}\qquad \begin{array}{rcl}\operatorname{Im}(\dot c) & \xrightarrow{\overset{*}{u}(c,\dot c,\ddot c)} & (\rho,\eta)\,T\overset{*}{E}_{|\operatorname{Im}(\dot c)}\\ \dot c(t) & \longmapsto & g^{\alpha a}(\eta\circ h\circ c(t))\cdot p_a(t)\,\overset{*}{\tilde\delta}_\alpha(\dot c(t))+\dfrac{dp_a(t)}{dt}\overset{\cdot\, a}{\tilde\partial}(\dot c(t))\end{array}$$

will be called the *canonical section associated to the triple* $(c,\dot c,\ddot c)$.

Remark 6.5.1 For any $t\in I$, we obtain:

$$\text{(6.5.4)}\qquad \begin{array}{c}\overset{*}{u}(c,\dot c,\ddot c)(\dot c(t))=g^{\alpha a}(\eta\circ h\circ c(t))\cdot p_a(t)\,\overset{*}{\tilde\delta}_\alpha(\dot c(t))+\dfrac{dp_b(t)}{dt}\overset{\cdot\, b}{\tilde\partial}(\dot c(t))\\ +(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\circ\overset{*}{u}(c,\dot c)\circ\eta\circ h\circ c(t)\cdot g^{\alpha a}\circ h\circ c(t)\cdot p_a(t)\overset{\cdot\, b}{\tilde\partial}(\dot c(t)).\end{array}$$

We observe easily that $\overset{*}{u}(c,\dot c,\ddot c)(\dot c(t))\in H(\rho,\eta)\,T\overset{*}{E}_{|\operatorname{Im}(\dot c)}$if and only if the components functions $\left(p_a,\ a\in\overline{1,r}\right)$ are solutions for the differentiable equations

$$\text{(6.5.5)}\qquad \frac{du_b}{dt}+(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\circ\overset{*}{u}(c,\dot c)\circ\eta\circ h\circ c\cdot g^{\alpha a}\circ h\circ c\cdot u_a,\ a\in\overline{1,r}.$$

Remark 6.5.2 In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, then, using the differentiable (g,Id_M)-lift

$$\text{(6.5.6)}\qquad \begin{array}{rcl} I & \xrightarrow{\dot c} & \overset{*}{TM}\\ t & \longmapsto & \tilde g_{ji}(c(t))\dfrac{dc^j(t)}{dt}\cdot dx^i(c(t)),\end{array}$$

we obtain the (g,Id_M)-lift of accelerations of the differentiable curve c as being

$$\text{(6.5.7)}\qquad \begin{array}{rcl} I & \xrightarrow{\ddot c} & (Id_{TM},Id_M)\,T\overset{*}{E}_{|\operatorname{Im}(\dot c)}\\ t & \longmapsto & \dfrac{dc^i(t)}{dt}\cdot\dfrac{\partial}{\partial\tilde z^i}(\dot c(t))+\tilde g_{ji}(c(t))\dfrac{dc^j(t)}{dt}\cdot\dfrac{\partial}{\partial\tilde p_i}(\dot c(t))\end{array}$$

Definition 6.5.2 If the component functions

$$\left((g^{\alpha a}\circ h\circ c)\cdot p_a,\ a\in\overline{1,r}\right)$$

are solutions for the differentiable system of equations

$$\text{(6.5.8)}\qquad \frac{dz^\alpha}{dt}+(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\gamma}\circ\overset{*}{u}(c,\dot c)\circ\eta\circ h\circ c\cdot z^\beta\cdot z^\gamma=0,\ \alpha\in\overline{1,p},$$

then the differentiable curve $\dot c$ will be called *horizontal parallel with respect to the distinguished linear (ρ,η)-connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$.

If the component functions $\left(p_a,\ a\in\overline{1,r}\right)$ are solutions for the differentiable system of equations

$$\text{(6.5.9)}\qquad \frac{du_b}{dt}-(\rho,\eta)\overset{*}{V}{}_b^{\ ac}\circ\overset{*}{u}(c,\dot c)\circ\eta\circ h\circ c\cdot u_a\cdot u_c=0,\ b\in\overline{1,r}.$$

then the differentiable curve $\dot{c}$ will be called *vertical parallel with respect to the distinguished linear (ρ,η)-connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$.

Remark 6.5.3 In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, then the differentiable (g,Id_M)-lift (6.5.6) is horizontal parallel with respect to the distinguished linear connection $\left(\overset{*}{H},\overset{*}{V}\right)$ if the component functions $\left(\frac{dc^j(t)}{dt},\ i\in\overline{1,m}\right)$ are solutions for the differentiable system of equations

$$\frac{dz^i(t)}{dt}+\overset{*}{H}{}^i_{jk}\circ\overset{*}{u}(c,\dot{c})\circ c\cdot z^j\cdot z^k=0,\ i\in\overline{1,m}. \tag{6.5.11}$$

Moreover, the differentiable (g,Id_M)-lift (4.5.6) is vertical parallel with respect to the distinguished linear connection $\left(\overset{*}{H},\overset{*}{V}\right)$ if the component functions $\left(\tilde{g}_{ji}\circ c\cdot\frac{dc^j}{dt},\ i\in\overline{1,m}\right)$ are solutions for the differentiable system of equations

$$\frac{du_j}{dt}+\overset{*}{V}{}^{ik}_{j}\circ\overset{*}{u}(c,\dot{c})\circ c\cdot u_i\cdot u_k=0,\ j\in\overline{1,m}. \tag{6.5.12}$$

6.6 The (ρ,η,h)-torsion and the (ρ,η,h)-curvature of a distinguished linear (ρ,η)-connection

We consider the following diagram:

$$\begin{array}{ccc}\overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\ \overset{*}{\pi}\downarrow & & \downarrow\nu\\ M & \xrightarrow{h} & N\end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,M),[,]_{F,h},(\rho,\eta)\right)\in|\mathbf{GLA}|$.

Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and let

$$\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$$

be a distinguished linear (ρ,η)-connection for the generalized tangent bundle

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right).$$

Definition 6.6.1 The application

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)^2 & \xrightarrow{(\rho,\eta,h)\mathbb{T}} & \Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ (X,Y) & \longmapsto & (\rho,\eta,h)\overset{*}{\mathbb{T}}(X,Y)\end{array}$$

defined by

$$(\rho,\eta,h)\overset{*}{\mathbb{T}}(X,Y)=(\rho,\eta)\overset{*}{D}_XY-(\rho,\eta)\overset{*}{D}_YX-[X,Y]_{(\rho,\eta)T\overset{*}{E}}, \tag{6.6.1}$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$, will be called the (ρ,η,h)*-torsion associated to distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$.

The applications

$$\overset{*}{\mathcal{H}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{H}}(\cdot),\overset{*}{\mathcal{H}}(\cdot)\right),\ \overset{*}{\mathcal{H}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{H}}(\cdot),\overset{*}{\mathcal{H}}(\cdot)\right),...,\overset{*}{\mathcal{V}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{V}}(\cdot),\overset{*}{\mathcal{V}}(\cdot)\right)$$

are called $\overset{*}{\mathcal{H}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right),\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right),...,\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{V}}\overset{*}{\mathcal{V}}\right)$ (ρ,η,h)*-torsions associated to distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$.

Proposition 6.6.1 *The* (ρ,η,h)*-torsion* $(\rho,\eta,h)\overset{*}{\mathbb{T}}$ *associated to distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$*, is* $\mathbb{R}$*-bilinear and antisymmetric in the lower indices.*

Using the notations:

$$\begin{array}{ll}\overset{*}{\mathcal{H}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\tilde{\delta}}_\gamma,\overset{*}{\tilde{\delta}}_\beta\right) & =(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{\alpha}{}_{\beta\gamma}\overset{*}{\tilde{\delta}}_\alpha,\\ \overset{*}{\mathcal{V}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\tilde{\delta}}_\gamma,\overset{*}{\tilde{\delta}}_\beta\right) & =(\rho,\eta,h)\overset{*}{\mathbb{T}}_{b\beta\gamma}\overset{\cdot\, b}{\tilde{\partial}},\\ \overset{*}{\mathcal{H}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{\cdot\, c}{\tilde{\partial}},\overset{*}{\tilde{\delta}}_\beta\right) & =(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\alpha}{}_{\beta}{}^{c}\overset{*}{\tilde{\delta}}_\alpha,\\ \overset{*}{\mathcal{V}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{\cdot\, c}{\tilde{\partial}},\overset{*}{\tilde{\delta}}_\beta\right) & =(\rho,\eta,h)\overset{*}{\mathbb{P}}_{b\beta}{}^{c}\overset{\cdot\, b}{\tilde{\partial}},\\ \overset{*}{\mathcal{V}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{\cdot\, c}{\tilde{\partial}},\overset{\cdot\, b}{\tilde{\partial}}\right) & =(\rho,\eta,h)\overset{*}{\mathbb{S}}_a{}^{bc}\overset{\cdot\, a}{\tilde{\partial}},\end{array} \tag{6.6.2}$$

we can easily prove the following

Theorem 6.6.1 *The* (ρ,η,h)*-torsion* $(\rho,\eta,h)\overset{*}{\mathbb{T}}$ *associated to the distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$*, is characterized by the tensor fields with local*

components:

$$
(6.6.3)\qquad
\begin{aligned}
(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{\alpha}{}_{\beta\gamma} &= (\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\gamma} - (\rho,\eta)\overset{*}{H}{}^{\alpha}_{\gamma\beta} - L^{\alpha}_{\beta\gamma}\circ h\circ\overset{*}{\pi},\\
(\rho,\eta,h)\overset{*}{\mathbb{T}}_{b\beta\gamma} &= -\,(\rho,\eta,h)\overset{*}{\mathbb{R}}_{b\ \beta\gamma},\\
(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\alpha\ c}_{\ \beta} &= (\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\beta}\ ,\\
(\rho,\eta,h)\overset{*}{\mathbb{P}}_{b\beta}{}^{c} &= \frac{\partial}{\partial p_c}\left((\rho,\eta)\overset{*}{\Gamma}_{b\beta}\right) - (\rho,\eta)\overset{*}{H}{}^{c}_{b\beta},\\
(\rho,\eta,h)\overset{*}{\mathbb{S}}_{a}{}^{bc} &= (\rho,\eta)\overset{*}{V}{}^{bc}_{a} - (\rho,\eta)\overset{*}{V}{}^{cb}_{a}.
\end{aligned}
$$

In particular, when $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, we regain the local components of torsion associated to distinguished linear connection $\left(\overset{*}{H},\overset{*}{V}\right)$*:*

$$
(6.6.3)'\qquad
\begin{aligned}
\overset{*}{\mathbb{T}}{}^{i}{}_{jk} &= \overset{*}{H}{}^{i}_{jk} - \overset{*}{H}{}^{i}_{kj}, & \overset{*}{\mathbb{T}}_{bjk} &= -\overset{*}{\mathbb{R}}_{b\ jk},\\
\overset{*}{\mathbb{P}}{}^{i\ c}_{\ j} &= \overset{*}{V}{}^{ic}_{j}\ , & \overset{*}{\mathbb{P}}_{b\ k}^{\ c} &= \frac{\partial\overset{*}{\Gamma}_{bk}}{\partial p_c} - \overset{*}{H}{}^{c}_{bk},\\
\overset{*}{\mathbb{S}}_{a}{}^{bc} &= \overset{*}{V}{}^{bc}_{a} - \overset{*}{V}{}^{bc}_{a}.
\end{aligned}
$$

Definition 6.6.2 The application

$$
\begin{array}{ccc}
\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\right)^3 & \xrightarrow{(\rho,\eta,h)\overset{*}{\mathbb{R}}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\
\left(\left(\tilde{Y},\tilde{Z}\right),\tilde{X}\right) & \longmapsto & (\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\tilde{Y},\tilde{Z}\right)\tilde{X}
\end{array}
$$

defined by:

$$
(6.6.4)\qquad
\begin{aligned}
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\tilde{Y},\tilde{Z}\right)\tilde{X} &= (\rho,\eta)\overset{*}{D}_{\tilde{Y}}\left((\rho,\eta)\overset{*}{D}_{\tilde{Z}}\tilde{X}\right) -\\
&-(\rho,\eta)\overset{*}{D}_{\tilde{Z}}\left((\rho,\eta)\overset{*}{D}_{\tilde{Y}}\tilde{X}\right) - (\rho,\eta)\overset{*}{D}_{\left[\tilde{Y},\tilde{Z}\right]_{(\rho,\eta)T\overset{*}{E}}}\tilde{X},
\end{aligned}
$$

for any $\tilde{X},\tilde{Y},\tilde{Z}\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$, will be called the (ρ,η,h)-*curvature associated to distinguished linear (ρ,η)-connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$.

Proposition 6.6.2 *The (ρ,η,h)-curvature $(\rho,\eta,h)\overset{*}{\mathbb{R}}$ associated to distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$, is $\mathbb{R}$-linear in each argument and antisymmetric in the first two arguments.*

Using the notations:

$$(6.6.5)\qquad \begin{aligned}
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\tilde{\overset{*}{\delta}}_{\varepsilon},\tilde{\overset{*}{\delta}}_{\gamma}\right)\tilde{\overset{*}{\delta}}_{\beta} &= (\rho,\eta,h)\overset{*}{\mathbb{R}}{}^{\alpha}_{\beta\ \gamma\varepsilon}\tilde{\overset{*}{\delta}}_{\alpha},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\tilde{\overset{*}{\delta}}_{\varepsilon},\tilde{\overset{*}{\delta}}_{\gamma}\right)\tilde{\dot{\partial}}{}^{a} &= (\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{b\ \gamma\varepsilon}^{\ a}\tilde{\dot{\partial}}{}^{b},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\tilde{\overset{*}{\delta}}_{\gamma},\tilde{\dot{\partial}}{}^{b}\right)\tilde{\overset{*}{\delta}}_{\varepsilon} &= (\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\alpha\ b}_{\ \varepsilon\ \gamma}\tilde{\overset{*}{\delta}}_{\alpha},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\tilde{\overset{*}{\delta}}_{\gamma},\tilde{\dot{\partial}}{}^{b}\right)\tilde{\dot{\partial}}{}^{a} &= (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{c\ \ \gamma}^{\ ab}\tilde{\dot{\partial}}{}^{c},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\tilde{\dot{\partial}}{}^{c},\tilde{\dot{\partial}}{}^{b}\right)\tilde{\overset{*}{\delta}}_{\beta} &= (\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{\alpha\ bc}_{\ \beta}\tilde{\overset{*}{\delta}}_{\alpha},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\tilde{\dot{\partial}}{}^{c},\tilde{\dot{\partial}}{}^{b}\right)\tilde{\dot{\partial}}{}^{a} &= (\rho,\eta,h)\overset{*}{\mathbb{S}}{}_{d}^{\ \ abc}\tilde{\dot{\partial}}{}^{d}.
\end{aligned}$$

we can easily prove the following

Theorem 6.6.2 *The (ρ,η,h)-curvature $(\rho,\eta,h)\overset{*}{\mathbb{R}}$ associated to distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$, is characterized by the tensor fields with local components:*

$$(6.6.6)\qquad \left\{\begin{aligned}
(\rho,\eta,h)\overset{*}{\mathbb{R}}{}^{\alpha}_{\beta\ \gamma\varepsilon} =&\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\tilde{\overset{*}{\delta}}_{\varepsilon}\right)(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\gamma}-\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\tilde{\overset{*}{\delta}}_{\gamma}\right)(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\varepsilon}\\
&+(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\theta\varepsilon}(\rho,\eta)\overset{*}{H}{}^{\theta}_{\beta\gamma}-(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\theta\gamma}(\rho,\eta)\overset{*}{H}{}^{\theta}_{\beta\varepsilon}\\
&-(\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{b\ \varepsilon\gamma}(\rho,\eta)\overset{*}{V}{}^{\alpha b}_{\beta}-L^{\theta}_{\varepsilon\gamma}\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\theta},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{b\ \gamma\varepsilon}^{\ a} =&\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\tilde{\overset{*}{\delta}}_{\varepsilon}\right)(\rho,\eta)\overset{*}{H}{}^{a}_{b\gamma}-(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\tilde{\overset{*}{\delta}}_{\gamma}\right)(\rho,\eta)\overset{*}{H}{}^{a}_{b\varepsilon}\\
&+(\rho,\eta)\overset{*}{H}{}^{c}_{b\varepsilon}(\rho,\eta)\overset{*}{H}{}^{a}_{c\gamma}-(\rho,\eta)\overset{*}{H}{}^{c}_{b\gamma}(\rho,\eta)\overset{*}{H}{}^{a}_{c\varepsilon}\\
&-(\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{c\ \varepsilon\gamma}(\rho,\eta)\overset{*}{V}{}^{ac}_{b}-L^{\theta}_{\varepsilon\gamma}\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{H}{}^{a}_{b\theta},
\end{aligned}\right.$$

$$(6.6.7)\qquad \left\{\begin{aligned}
(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\alpha\ b}_{\ \varepsilon\ \gamma} =&\ \Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})(\tilde{\overset{*}{\delta}}_{\gamma})(\rho,\eta)\overset{*}{V}{}^{\alpha b}_{\varepsilon}-\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\tilde{\dot{\partial}}{}^{b}\right)(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\varepsilon\gamma}\\
&+(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\theta\gamma}\cdot(\rho,\eta)\overset{*}{V}{}^{\theta b}_{\varepsilon}-(\rho,\eta)\overset{*}{V}{}^{\alpha b}_{\theta}\cdot(\rho,\eta)\overset{*}{H}{}^{\theta}_{\varepsilon\gamma}\\
&+\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\tilde{\dot{\partial}}{}^{b}\right)\left((\rho,\eta)\overset{*}{\Gamma}_{c\gamma}\right)\cdot(\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\varepsilon},\\
(\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{c\ \ \gamma}^{\ ab} =&\ \Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\tilde{\overset{*}{\delta}}_{\gamma}\right)(\rho,\eta)\overset{*}{V}{}^{ab}_{c}-\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\tilde{\dot{\partial}}{}^{b}\right)(\rho,\eta)\overset{*}{H}{}^{a}_{c\gamma}\\
&+(\rho,\eta)\overset{*}{H}{}^{d}_{c\gamma}(\rho,\eta)\overset{*}{V}{}^{ab}_{d}-(\rho,\eta)\overset{*}{V}{}^{db}_{c}(\rho,\eta)\overset{*}{H}{}^{a}_{d\gamma}\\
&+\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\tilde{\dot{\partial}}{}^{b}\right)\left((\rho,\eta)\overset{*}{\Gamma}_{d\gamma}\right)(\rho,\eta)\overset{*}{V}{}^{ad}_{c},
\end{aligned}\right.$$

$$(6.6.8)\quad \begin{cases} (\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{\alpha\ bc}_{\ \beta} &= \Gamma(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}})\left(\overset{\cdot\,c}{\tilde{\partial}}\right)(\rho,\eta)\overset{*}{V}{}^{\alpha b}_{\beta} - \Gamma(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}})\left(\overset{\cdot\,b}{\tilde{\partial}}\right)(\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\beta} \\ &+ (\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\theta}\,(\rho,\eta)\overset{*}{V}{}^{\theta b}_{\beta} - (\rho,\eta)\overset{*}{V}{}^{\alpha b}_{\theta}\,(\rho,\eta)\overset{*}{V}{}^{\theta c}_{\beta}, \\ (\rho,\eta,h)\overset{*}{\mathbb{S}}{}_{d}^{\ abc} &= \Gamma(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}})\left(\overset{\cdot\,c}{\tilde{\partial}}\right)(\rho,\eta)\overset{*}{V}{}^{ab}_{d} - \Gamma(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}})\left(\overset{\cdot\,b}{\tilde{\partial}}\right)(\rho,\eta)\overset{*}{V}{}^{ac}_{d} \\ &+ (\rho,\eta)\overset{*}{V}{}^{ec}_{d}\,(\rho,\eta)\overset{*}{V}{}^{ab}_{e} - (\rho,\eta)\overset{*}{V}{}^{e}_{bd}\,(\rho,\eta)\overset{*}{V}{}^{ac}_{e}. \end{cases}$$

In particular, when $(\rho,\eta,h) = (Id_{TM}, Id_M, Id_M)$, *we see the local components of the curvature associated to distinguished linear connection* $\left(\overset{*}{H},\overset{*}{V}\right)$ *in the followings:*

$$(6.6.6)'\quad \begin{aligned} \overset{*}{\mathbb{R}}{}^{i}_{\ j\ kl} &= \overset{*}{\delta}_l\left(\overset{*}{H}{}^{i}_{jk}\right) - \overset{*}{\delta}_k\left(\overset{*}{H}{}^{i}_{jl}\right) \\ &+\overset{*}{H}{}^{i}_{hl}\cdot\overset{*}{H}{}^{h}_{jk} - \overset{*}{H}{}^{i}_{hk}\cdot\overset{*}{H}{}^{h}_{jl} - \overset{*}{\mathbb{R}}_{b\ lk}\cdot\overset{*}{V}{}^{ib}_{j}, \\ \overset{*}{\mathbb{R}}{}_{b\ kl}^{\ a} &= \overset{*}{\delta}_l\left(\overset{*}{H}{}^{a}_{bk}\right) - \overset{*}{\delta}_k\left(\overset{*}{H}{}^{a}_{bl}\right) \\ &+\overset{*}{H}{}^{c}_{bl}\cdot\overset{*}{H}{}^{a}_{ck} - \overset{*}{H}{}^{c}_{bk}\cdot\overset{*}{H}{}^{a}_{cl} - \overset{*}{\mathbb{R}}_{c\ lk}\cdot\overset{*}{V}{}^{ac}_{b}, \end{aligned}$$

$$(6.6.7)'\quad \begin{aligned} \overset{*}{\mathbb{P}}{}^{i\ b}_{\ l\ k} &= \overset{*}{\delta}_k\left(\overset{*}{V}{}^{ib}_{l}\right) - \overset{\cdot\,b}{\tilde{\partial}}\left(\overset{*}{H}{}^{i}_{lk}\right) \\ &+\overset{*}{H}{}^{i}_{hk}\cdot\overset{*}{V}{}^{hb}_{l} - \overset{*}{V}{}^{ib}_{h}\cdot\overset{*}{H}{}^{h}_{lk} + \overset{\cdot\,b}{\tilde{\partial}}\left(\overset{*}{\Gamma}_{ck}\right)\cdot\overset{*}{V}{}^{ic}_{l}, \\ \overset{*}{\mathbb{P}}{}_{c\ \ k}^{\ ab} &= \overset{*}{\delta}_k\left(\overset{*}{V}{}^{ab}_{c}\right) - \overset{\cdot\,b}{\tilde{\partial}}\left(\overset{*}{H}{}^{a}_{ck}\right) \\ &+\overset{*}{H}{}^{d}_{ck}\cdot\overset{*}{V}{}^{ab}_{d} - \overset{*}{V}{}^{db}_{c}\cdot\overset{*}{H}{}^{a}_{dk} + \overset{\cdot\,b}{\tilde{\partial}}\left(\overset{*}{\Gamma}_{dk}\right)\cdot\overset{*}{V}{}^{ad}_{c}, \end{aligned}$$

$$(6.6.8)'\quad \begin{aligned} \overset{*}{\mathbb{S}}{}^{i\ bc}_{\ j} &= \overset{\cdot\,c}{\tilde{\partial}}\left(\overset{*}{V}{}^{ib}_{j}\right) - \overset{\cdot\,b}{\tilde{\partial}}\left(\overset{*}{V}{}^{ic}_{j}\right) + \overset{*}{V}{}^{ic}_{h}\cdot\overset{*}{V}{}^{hb}_{j} - \overset{*}{V}{}^{ib}_{h}\cdot\overset{*}{V}{}^{hc}_{j}, \\ \overset{*}{\mathbb{S}}{}_{d}^{\ abc} &= \overset{\cdot\,c}{\tilde{\partial}}\left(\overset{*}{V}{}^{ab}_{d}\right) - \overset{\cdot\,b}{\tilde{\partial}}\left(\overset{*}{V}{}^{ac}_{d}\right) + \overset{*}{V}{}^{ae}_{d}\cdot\overset{*}{V}{}^{ab}_{e} - \overset{*}{V}{}^{eb}_{d}\cdot\overset{*}{V}{}^{ac}_{d}. \end{aligned}$$

Definition 6.6.3 The tensor field

$$(6.6.9)\quad \begin{aligned} &\mathbf{Ric}\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right) = \\ &= (\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{\alpha\ \beta}d\overset{*}{\tilde{z}}{}^{\alpha}\otimes d\overset{*}{\tilde{z}}{}^{\beta} + (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{\alpha}^{\ \ b}d\overset{*}{\tilde{z}}{}^{\alpha}\otimes\delta\tilde{p}_b \\ &+ (\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{a}_{\ \ \beta}\delta\tilde{p}_a\otimes d\overset{*}{\tilde{z}}{}^{\beta} + (\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{a\ b}\delta\tilde{p}_a\otimes\delta\tilde{p}_b, \end{aligned}$$

$$(6.6.10)\quad \begin{aligned} &(\rho,\eta,h)\overset{*}{\mathbb{R}}_{\alpha\beta} = (\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{\alpha\ \beta\gamma}^{\ \gamma} \quad (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{\alpha}^{\ \ b} = (\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\beta\ b}_{\ \alpha\ \beta}, \\ &(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{a}_{\ \ \beta} = (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{c\ \ \beta}^{\ ac} \quad (\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{ab} = (\rho,\eta,h)\overset{*}{\mathbb{S}}{}_{c}^{\ acb} \end{aligned}$$

will be called *the Ricci tensor field associated to distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$.

This tensor field will be used for writing the Einstein equations in Subsection 6.10.

6.7 Formulas of Ricci type. Identities of Cartan and Bianchi type

We consider the following diagram:

$$\begin{array}{ccc} \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \overset{*}{\pi}\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $((F,\nu,M),[,]_{F,h},(\rho,\eta))\in|\mathbf{GLA}|$.

Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and let

$$\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$$

be a distinguished linear (ρ,η)-connection for the generalized tangent bundle

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right).$$

Theorem 6.7.1 Using the definition of (ρ,η,h)-curvature associated to the distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$, it results the following formulas:

$$(\mathcal{B}_1)\qquad\left\{\begin{array}{l}
(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}\overset{*}{\mathcal{H}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}X}\overset{*}{\mathcal{H}}Z\\
\quad=(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right)\overset{*}{\mathcal{H}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}\left[\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{H}}Z\\
\quad+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{H}}Z,\\
(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}\overset{*}{\mathcal{H}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}\overset{*}{\mathcal{H}}Z\\
\quad=(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right)\overset{*}{\mathcal{H}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{H}}Z\\
\quad+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{H}}Z,\\
(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}Y}\overset{*}{\mathcal{H}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}\overset{*}{\mathcal{H}}Z\\
\quad=(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{V}}Y\right)\overset{*}{\mathcal{H}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{V}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{H}}Z,
\end{array}\right.$$

and

$$(\mathcal{B}_2)\quad\begin{cases}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}\overset{*}{\mathcal{V}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}X}\overset{*}{\mathcal{V}}Z\\ \quad=(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right)\overset{*}{\mathcal{V}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}\left[\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{V}}Z\\ \quad+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{V}}Z,\\ (\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}\overset{*}{\mathcal{V}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}\overset{*}{\mathcal{V}}Z\\ \quad=(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right)\overset{*}{\mathcal{V}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{V}}Z\\ \quad+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{V}}Z,\\ (\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}Y}\overset{*}{\mathcal{V}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}\overset{*}{\mathcal{V}}Z\\ \quad=(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{V}}Y\right)\overset{*}{\mathcal{V}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{V}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{V}}Z.\end{cases}$$

Using the previous theorem, the horizontal and vertical sections of adapted base and an arbitrary section

$$\tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y^a\frac{\partial}{\partial\tilde{y}^a}\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

we can propose the following

Theorem 6.7.2 *We obtain the following formulas of Ricci type:*

$$(\mathcal{R}_1)\quad\begin{cases}\tilde{Z}^\alpha{}_{|\gamma|\beta}-\tilde{Z}^\alpha{}_{|\beta|\gamma}&=(\rho,\eta,h)\overset{*}{\mathbb{R}}{}^{\alpha}{}_{\theta\,\gamma\beta}\cdot\tilde{Z}^\theta-\left(L^\theta_{\gamma\beta}\circ h\circ\overset{*}{\pi}\right)\cdot\tilde{Z}^\alpha{}_{|\theta}\\ &\quad-(\rho,\eta,h)\overset{*}{\mathbb{T}}_{b\gamma\beta}\cdot\tilde{Z}^\alpha|^b-(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{\theta}{}_{\gamma\beta}\cdot\tilde{Z}^\alpha{}_{|\theta},\\ \tilde{Z}^\alpha{}_{|\gamma}|^b-\tilde{Z}^\alpha|^b{}_{|\gamma}&=(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\alpha}{}_{\theta}{}^{b}{}_{\gamma}\cdot\tilde{Z}^\theta-(\rho,\eta,h)\overset{*}{\mathbb{P}}_{c\gamma}{}^{b}\cdot\tilde{Z}^\alpha|^c\\ &\quad-(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\theta}{}^{b}{}_{\gamma}\cdot\tilde{Z}^\alpha{}_{|\theta},\\ \tilde{Z}^\alpha|^c|^b-\tilde{Z}^\alpha|^b|^c&=(\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{\alpha}{}_{\theta}{}^{cb}\cdot\tilde{Z}^\theta-(\rho,\eta,h)\overset{*}{\mathbb{S}}_a{}^{bc}\cdot\tilde{Z}^\alpha|^a,\end{cases}$$

and

$$(\mathcal{R}_2)\quad\begin{cases}Y_{a\,|\gamma|\beta}-Y_{a\,|\beta|\gamma}&=(\rho,\eta,h)\overset{*}{\mathbb{R}}{}_a{}^{c}{}_{\gamma\beta}\cdot Y_c-\left(L^\theta_{\gamma\beta}\circ h\circ\pi\right)\cdot Y_{a\,|\theta}\\ &\quad-(\rho,\eta)\overset{*}{\mathbb{T}}_{c\gamma\beta}\cdot Y_a|^c-(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{\theta}{}_{\gamma\beta}\cdot Y_{a\,|\theta},\\ Y_{a\,|\gamma}|^b-Y_a|^b{}_{|\gamma}&=(\rho,\eta,h)\overset{*}{\mathbb{P}}{}_a{}^{cb}{}_{\gamma}\cdot Y_c-(\rho,\eta,h)\overset{*}{\mathbb{P}}_{c\gamma}{}^{b}\cdot Y_a|^c\\ &\quad-(\rho,\eta)\overset{*}{\mathbb{P}}{}^{\theta}{}^{b}{}_{\gamma}\cdot Y_{a\,|\theta},\\ Y_a|^c|^b-Y_a|^b|^c&=(\rho,\eta,h)\overset{*}{\mathbb{S}}{}_a{}^{dbc}\cdot Y_d-(\rho,\eta,h)\overset{*}{\mathbb{S}}_d{}^{bc}\cdot Y_a|^d.\end{cases}$$

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,id_M)$ *and the Lie bracket* $[,]_{TM}$ *is the usual Lie bracket, then the formulas of Ricci type* $(\mathcal{R}_1)$ *and* $(\mathcal{R}_2)$ *become:*

$$(\mathcal{R}_1)'\quad\left\{\begin{array}{ll}\tilde{Z}^i{}_{|k|j}-\tilde{Z}^i{}_{|j|k} & =\overset{*}{\mathbb{R}}{}^i{}_{h\ kj}\cdot\tilde{Z}^h-\overset{*}{\mathbb{T}}{}_{bkj}\cdot\tilde{Z}^i|^b-\overset{*}{\mathbb{T}}{}^h{}_{kj}\cdot\tilde{Z}^i{}_{|h},\\ \tilde{Z}^i{}_{|k}|^b-\tilde{Z}^i|^b{}_{|k} & =\overset{*}{\mathbb{P}}{}^i{}_{h\ k}^{\ b}\cdot\tilde{Z}^h-\overset{*}{\mathbb{P}}{}_{ck}^{\ \ b}\cdot\tilde{Z}^i|^c-\overset{*}{\mathbb{P}}{}^h{}_k^{\ b}\cdot\tilde{Z}^i{}_{|h},\\ \tilde{Z}^i|^c|^b-\tilde{Z}^i|^b|^c & =\overset{*}{\mathbb{S}}{}^i{}_h^{\ cb}\cdot\tilde{Z}^h-\overset{*}{\mathbb{S}}{}_a^{\ bc}\cdot\tilde{Z}^i|^a\end{array}\right.$$

and

$$(\mathcal{R}_2)'\quad\left\{\begin{array}{ll}Y_{a\ |k|j}-Y_{a\ |j|k} & =\overset{*}{\mathbb{R}}{}_a{}^c{}_{kj}\cdot Y_c-\overset{*}{\mathbb{T}}{}_{ckj}\cdot Y_a|^c-\overset{*}{\mathbb{T}}{}^\theta{}_{kj}\cdot Y_{a\ |\theta},\\ Y_{a\ |k}|^b-Y_a|^b{}_{|k} & =\overset{*}{\mathbb{P}}{}_a{}^{cb}{}_k\cdot Y_c-\overset{*}{\mathbb{P}}{}_{ck}^{\ \ b}\cdot Y_a|^c-\overset{*}{\mathbb{P}}{}^h{}_k^{\ b}\cdot Y_{a\ |h},\\ Y_a|^c|^b-Y_a|^b|^c & =\overset{*}{\mathbb{S}}{}_a^{\ dbc}\cdot Y_d-\overset{*}{\mathbb{S}}{}_d^{\ bc}\cdot Y_a|^d.\end{array}\right.$$

Using the 1-forms associated to distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$

$$\begin{array}{l}(\rho,\eta)\overset{*}{\omega}{}^\alpha_\beta=(\rho,\eta)\overset{*}{H}{}^\alpha_{\beta\gamma}d\tilde{z}^\gamma+(\rho,\eta)\overset{*}{V}{}^{\alpha c}_\beta\,\delta\tilde{p}_c,\\ (\rho,\eta)\overset{*}{\omega}{}^a_b=(\rho,\eta)\overset{*}{H}{}^a_{b\gamma}d\tilde{z}^\gamma+(\rho,\eta)\overset{*}{V}{}^{ac}_b\,\delta\tilde{p}_c,\end{array}\tag{6.7.1}$$

the torsion 2-forms

$$\left\{\begin{array}{ll}(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^\alpha & =\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^\alpha{}_{\beta\gamma}d\tilde{z}^\beta\wedge d\tilde{z}^\gamma+(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^\alpha{}_\beta^{\ c}d\tilde{z}^\beta\wedge\delta\tilde{p}_c,\\ (\rho,\eta,h)\overset{*}{\mathbb{T}}{}_b & =\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{b\beta\gamma}d\tilde{z}^\beta\wedge d\tilde{z}^\gamma+(\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{b\beta}^{\ \ c}d\tilde{z}^\beta\wedge\delta\tilde{p}_c\\ & +\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{S}}{}_b^{\ ac}\delta\tilde{p}_a\wedge\delta\tilde{p}_c\end{array}\right.\tag{6.7.2}$$

and the curvature 2-forms

$$\left\{\begin{array}{ll}(\rho,\eta,h)\overset{*}{\mathbb{R}}{}^\alpha{}_\beta & =\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{R}}{}^\alpha{}_{\beta\ \gamma\theta}d\tilde{z}^\gamma\wedge d\tilde{z}^\theta+(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^\alpha{}_\beta{}^c{}_\gamma d\tilde{z}^\gamma\wedge\delta\tilde{p}_c\\ & +\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{S}}{}^\alpha{}_\beta^{\ bc}\delta\tilde{p}_b\wedge\delta\tilde{p}_c,\\ (\rho,\eta,h)\overset{*}{\mathbb{R}}{}_b^{\ a} & =\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{R}}{}_b{}^a{}_{\gamma\theta}d\tilde{z}^\gamma\wedge d\tilde{z}^\theta+(\rho,\eta,h)\overset{*}{\mathbb{P}}{}_b^{\ ac}{}_\gamma d\tilde{z}^\gamma\wedge\delta\tilde{p}_c\\ & +\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{S}}{}^a{}_b^{\ cd}\delta\tilde{p}_c\wedge\delta\tilde{p}_d,\end{array}\right.\tag{6.7.3}$$

we obtain the following

Theorem 4.7.3 *Identities of Cartan type hold good:*

$$\begin{array}{c}(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^\alpha=d^{(\rho,\eta)T\overset{*}{E}}(d\tilde{z}^\alpha)+(\rho,\eta)\overset{*}{\omega}{}^\alpha_\beta\wedge d\tilde{z}^\beta\\ (\rho,\eta,h)\overset{*}{\mathbb{T}}{}_b=d^{(\rho,\eta)T\overset{*}{E}}(\delta\tilde{p}_b)+(\rho,\eta)\overset{*}{\omega}{}^a_b\wedge\delta\tilde{p}_a\end{array},\tag{$\mathcal{C}_1$}$$

$$\begin{array}{c}(\rho,\eta,h)\overset{*}{\mathbb{R}}{}^\alpha{}_\beta=d^{(\rho,\eta)T\overset{*}{E}}\left((\rho,\eta)\overset{*}{\omega}{}^\alpha_\beta\right)+(\rho,\eta)\overset{*}{\omega}{}^\alpha_\gamma\wedge(\rho,\eta)\overset{*}{\omega}{}^\gamma_\beta\\ (\rho,\eta,h)\overset{*}{\mathbb{R}}{}_b^{\ a}=d^{(\rho,\eta)T\overset{*}{E}}\left((\rho,\eta)\overset{*}{\omega}{}^a_b\right)+(\rho,\eta)\overset{*}{\omega}{}^a_c\wedge(\rho,\eta)\overset{*}{\omega}{}^c_b.\end{array}\tag{$\mathcal{C}_2$}$$

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$ and the Lie bracket $[,]_{TM}$ is the usual Lie bracket, then the identities of Cartan type $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ become:

$$(\mathcal{C}_1)' \qquad \begin{aligned} \overset{*}{\mathbb{T}}{}^i &= d^{\left(Id_{T\overset{*}{E}},Id_{\overset{*}{E}}\right)T\overset{*}{E}}\left(d\tilde{z}^i\right)+\overset{*}{\omega}{}^i_j\wedge d\tilde{z}^j \\ \overset{*}{\mathbb{T}}_b &= d^{\left(Id_{T\overset{*}{E}},Id_{\overset{*}{E}}\right)T\overset{*}{E}}\left(\delta\tilde{p}_b\right)+\omega^a_b\wedge\delta\tilde{p}_a \end{aligned}$$

and

$$(\mathcal{C}_2)' \qquad \begin{aligned} \overset{*}{\mathbb{R}}{}^i{}_j &= d^{\left(Id_{T\overset{*}{E}},Id_{\overset{*}{E}}\right)T\overset{*}{E}}\left(\overset{*}{\omega}{}^i_j\right)+\overset{*}{\omega}{}^i_k\wedge\overset{*}{\omega}{}^k_j \\ \overset{*}{\mathbb{R}}{}_b{}^a &= d^{\left(Id_{T\overset{*}{E}},Id_{\overset{*}{E}}\right)T\overset{*}{E}}\left(\overset{*}{\omega}{}^a_b\right)+\overset{*}{\omega}{}^a_c\wedge\overset{*}{\omega}{}^c_b. \end{aligned}$$

Remark 6.7.1 For any

$$X,Y,Z\in\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

the following identities hold good:

$$(6.7.4) \qquad \begin{aligned} \overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)\overset{*}{\mathcal{H}}Z &= 0 \\ \overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)\overset{*}{\mathcal{V}}Z &= 0, \end{aligned}$$

$$(6.7.5) \qquad \begin{aligned} \overset{*}{\mathcal{V}}D_X\left(\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(Y,Z\right)\overset{*}{\mathcal{H}}U\right) &= 0 \\ \overset{*}{\mathcal{H}}D_X\left(\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(Y,Z\right)\overset{*}{\mathcal{V}}U\right) &= 0, \end{aligned}$$

and

$$(6.7.6) \qquad \left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)Z = \overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)\overset{*}{\mathcal{H}}Z+\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)\overset{*}{\mathcal{V}}Z.$$

Using the formulas of Bianchi type presented in Subsection 4.2 of our paper and the Remark 6.7.1 we obtain the following

Theorem 6.7.4 *The identities of Bianchi type:*

$$(\mathcal{B}_1) \qquad \left\{ \begin{aligned} &\sum_{cyclic(X,Y,Z)}\left\{\overset{*}{\mathcal{H}}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(Y,Z\right)\right)-\overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)Z\right. \\ &\quad+\overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(X,Y\right),Z\right) \\ &\quad\left.+\overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(X,Y\right),Z\right)\right\}=0, \\ &\sum_{cyclic(X,Y,Z)}\left\{\overset{*}{\mathcal{V}}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(Y,Z\right)\right)-\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)Z\right. \\ &\quad+\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(X,Y\right),Z\right) \\ &\quad\left.+\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(X,Y\right),Z\right)\right\}=0. \end{aligned} \right.$$

and

$$(\mathcal{B}_2)\qquad \left\{\begin{array}{l} \displaystyle\sum_{cyclic(X,Y,Z,U)} \Big\{ \overset{*}{\mathcal{H}}(\rho,\eta)\, D_X \Big((\rho,\eta,h)\, \overset{*}{\mathbb{R}}(Y,Z)\, U \Big) \\ \qquad -\overset{*}{\mathcal{H}}(\rho,\eta,h)\, \overset{*}{\mathbb{R}} \Big(\overset{*}{\mathcal{H}}(\rho,\eta,h)\, \overset{*}{\mathbb{T}}(X,Y), Z \Big) U \\ \qquad -\overset{*}{\mathcal{H}}(\rho,\eta,h)\, \overset{*}{\mathbb{R}} \Big(\overset{*}{\mathcal{V}}(\rho,\eta,h)\, \overset{*}{\mathbb{T}}(X,Y), Z \Big) U \Big\} = 0, \\ \displaystyle\sum_{cyclic(X,Y,Z,U)} \Big\{ \overset{*}{\mathcal{V}}(\rho,\eta)\, D_X \Big((\rho,\eta,h)\, \overset{*}{\mathbb{R}}(Y,Z)\, U \Big) \\ \qquad -\overset{*}{\mathcal{V}}(\rho,\eta,h)\, \overset{*}{\mathbb{R}} \Big(\overset{*}{\mathcal{H}}(\rho,\eta,h)\, \overset{*}{\mathbb{T}}(X,Y), Z \Big) U \\ \qquad -\overset{*}{\mathcal{V}}(\rho,\eta,h)\, \overset{*}{\mathbb{R}} \Big(\overset{*}{\mathcal{V}}(\rho,\eta,h)\, \overset{*}{\mathbb{T}}(X,Y), Z \Big) U \Big\} = 0, \end{array}\right.$$

hold good for any $X, Y, Z \in \Gamma\left((\rho,\eta)\, T\overset{*}{E}, (\rho,\eta)\, \tau_{\overset{*}{E}}, \overset{*}{E} \right)$.

Corollary 6.7.1 *Using the following sections* $(\delta_\theta, \delta_\gamma, \delta_\beta)$, *the identities* $(\mathcal{B}_1)$ *become:*

$$(\mathcal{B}_1)'\qquad \left\{\begin{array}{l} \displaystyle\sum_{cyclic(\beta,\gamma,\theta)} \Big\{ (\rho,\eta,h)\, \overset{*}{\mathbb{T}}{}^{\alpha}_{\ \beta\gamma_{|\theta}} - (\rho,\eta,h)\, \overset{*}{\mathbb{R}}{}^{\alpha}_{\ \beta\ \gamma\theta} \\ \qquad + (\rho,\eta,h)\, \overset{*}{\mathbb{T}}{}^{\lambda}_{\ \gamma\theta}\, (\rho,\eta,h)\, \overset{*}{\mathbb{T}}{}^{\alpha}_{\ \beta\gamma} + (\rho,\eta,h)\, \overset{*}{\mathbb{T}}_{b\gamma\theta}\, (\rho,\eta,h)\, \overset{*}{\mathbb{P}}{}^{\alpha\ b}_{\ \beta} \Big\} = 0, \\ \displaystyle\sum_{cyclic(\beta,\gamma,\theta)} \Big\{ (\rho,\eta,h)\, \overset{*}{\mathbb{T}}_{b\beta\gamma\ |\theta} + (\rho,\eta,h)\, \overset{*}{\mathbb{T}}{}^{\alpha}_{\ \gamma\theta}\, (\rho,\eta,h)\, \overset{*}{\mathbb{T}}_{b\beta\alpha} \\ \qquad + (\rho,\eta,h)\, \overset{*}{\mathbb{T}}_{c\gamma\theta}\, (\rho,\eta,h)\, \overset{*}{\mathbb{P}}_{b\beta}{}^{c} \Big\} = 0, \end{array}\right.$$

and using the following sections $(\delta_\lambda, \delta_\theta, \delta_\gamma, \delta_\beta)$, *the identities* $(\mathcal{B}_2)$ *become:*

$$(\mathcal{B}_2)'\qquad \left\{\begin{array}{l} \displaystyle\sum_{cyclic(\beta,\gamma,\theta,\lambda)} \Big\{ (\rho,\eta,h)\, \overset{*}{\mathbb{R}}{}^{\alpha}_{\ \beta\ \gamma\theta_{|\lambda}} - (\rho,\eta,h)\, \overset{*}{\mathbb{T}}{}^{\mu}_{\ \theta\lambda}\, (\rho,\eta,h)\, \overset{*}{\mathbb{T}}{}^{\alpha}_{\ \beta\gamma\mu} \\ \qquad - (\rho,\eta,h)\, \overset{*}{\mathbb{T}}_{b\theta\lambda}\, (\rho,\eta,h)\, \overset{*}{\mathbb{P}}{}^{\alpha\ b}_{\ \beta\ \gamma} \Big\} = 0, \\ \displaystyle\sum_{cyclic(\gamma,\theta,\lambda)} \Big\{ (\rho,\eta,h)\, \overset{*}{\mathbb{R}}_{b\ \gamma\theta_{|\lambda}}^{\ a} - (\rho,\eta,h)\, \overset{*}{\mathbb{T}}{}^{\mu}_{\ \theta\lambda}\, (\rho,\eta,h)\, \overset{*}{\mathbb{R}}_{b\ \gamma\mu}^{\ a} \\ \qquad - (\rho,\eta,h)\, \overset{*}{\mathbb{T}}_{c\theta\lambda}\, (\rho,\eta,h)\, \overset{*}{\mathbb{P}}_{b\ \ \gamma}^{\ ac} \Big\} = 0. \end{array}\right.$$

Using another base of sections, we shall obtain new identities of Bianchi type necessary in the applications.

6.8 The (ρ, η)-(pseudo)metrizability

We consider the following diagram:

$$\begin{tikzcd} \overset{*}{E} \arrow[d, "\overset{*}{\pi}"'] & \left(F, [,]_{F,h}, (\rho, \eta)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E, \pi, M) \in |\mathbf{B^v}|$ and $((F, \nu, M), [,]_{F,h}, (\rho, \eta)) \in |\mathbf{GLA}|$. Let $(\rho, \eta)\overset{*}{\Gamma}$ be a (ρ, η)-connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ and let $\left((\rho, \eta)\overset{*}{H}, (\rho, \eta)\overset{*}{V}\right)$ be a distinguished linear (ρ, η)-connection for the generalized tangent bundle

$$\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right).$$

Definition 6.8.1 A tensor d-field

$$\overset{*}{G} = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g^{ab}\delta\tilde{p}_a \otimes \delta\tilde{p}_b \in \mathcal{DT}^{02}_{20}\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

will be called a *pseudometrical structure* if its components are symmetric and the matrices $\left\|g_{\alpha\beta}\left(\overset{*}{u}_x\right)\right\|$and $\left\|g^{ab}\left(\overset{*}{u}_x\right)\right\|$ are nondegenerate, for any point $\overset{*}{u}_x \in \overset{*}{E}$.

Moreover, if the matrices $\left\|g_{\alpha\beta}\left(\overset{*}{u}_x\right)\right\|$ and $\left\|g^{ab}\left(\overset{*}{u}_x\right)\right\|$ has constant signature, then the tensor d-field $\overset{*}{G}$ will be called *metrical structure.*

Let

$$\overset{*}{G} = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g^{ab}\delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

be a (pseudo)metrical structure. If $\alpha, \beta \in \overline{1,p}$ and $a, b \in \overline{1,r}$, then for any vector local $(m+r)$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, we consider the real functions

$$\overset{*}{\pi}^{-1}(U) \xrightarrow{\tilde{g}^{\beta\alpha}} \mathbb{R}$$

and

$$\overset{*}{\pi}^{-1}(U) \xrightarrow{\tilde{g}_{ba}} \mathbb{R}$$

such that

$$\tilde{g}^{\beta\alpha}\left(\overset{*}{u}_x\right) \cdot g_{\alpha\gamma}\left(\overset{*}{u}_x\right) = \delta^{\beta}_{\gamma},\ \forall \overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U) \setminus \{0_x\}$$

and

$$\tilde{g}_{ba}\left(\overset{*}{u}_x\right) \cdot g^{ac}\left(\overset{*}{u}_x\right) = \delta^{c}_{b},\ \forall \overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U) \setminus \{0_x\}$$

respectively.

Definition 6.8.2 We will say that the *(pseudo)metrical structure*

$$\overset{*}{G} = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g^{ab}\delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

is Riemannian (pseudo)metrical structure if around each point $x\in M$ it exists a local vector $m+r$-chart $\left(U,\overset{*}{s}_U\right)$ and a local m-chart (U,ξ_U) such that $g_{\alpha\beta}\circ s_U^{-1}\circ\left(\xi_U^{-1}\times Id_{\mathbb{R}^m}\right)(x,p)$ and $g^{ab}\circ s_U^{-1}\circ\left(\xi_U^{-1}\times Id_{\mathbb{R}^m}\right)(x,p)$ depends only on x, for any $\overset{*}{u}_x\in\overset{*}{\pi}^{-1}(U)$.

If only the condition is verified: *"$g_{\alpha\beta}\circ s_U^{-1}\circ\left(\xi_U^{-1}\times Id_{\mathbb{R}^m}\right)(x,p)$ depends only on x, for any $\overset{*}{u}_x\in\overset{*}{\pi}^{-1}(U)$"* (respectively *"$g^{ab}\circ s_U^{-1}\circ\left(\xi_U^{-1}\times Id_{\mathbb{R}^m}\right)(x,p)$ depend only on x, for any $\overset{*}{u}_x\in\overset{*}{\pi}^{-1}(U)$"*), then we will say that the *(pseudo)metrical structure G is a Riemannian $\overset{*}{\mathcal{H}}$-(pseudo)metrical structure* respectively a *Riemannian $\overset{*}{\mathcal{V}}$-(pseudo)metrical structure.*

Definition 6.8.3 We will say that the *(pseudo)metrical structure*

$$\overset{*}{G}=g_{\alpha\beta}d\tilde{z}^\alpha\otimes d\tilde{z}^\beta+g^{ab}\delta\tilde{p}_a\otimes\delta\tilde{p}_b$$

is locally Minkowski structure if around each point $x\in M$ it exists a local vector $m+r$-chart $\left(U,\overset{*}{s}_U\right)$ and a local m-chart (U,ξ_U) such that $g_{\alpha\beta}\circ s_U^{-1}\circ\left(\xi_U^{-1}\times Id_{\mathbb{R}^m}\right)(x,p)$ and $g^{ab}\circ s_U^{-1}\circ\left(\xi_U^{-1}\times Id_{\mathbb{R}^m}\right)(x,p)$ depends only on p, for any $\overset{*}{u}_x\in\overset{*}{\pi}^{-1}(U)$.

If only the condition is verified: "$g_{\alpha\beta}\circ s_U^{-1}\circ\left(\xi_U^{-1}\times Id_{\mathbb{R}^m}\right)(x,p)$ depend only on p, for any $\overset{*}{u}_x\in\overset{*}{\pi}^{-1}(U)$" respectively "$g^{ab}\circ s_U^{-1}\circ\left(\xi_U^{-1}\times Id_{\mathbb{R}^m}\right)(x,p)$ depends only on p, for any $\overset{*}{u}_x\in\overset{*}{\pi}^{-1}(U)$", then we will say that *the (pseudo)metrical structure $\overset{*}{G}$ is a (pseudo)metrical structure $\overset{*}{\mathcal{H}}$-locally Minkowski* and *$\overset{*}{\mathcal{V}}$-locally Minkowski, respectively.*

Definition 6.8.4 The generalized tangent bundle $\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$ will be called (ρ,η)-*(pseudo)metrizable* if it exists a (pseudo)metrical structure

$$\overset{*}{G}=g_{\alpha\beta}d\tilde{z}^\alpha\otimes d\tilde{z}^\beta+g^{ab}\delta\tilde{p}_a\otimes\delta\tilde{p}_b$$

and a distinguished linear (ρ,η)-connection

$$\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$$

such that

$$(\rho,\eta)D_XG=0,\ \forall X\in\Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right). \tag{6.8.1}$$

The condition (6.8.1) is equivalent with the following equalities:

$$g_{\alpha\beta|\gamma}=0,\ g^{ab}{}_{|\gamma}=0,\ g_{\alpha\beta}|^c=0\,,\ g^{ab}|^c=0. \tag{6.8.2}$$

If $g_{\alpha\beta|\gamma}$=0 and $g^{ab}{}_{|\gamma}$=0, then we will say that *the vector bundle*

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is $\overset{*}{\mathcal{H}}$-(ρ,η)-(pseudo)metrizable.

If $g_{\alpha\beta}|^{c}=0$ and $g^{ab}|^{c}=0$, then we will say that *the vector bundle*

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is $\overset{*}{\mathcal{V}}$*-*(ρ,η)*-(pseudo)metrizable.*

Theorem 6.8.1 *If*

$$\left((\rho,\eta)\overset{*}{\mathring{H}},(\rho,\eta)\overset{*}{\mathring{V}}\right)$$

is a distinguished linear (ρ,η)*-connection for the generalized tangent bundle*

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and

$$\overset{*}{G}=g_{\alpha\beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+g^{ab}\delta\tilde{p}_{a}\otimes\delta\tilde{p}_{b}$$

is a (pseudo)metrical structure, then the real local functions:

$$\begin{array}{ll}(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\gamma} & =\dfrac{1}{2}\tilde{g}^{\alpha\varepsilon}\left(\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)g_{\varepsilon\beta}\right.\\ & +\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\beta}\right)g_{\varepsilon\gamma}-\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\varepsilon}\right)g_{\beta\gamma}+\\ & \left.+g_{\theta\varepsilon}L^{\theta}_{\gamma\beta}\circ h\circ\overset{*}{\pi}-g_{\beta\theta}L^{\theta}_{\gamma\varepsilon}\circ h\circ\overset{*}{\pi}-g_{\theta\gamma}L^{\theta}_{\beta\varepsilon}\circ h\circ\overset{*}{\pi}\right),\\ (\rho,\eta)\overset{*}{H}{}^{a}_{b\gamma} & =(\rho,\eta)\overset{*}{\mathring{H}}{}^{a}_{b\gamma}+\dfrac{1}{2}\tilde{g}_{bc}g^{ac}{}_{\underset{|\gamma}{0}},\\ (\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\beta} & =(\rho,\eta)\overset{*}{\mathring{V}}{}^{\alpha c}_{\beta}+\dfrac{1}{2}g_{\beta\varepsilon}\tilde{g}^{\alpha\varepsilon}\overset{0^{c}}{|},\\ (\rho,\eta)\overset{*}{V}{}^{ac}_{b} & =\dfrac{1}{2}\tilde{g}_{be}\left(\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\dot{\partial}}^{c}\right)g^{ea}\right.\\ & \left.+\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\dot{\partial}}^{a}\right)g^{ec}-\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\dot{\partial}}^{e}\right)g^{ac}\right)\end{array}\tag{6.8.3}$$

are components of a distinguished linear (ρ,η)*-connection such that the generalized tangent bundle*

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

becomes (ρ,η)*-(pseudo)metrizable.*

Corollary 6.8.1 *If the distinguished linear* (ρ,η)*-connection*

$$\left((\rho,\eta)\overset{*}{\mathring{H}},(\rho,\eta)\overset{*}{\mathring{V}}\right)$$

coincide with the Berwald linear (ρ,η)-connection, then the local real functions:

$$
\begin{aligned}
(\rho,\eta)\overset{c\ \alpha}{\overset{*}{H}}{}_{\beta\gamma} &= \frac{1}{2}\tilde{g}^{\alpha\varepsilon}\left(\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)g_{\varepsilon\beta}\right.\\
&\quad +\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\beta}\right)g_{\varepsilon\gamma} - \Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\varepsilon}\right)g_{\beta\gamma}\\
&\quad \left.+g_{\theta\varepsilon}L^{\theta}_{\gamma\beta}\circ h\circ\overset{*}{\pi} - g_{\beta\theta}L^{\theta}_{\gamma\varepsilon}\circ h\circ\overset{*}{\pi} - g_{\theta\gamma}L^{\theta}_{\beta\varepsilon}\circ h\circ\overset{*}{\pi}\right),\\
(\rho,\eta)\overset{c\ a}{\overset{*}{H}}{}_{b\gamma} &= \frac{\partial\,(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}}{\partial p_a} + \frac{1}{2}\tilde{g}_{bc}g^{ac}{}_{\overset{0}{|\gamma}},\\
(\rho,\eta)\overset{c\ \alpha c}{\overset{*}{V}}{}_{\beta} &= \frac{1}{2}g_{\beta\varepsilon}\frac{\partial\tilde{g}^{\alpha\varepsilon}}{\partial p_c},\\
(\rho,\eta)\overset{c\ bc}{\overset{*}{V}}{}_{a} &= \frac{1}{2}\tilde{g}_{ae}\left(\frac{\partial g^{eb}}{\partial p_c} + \frac{\partial g^{ec}}{\partial p_b} - \frac{\partial g^{bc}}{\partial p_e}\right)
\end{aligned}
\tag{6.8.4}
$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle

$$\left((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

becomes (ρ,η)-(pseudo)metrizable.

Moreover, if the (pseudo)metrical structure $\overset{*}{G}$ is $\mathcal{H}$- and $\mathcal{V}$-Riemannian, then the local real functions:

$$
\begin{aligned}
(\rho,\eta)\overset{c\ \alpha}{\overset{*}{H}}{}_{\beta\gamma} &= \frac{1}{2}g^{\alpha\varepsilon}\left(\rho^{k}_{\gamma}\circ h\circ\pi\frac{\partial g_{\varepsilon\beta}}{\partial x^{k}} + \rho^{j}_{\beta}\circ h\circ\pi\frac{\partial g_{\varepsilon\gamma}}{\partial x^{j}} - \rho^{e}_{\varepsilon}\circ h\circ\pi\frac{\partial g_{\beta\gamma}}{\partial x^{e}} +\right.\\
&\qquad \left.+g_{\theta\varepsilon}L^{\theta}_{\gamma\beta}\circ h\circ\pi - g_{\beta\theta}L^{\theta}_{\gamma\varepsilon}\circ h\circ\pi - g_{\theta\gamma}L^{\theta}_{\beta\varepsilon}\circ h\circ\pi\right),\\
(\rho,\eta)\overset{c\ a}{\overset{*}{H}}{}_{b\gamma} &= \frac{\partial\,(\rho,\eta)\,\Gamma^{a}_{\gamma}}{\partial y^{b}} + \frac{1}{2}g^{ac}\left(\rho^{i}_{\gamma}\circ h\circ\pi\frac{\partial g_{bc}}{\partial x^{i}} - \frac{\partial\rho\Gamma^{e}_{\gamma}}{\partial y^{b}}g_{ec} - \frac{\partial\rho\Gamma^{e}_{\gamma}}{\partial y^{c}}g_{eb}\right),\\
(\rho,\eta)\overset{c\ \alpha}{\overset{*}{V}}{}_{\beta c} &= 0,\ \ (\rho,\eta)\overset{c\ a}{\overset{*}{V}}{}_{bc} = 0
\end{aligned}
\tag{4.8.5}
$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle

$$\left((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

becomes (ρ,η)-(pseudo)metrizable.

Theorem 6.8.2 *Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$. Let*

$$\left((\rho,\eta)\overset{*}{\overset{\circ}{H}}, (\rho,\eta)\overset{*}{\overset{\circ}{V}}\right)$$

be a distinguished linear (ρ,η)-connection for

$$\left((\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

and let

$$\overset{*}{G} = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

be a (pseudo)metrical structure.

Let

$$\begin{array}{ll} O_{\beta\gamma}^{\alpha\varepsilon} = \frac{1}{2}\left(\delta_{\beta}^{\alpha}\delta_{\gamma}^{\varepsilon} - g_{\beta\gamma}\tilde{g}^{\alpha\varepsilon}\right), & O_{\beta\gamma}^{*\alpha\varepsilon} = \dfrac{1}{2}\left(\delta_{\beta}^{\alpha}\delta_{\gamma}^{\varepsilon} + g_{\beta\gamma}\tilde{g}^{\alpha\varepsilon}\right), \\ O_{bc}^{ae} = \dfrac{1}{2}\left(\delta_{b}^{a}\delta_{c}^{e} - \tilde{g}_{bc}g^{ae}\right), & O_{bc}^{*ae} = \frac{1}{2}\left(\delta_{b}^{a}\delta_{c}^{e} + \tilde{g}_{bc}g^{ae}\right), \end{array} \tag{6.8.6}$$

be the Obata operators

If the real local functions $X_{\beta\gamma}^{\alpha}, X_{\beta}^{\alpha c}, Y_{b\gamma}^{a}, Y_{b}^{ac}$ *are components of tensor fields, then the local real functions are given in the following:*

$$\begin{aligned} (\rho,\eta)\overset{*}{H}{}_{\beta\gamma}^{\alpha} &= (\rho,\eta)\overset{c}{\overset{*}{H}}{}_{\beta\gamma}^{\alpha} + O_{\gamma\eta}^{\alpha\varepsilon}X_{\varepsilon\beta}^{\eta}, \\ (\rho,\eta)\overset{*}{H}{}_{b\gamma}^{a} &= (\rho,\eta)\overset{c}{\overset{*}{H}}{}_{b\gamma}^{a} + O_{bd}^{ae}Y_{e\gamma}^{d}, \\ (\rho,\eta)\overset{*}{V}{}_{\beta}^{\alpha c} &= (\rho,\eta)\overset{c}{\overset{*}{V}}{}_{\beta}^{\alpha c} + O_{\beta\eta}^{*\alpha\varepsilon}X_{\varepsilon}^{\eta c}, \\ (\rho,\eta)\overset{*}{V}{}_{b}^{ac} &= (\rho,\eta)\overset{c}{\overset{*}{V}}{}_{b}^{ac} + O_{bd}^{*ae}Y_{e}^{dc}, \end{aligned} \tag{6.8.7}$$

are the components of a distinguished linear (ρ,η)*-connection such that the generalized tangent bundle*

$$\left((\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

becomes (ρ,η)*-(pseudo)metrizable.*

Theorem 6.8.3 *Let* $(\rho,\eta)\overset{*}{\Gamma}$ *be a* (ρ,η)*-connection for the vector bundle* $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$. *If*

$$\left((\rho,\eta)\,\overset{*}{\mathring{H}}, (\rho,\eta)\,\overset{*}{\mathring{V}}\right)$$

is a distinguished linear (ρ,η)*-connection for the generalized tangent bundle*

$$\left((\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

and

$$\overset{*}{G} = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

is a (pseudo)metrical structure, then the real local functions:

$$\begin{aligned} (\rho,\eta)\overset{*}{H}{}_{\beta\gamma}^{\alpha} &= (\rho,\eta)\overset{*}{\mathring{H}}{}_{\beta\gamma}^{\alpha} + \tfrac{1}{2}g_{\beta\varepsilon}\tilde{g}^{\varepsilon\alpha}{}_{\overset{0}{|}\gamma}, \\ (\rho,\eta)\overset{*}{H}{}_{b\gamma}^{a} &= (\rho,\eta)\overset{*}{\mathring{H}}{}_{b\gamma}^{a} + \tfrac{1}{2}\tilde{g}_{be}g^{ea}{}_{\overset{0}{|}\gamma}, \\ (\rho,\eta)\overset{*}{V}{}_{\beta}^{\alpha c} &= (\rho,\eta)\overset{*}{\mathring{V}}{}_{\beta}^{\alpha c} + \tfrac{1}{2}g_{\beta\varepsilon}\tilde{g}^{\varepsilon\alpha}\overset{0^{c}}{|}, \\ (\rho,\eta)\overset{*}{V}{}_{b}^{ac} &= (\rho,\eta)\overset{*}{\mathring{V}}{}_{b}^{ac} + \tfrac{1}{2}\tilde{g}_{be}g^{ea}\overset{0^{c}}{|} \end{aligned} \tag{6.8.8}$$

are the components of a distinguished linear (ρ, η)-connection such that the generalized tangent bundle

$$\left((\rho, \eta)\, T\overset{*}{E}, (\rho, \eta)\, \tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

becomes (ρ, η)-(pseudo)metrizable.

6.9 Generalized Hamilton (ρ, η)-spaces, Hamilton (ρ, η)-spaces and Cartan (ρ, η)-spaces

We consider the following diagram:

$$\begin{array}{ccc} \overset{*}{E} & & \left(F, [,]_{F,h}, (\rho, \eta)\right) \\ {\scriptstyle \overset{*}{\pi}} \downarrow & & \downarrow {\scriptstyle \nu} \\ M & \xrightarrow{h} & N \end{array}$$

such that $(E, \pi, M) = (F, \nu, N)$ and the generalized tangent bundle

$$\left((\rho, \eta)\, T\overset{*}{E}, (\rho, \eta)\, \tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

is (ρ, η)-(pseudo)metrizable.

Definition 6.9.1 A smooth *Hamilton fundamental function* on the dual vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ is a mapping

$$\overset{*}{E} \xrightarrow{H} \mathbb{R}$$

which satisfies the following conditions:

1. $H \circ \overset{*}{u} \in C^{\infty}(M)$, for any $\overset{*}{u} \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right) \setminus \{0\}$;
2. $H \circ 0 \in C^{0}(M)$, where 0 means the null section of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$.

Let H be a differentiable Hamiltonian defined on the total space of the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$.

If $\left(U, \overset{*}{s}_U\right)$ is a local vector $(m+r)$-chart for $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, then we obtain the following real functions defined on $\overset{*}{\pi}^{-1}(U)$:

$$\begin{array}{ll} H_i \overset{put}{=} \dfrac{\partial H}{\partial x^i} \overset{put}{=} \dfrac{\partial}{\partial x^i}(H) & H_i^b \overset{put}{=} \dfrac{\partial^2 H}{\partial x^i \partial p_b} \overset{put}{=} \dfrac{\partial}{\partial x^i}\left(\dfrac{\partial}{\partial p_b}(H)\right) \\ H^a \overset{put}{=} \dfrac{\partial H}{\partial p_a} \overset{put}{=} \dfrac{\partial}{\partial p_a}(H) & H^{ab} \overset{put}{=} \dfrac{\partial^2 H}{\partial p_a \partial p_b} \overset{put}{=} \dfrac{\partial}{\partial p_a}\left(\dfrac{\partial}{\partial p_b}(H)\right) \end{array}. \tag{6.9.1}$$

Definition 6.9.2 If for any local vector $m+r$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, we have:

$$rank \left\| H^{ab}\left(\overset{*}{u}_x\right) \right\| = r, \tag{6.9.2}$$

for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U) \setminus \{0_x\}$, then we say that *the Hamiltonian H is regular.*

Proposition 6.9.1 If the Hamiltonian H is regular, then for any local vector $m+r$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, we obtain the real functions H_{ba} locally defined by

$$\begin{array}{ccc} \overset{*}{\pi}^{-1}(U) & \xrightarrow{\tilde{H}_{ba}} & \mathbb{R} \\ \overset{*}{u}_x & \longmapsto & H_{ba}\left(\overset{*}{u}_x\right) \end{array} \tag{6.9.3}$$

where $\left\|\tilde{H}_{ba}\left(\overset{*}{u}_x\right)\right\| = \left\|H^{ab}\left(\overset{*}{u}_x\right)\right\|^{-1}$, for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$.

Definition 6.9.3 A smooth *Cartan fundamental function* on the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ is a mapping

$$\overset{*}{E} \xrightarrow{K} \mathbb{R}_+$$

which satisfies the following conditions:

1. $K \circ \overset{*}{u} \in C^\infty(M)$, for any $\overset{*}{u} \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right) \setminus \{0\}$;
2. $K \circ 0 \in C^0(M)$, where 0 means the null section of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$;
3. K is positively 1-homogenous on the fibres of vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$;
4. For any local vector $m+r$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, the hessian:

$$\left\|K^{2\ ab}\left(\overset{*}{u}_x\right)\right\| \tag{6.9.4}$$

is positively define for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U) \setminus \{0_x\}$.

Definition 6.9.4 If the (pseudo)metrical structure $\overset{*}{G}$ is determined by a (pseudo)metrical structure

$$g \in \mathcal{T}^{\,2}_{\,0}\left(V(\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right),$$

then the (ρ,η)-(pseudo)metrizable vector bundle

$$\left((\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

will be called the *generalized Hamilton (ρ,η)-space.*

In particular, if the (pseudo)metrical structure g is determined with the help of a Hamilton fundamental function or Cartan fundamental function, then the (ρ,η)-(pseudo)metrizable vector bundle

$$\left((\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

will be called the *Hamilton (ρ,η)-space* or the *Cartan (ρ,η)-space, respectively.*

The generalized Hamilton (Id_{TM}, Id_M)-space, the Hamilton (Id_{TM}, Id_M)-space, and the Cartan (Id_{TM}, Id_M)-space will be called the *generalized Hamilton space, Hamilton space, Cartan space.*

Definition 6.9.5 The normal distinguished linear (ρ,η)-connections of a Hamilton or Cartan (ρ,η)-space will be called *Hamilton* and *Cartan linear* (ρ,η)*-connections.*

The Hamilton and Cartan linear (Id_{TM}, Id_M)-connections will be called *Hamilton* and *Cartan linear connections*, respectively.

Theorem 6.9.1 *If the (pseudo)metrical structure* $\overset{*}{G}$ *is determined by a (pseudo)metrical structure*

$$g \in \mathcal{T}\,{}^2_0\left(V\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

then the real local functions:

$$\begin{array}{rl}
\left(\rho,\eta\right)\overset{*}{H}{}^a_{bc} & = \dfrac{1}{2}g^{ae}\left(\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_b\right)\tilde{g}_{ec}+\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_c\right)\tilde{g}_{be}\right.\\
& -\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_e\right)\tilde{g}_{bc}-\tilde{g}_{cd}\cdot L^d_{be}\circ h\circ\overset{*}{\pi}\\
& \left.+\tilde{g}_{bd}\cdot L^d_{ec}\circ h\circ\overset{*}{\pi}-\tilde{g}_{ed}\cdot L^d_{bc}\circ h\circ\overset{*}{\pi}\right),\\
\left(\rho,\eta\right)\overset{*}{V}{}_a^{bc} & = \dfrac{1}{2}\tilde{g}_{ae}\left(\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\partial}^{\cdot c}\right)g^{eb}\right.\\
& \left.+\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\partial}^{\cdot b}\right)g^{ec}-\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\partial}^{\cdot e}\right)g^{bc}\right)
\end{array}\tag{6.9.5}$$

are the components of a normal distinguished linear (ρ,η)*-connection with* (ρ,η)-$\overset{*}{\mathcal{H}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right)$ *and* (ρ,η)-$\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{V}}\overset{*}{\mathcal{V}}\right)$ *torsions free such that the generalized tangent bundle*

$$\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

derives generalized Hamilton (ρ,η)*-space.*

This normal distinguished linear (ρ,η)-connection will be called *generalized linear* (ρ,η)*-connection of Levi-Civita type.*

If the *(pseudo)metrical structure* g is determined with the help of a Hamilton or Cartan fundamental function, then the Hamilton and the Cartan linear (ρ,η)-connections will be called *canonical Hamilton* and *Cartan linear* (ρ,η)*-connection, respectively.*

The canonical Hamilton and Cartan linear (Id_{TM}, Id_M)-connection will be called the *canonical Hamilton* and *Cartan linear connection* respectively.

Theorem 6.9.2 *Let* $\left(\left(\rho,\eta\right)\overset{*}{H},\left(\rho,\eta\right)\overset{*}{V}\right)$ *be the normal distinguished linear* (ρ,η)*-connection presented in the previous theorem.*

If

$$\overset{*}{\mathbb{T}}{}^a_{bc}\overset{*}{\tilde{\delta}}_a\otimes d\tilde{z}^b\otimes d\tilde{z}^c\in\mathcal{T}^{10}_{20}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and
$$\overset{*}{\mathbb{S}}{}_{a}^{bc}\,\dot{\tilde{\partial}}^{a}\otimes\delta\tilde{p}_b\otimes\delta\tilde{p}_c\in\mathcal{T}_{01}^{02}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

such that they satisfy the following conditions:

$$\overset{*}{\mathbb{T}}{}_{bc}^{a}=-\overset{*}{\mathbb{T}}{}_{cb}^{a},\ \overset{*}{\mathbb{S}}{}_{a}^{bc}=-\overset{*}{\mathbb{S}}{}_{a}^{bc},\ \forall b,c\in\overline{1,r},$$

then the real local functions:

$$\begin{aligned}(\rho,\eta)\,\overset{*}{\tilde{H}}{}_{bc}^{a}&=(\rho,\eta)\,\overset{*}{H}{}_{bc}^{a}+\frac{1}{2}g^{ae}\cdot\left(\tilde{g}_{ed}\overset{*}{\mathbb{T}}{}_{bc}^{d}-\tilde{g}_{bd}\overset{*}{\mathbb{T}}{}_{ec}^{d}+\tilde{g}_{cd}\overset{*}{\mathbb{T}}{}_{be}^{d}\right),\\(\rho,\eta)\,\overset{*}{\tilde{V}}{}_{a}^{bc}&=(\rho,\eta)\,\overset{*}{V}{}_{a}^{bc}+\frac{1}{2}\tilde{g}_{ae}\cdot\left(g^{ed}\overset{*}{\mathbb{S}}{}_{d}^{bc}-g^{bd}\overset{*}{\mathbb{S}}{}_{d}^{ec}+g^{cd}\overset{*}{\mathbb{S}}{}_{d}^{be}\right)\end{aligned}\tag{6.9.6}$$

are the components of a normal distinguished linear (ρ,η)-connection with (ρ,η)-$\overset{*}{\mathcal{H}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right)$ and (ρ,η)-$\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{V}}\overset{*}{\mathcal{V}}\right)$ torsions a priori given such that the generalized tangent bundle

$$\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

derives generalized Hamilton (ρ,η)-space.

Moreover, we obtain:

$$\begin{aligned}\overset{*}{\mathbb{T}}{}_{bc}^{a}&=(\rho,\eta)\,\overset{*}{H}{}_{bc}^{a}-(\rho,\eta)\,\overset{*}{H}{}_{cb}^{a}-L_{bc}^{a}\circ h\circ\overset{*}{\pi},\\\overset{*}{\mathbb{S}}{}_{a}^{bc}&=(\rho,\eta)\,\overset{*}{V}{}_{a}^{bc}-(\rho,\eta)\,\overset{*}{V}{}_{a}^{cb}.\end{aligned}\tag{6.9.7}$$

6.10 Einstein equations

We shall consider a metric structure

$$\overset{*}{G}=g_{\alpha\beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+g^{ab}\delta\tilde{p}_a\otimes\delta\tilde{p}_b$$

and a distinguished linear (ρ,η)-connection $\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$ compatible with the structure metric $\overset{*}{G}$ having $\overset{*}{\mathcal{H}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right)$ and $\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{V}}\overset{*}{\mathcal{V}}\right)$-torsions prescribed.

Definition 6.10.1 If $(\rho,\eta,h)\,\overset{*}{\mathbb{R}}{}_{\alpha\ \beta}$ and $(\rho,\eta,h)\,\overset{*}{\mathbb{S}}{}^{a\ b}$ are the components of tensor Ricci associated to distinguished linear (ρ,η)-connection

$$\left((\rho,\eta)\,H,(\rho,\eta)\,V\right),$$

then the scalar

$$(\rho,\eta,h)\,\overset{*}{\mathbb{R}}=(\rho,\eta,h)\,\overset{*}{\mathbb{R}}{}_{\alpha\ \beta}g^{\alpha\beta}+(\rho,\eta,h)\,\overset{*}{\mathbb{S}}{}^{a\ b}\tilde{g}_{ab}\tag{6.10.1}$$

will be called the *scalar of curvature of distinguished linear (ρ,η)-connection* $\left((\rho,\eta)\,H,(\rho,\eta)\,V\right)$.

Definition 6.10.2 The tensor field

$$\begin{aligned}(\rho,\eta,h)\overset{*}{\mathbb{T}} &= (\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{\alpha\ \beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+(\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{\alpha}^{\ \ b}d\tilde{z}^{\alpha}\otimes\delta\tilde{p}_{b}\\ &+(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{a}_{\ \ \beta}\delta\tilde{p}_{a}\otimes d\tilde{z}^{\beta}+(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{a\ b}\delta\tilde{p}_{a}\otimes\delta\tilde{p}_{b}\end{aligned}\tag{6.10.2}$$

such that its components verify the following conditions:

$$\begin{aligned}\varkappa(\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{\alpha\ \beta} &= (\rho,\eta,h)\overset{*}{\mathbb{R}}_{\alpha\beta}-\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{R}}\cdot g_{\alpha\beta},\\ -\varkappa(\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{\alpha}^{\ \ b} &= (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{\alpha}^{\ \ b},\\ \varkappa(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{a}_{\ \ \beta} &= (\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{a}_{\ \ \beta},\\ \varkappa(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{a\ b} &= (\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{a\ b}-\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{R}}\cdot g^{ab},\end{aligned}\tag{6.10.3}$$

where $\varkappa$ is a constant, will be called *the energy-momentum tensor field associated to distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$ and metrical structure $\overset{*}{G}$.*

The equations (4.10.3) will be called *the Einstein equations associated to distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$ and metrical structure $\overset{*}{G}$.*

Formally, the Einstein equations will be written

$$\mathbf{Ric}\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)-\tfrac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{R}}\cdot\overset{*}{G}=\varkappa\cdot(\rho,\eta,h)\overset{*}{\mathbb{T}}.\tag{6.10.3$'$}$$

6.11 Dual mechanical systems

Using the diagram:

$$\begin{array}{ccc}\overset{*}{E} & & \left(E,[,]_{E,h},(\rho,\eta)\right)\\ \downarrow{\scriptstyle\overset{*}{\pi}} & & \downarrow{\scriptstyle\pi}\\ M & \xrightarrow{\ \ h\ \ } & M\end{array}\tag{6.11.1}$$

where $\left((E,\pi,M),[,]_{E,h},(\rho,\eta)\right)$ is a generalized Lie algebroid, we build the generalized tangent bundle

$$(((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}),[,]_{(\rho,\eta)T\overset{*}{E}},(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})).$$

Definition 6.11.1 A triple

$$\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_{e},(\rho,\eta)\overset{*}{\Gamma}\right),\tag{6.11.2}$$

where

$$\overset{*}{F}_{e}=F_{a}\dot{\tilde{\partial}}^{a}\in\Gamma\left(V(\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)\tag{6.11.3}$$

is an external force and $(\rho,\eta)\overset{*}{\Gamma}$ is a (ρ,η)-connection, will be called *dual mechanical (ρ,η)-system.*

A dual mechanical (ρ,η)-system

$$\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$$

endowed with a (pseudo)metrical structure $\overset{*}{G}$ determined with the help of a (pseudo)metrical structure

$$g=g^{ab}\delta\tilde{p}_a\otimes\delta\tilde{p}_a\in\mathcal{T}\,{}^{02}_{00}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

will be denoted

$$\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma},\overset{*}{G}\right). \tag{6.11.4}$$

and will be called *generalized Hamilton mechanical* (ρ,η)*-system.*

Any dual mechanical (Id_{TM},Id_M)-system and any generalized Hamilton mechanical (Id_{TM},Id_M)-system will be called *mechanical system* and *generalized Hamilton mechanical system,* respectively.

Definition 6.11.2 If H respectively K is a smooth Hamilton respectively Cartan function, then we put the triple

$$\left(\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right),$$

respectively

$$\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,K\right),$$

where

$$\overset{*}{F}_e=F_a\overset{\cdot\,a}{\tilde{\partial}}\in\Gamma\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is an external force. These are called *Hamilton mechanical* (ρ,η)*-system* and *Cartan mechanical* (ρ,η)*-system* respectively.

Any Hamilton mechanical (Id_{TM},Id_M)-system and any Cartan mechanical (Id_{TM},Id_M)-system will be called *Hamilton mechanical system* and *Cartan mechanical system, respectively.*

6.11.1 (ρ,η)-semisprays and (ρ,η)-sprays for dual mechanical (ρ,η)-systems

Let $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ be an arbitrary dual mechanical (ρ,η)-system.

Definition 6.11.1.1 The vertical section

$$\overset{*}{\mathbb{C}}=p_a\overset{\cdot\,a}{\tilde{\partial}}, \tag{6.11.1.1}$$

will be called *the Liouville section.*

Definition 6.11.1.2 The section $\overset{*}{S}\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ will be called (ρ,η)-*semispray* if there exists an almost tangent structure e such that

$$e\left(\overset{*}{S}\right)=\overset{*}{\mathbb{C}}. \tag{6.11.1.2}$$

Let (g,h) be a locally invertible $\mathbf{B^v}$-morphism of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ source and (E,π,M) target.

Theorem 6.11.1.1 *The section*

$$S=\left(g^{ab}\circ h\circ\overset{*}{\pi}\right)p_b\tfrac{\partial}{\partial\tilde{z}^a}-2\left(G_a-\tfrac{1}{4}F_a\right)\tfrac{\partial}{\partial\tilde{p}_a}\tag{6.11.1.3}$$

is a (ρ,η)-semispray such that the real local functions $G_a,\ a\in\overline{1,n}$, satisfy the following conditions

$$\begin{array}{cl}(\rho,\eta)\overset{*}{\Gamma}_{bc} & =-\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\right)\frac{\partial\left(G_c-\frac{1}{4}F_c\right)}{\partial p_e}\\ & +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)L_{db}^a\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ac}\circ h\circ\overset{*}{\pi},\ b,c\in\overline{1,r}.\end{array}\tag{6.11.1.4}$$

In addition, we remark that the local real functions

$$\begin{array}{cl}(\rho,\eta)\overset{\overset{*}{\circ}}{\Gamma}_{bc} & \overset{put}{=}-\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\right)\frac{\partial G_c}{\partial p_e}\\ & +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)L_{db}^a\circ h\circ\pi\cdot\tilde{g}_{ac}\circ h\circ\overset{*}{\pi},\ a,b\in\overline{1,r}\end{array}\tag{6.11.1.5}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\overset{\overset{*}{\circ}}{\Gamma}$ for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

The (ρ,η)-semispray $\overset{*}{S}$ will be called *the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible* $\mathbf{B^v}$-*morphism (g,h).*

Proof. We consider the **Mod**-endomorphism

$$\begin{array}{c}\Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)\ \xrightarrow{\overset{*}{\mathbb{P}}}\ \Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ X\ \longmapsto\overset{*}{\mathcal{J}}_{(g,h)}\left[\overset{*}{S},X\right]_{(\rho,\eta)T\overset{*}{E}}-\left[\overset{*}{S},\overset{*}{\mathcal{J}}_{(g,h)}X\right]_{(\rho,\eta)T\overset{*}{E}}.\end{array}$$

Let $X=\tilde{Z}^a\overset{*}{\tilde{\partial}}_a+Y_a\overset{\cdot\,a}{\tilde{\partial}}$ be an arbitrary section.
Since

$$\begin{array}{cl}\left[\overset{*}{S},X\right]_{(\rho,\eta)T\overset{*}{E}} & =\left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,\tilde{Z}^b\overset{*}{\tilde{\partial}}_b\right]_{(\rho,\eta)T\overset{*}{E}}\\ & +\left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,Y_b\overset{\cdot\,b}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}}\\ & -\left[2\left(G_a-\frac{1}{4}F_a\right)\overset{\cdot\,a}{\tilde{\partial}},\tilde{Z}^b\overset{*}{\tilde{\partial}}_b\right]_{(\rho,\eta)T\overset{*}{E}}\\ & -\left[2\left(G_a-\frac{1}{4}F_a\right)\overset{\cdot\,a}{\tilde{\partial}},Y_b\overset{\cdot\,b}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}}\end{array}$$

and

$$\begin{aligned}\left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,\tilde{Z}^b\overset{*}{\tilde{\partial}}_b\right]_{(\rho,\eta)T\overset{*}{E}} &= \left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial\tilde{Z}^c}{\partial x^i}\overset{*}{\tilde{\partial}}_c\\ &-\tilde{Z}^b\rho_b^j\circ h\circ\overset{*}{\pi}\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial x^j}\overset{*}{\tilde{\partial}}_c\\ &+\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\tilde{Z}^b\left(L_{ab}^c\circ h\circ\overset{*}{\pi}\right)\overset{*}{\tilde{\partial}}_c,\end{aligned}$$

$$\begin{aligned}\left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,Y_b\overset{\cdot\,b}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}} &= \left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial Y_c}{\partial x^i}\overset{\cdot\,c}{\tilde{\partial}}\\ &-Y_b\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_b}\overset{*}{\tilde{\partial}}_c,\end{aligned}$$

$$\begin{aligned}\left[2\left(G_a-\tfrac{1}{4}F_a\right)\overset{\cdot\,a}{\tilde{\partial}},\tilde{Z}^b\overset{*}{\tilde{\partial}}_b\right]_{(\rho,\eta)T\overset{*}{E}} &= 2\left(G_a-\tfrac{1}{4}F_a\right)\frac{\partial\tilde{Z}^c}{\partial p_a}\overset{*}{\tilde{\partial}}_c\\ &-2\tilde{Z}^b\rho_b^j\circ h\circ\overset{*}{\pi}\frac{\partial\left(G_c-\frac{1}{4}F_c\right)}{\partial x^j}\overset{\cdot\,c}{\tilde{\partial}},\end{aligned}$$

$$\begin{aligned}\left[2\left(G_a-\tfrac{1}{4}F_a\right)\overset{\cdot\,a}{\tilde{\partial}},Y_b\overset{\cdot\,b}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}} &= 2\left(G_a-\tfrac{1}{4}F_a\right)\frac{\partial Y_c}{\partial p_a}\overset{\cdot\,c}{\tilde{\partial}}\\ &-2Y_b\frac{\partial\left(G_c-\frac{1}{4}F_c\right)}{\partial p_b}\overset{\cdot\,c}{\tilde{\partial}},\end{aligned}$$

it results that

$$(P_1)\qquad\begin{aligned}\overset{*}{\mathcal{J}}_{(g,h)}\left[\overset{*}{S},X\right]_{(\rho,\eta)T\overset{*}{E}} &= \left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial\tilde{Z}^c}{\partial x^i}\cdot\left(\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\,d}{\tilde{\partial}}\\ &-\tilde{Z}^b\rho_b^j\circ h\circ\overset{*}{\pi}\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial x^j}\cdot\left(\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\,d}{\tilde{\partial}}\\ &+\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\tilde{Z}^bL_{ab}^c\cdot\left(\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\,d}{\tilde{\partial}}\\ &-Y^b\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial y^b}\cdot\left(\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\,d}{\tilde{\partial}}\\ &-2\left(G_a-\tfrac{1}{4}F_a\right)\frac{\partial\tilde{Z}^c}{\partial p_a}\cdot\left(\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\,d}{\tilde{\partial}}.\end{aligned}$$

Since

$$\begin{aligned}\left[\overset{*}{S},\overset{*}{\mathcal{J}}_{(g,h)}X\right]_{(\rho,\eta)T\overset{*}{E}} &= \left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,\left(\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^b\overset{\cdot\,c}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}}\\ &-\left[2\left(G_a-\tfrac{1}{4}F_a\right)\overset{\cdot\,a}{\tilde{\partial}},\left(\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^b\overset{\cdot\,c}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}}\end{aligned}$$

and

$$\begin{aligned}\left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,\left(\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^b\dot{\tilde{\partial}}_c\right]_{(\rho,\eta)T\overset{*}{E}} &= \left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial\left(\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\right)}{\partial x^i}\overset{\cdot\,d}{\tilde{\partial}}\\ &-\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\frac{\partial\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_c}\overset{*}{\tilde{\partial}}_d,\end{aligned}$$

$$\left[2\left(G_a-\tfrac{1}{4}F_a\right)\overset{\cdot\ a}{\tilde{\partial}},\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\tilde{Z}^b\overset{\cdot\ c}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}}=2\left(G_a-\tfrac{1}{4}F_a\right)\frac{\partial\left(\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\right)}{\partial p_a}\overset{\cdot\ d}{\tilde{\partial}}$$
$$-2\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\frac{\partial\left(G_d-\frac{1}{4}F_d\right)}{\partial p_c}\overset{\cdot\ d}{\tilde{\partial}}$$

it results that

$$(P_2)\quad \begin{aligned}\left[\overset{*}{S},\overset{*}{\mathcal{J}}_{(g,h)}X\right]_{(\rho,\eta)T\overset{*}{E}}&=\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\pi\frac{\partial\left(\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\right)}{\partial x^i}\overset{\cdot\ d}{\tilde{\partial}}\\&-\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\frac{\partial\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_c}\overset{*}{\tilde{\partial}}_d\\&-2\left(G_a-\tfrac{1}{4}F_a\right)\frac{\partial\left(\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\right)}{\partial p_a}\overset{\cdot\ d}{\tilde{\partial}}\\&+2\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\frac{\partial\left(G_d-\frac{1}{4}F_d\right)}{\partial p_c}\overset{\cdot\ d}{\tilde{\partial}}.\end{aligned}$$

We remark that

$$\begin{aligned}\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial\left(\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\right)}{\partial x^i}&=g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\left(\rho_a^i\circ h\circ\overset{*}{\pi}\right)\frac{\partial\tilde{Z}^c}{\partial x^i}\cdot\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\\&-\tilde{Z}^b\rho_b^j\circ h\circ\overset{*}{\pi}\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial x^j}\cdot\tilde{g}_{dc}\circ h\circ\overset{*}{\pi},\end{aligned}$$

$$Y_d=Y_b\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_b}\cdot\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}$$

and

$$\tilde{Z}^d=\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\frac{\partial\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_c}.$$

Using the equalities (P_1) and (P_2) we obtain:

$$\mathbb{P}\left(\tilde{Z}^a\overset{*}{\tilde{\partial}}_a+Y_a\overset{\cdot\ a}{\tilde{\partial}}\right)=\tilde{Z}^a\overset{*}{\tilde{\partial}}_a+$$
$$+\left(-Y_a-2\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\frac{\partial\left(G_a-\frac{1}{4}F_a\right)}{\partial p_c}\tilde{Z}^b+\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\tilde{Z}^bL_{db}^c\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ac}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\ a}{\tilde{\partial}}.$$

After some calculations, it results that $\mathbb{P}$ is an almost product structure.

Using the equality

$$\mathbb{P}=Id-2\left(\rho,\eta\right)\overset{*}{\Gamma},$$

we obtain that

$$\left(\rho,\eta\right)\Gamma\left(\tilde{Z}^a\overset{*}{\tilde{\partial}}_a+Y_a\overset{\cdot\ a}{\tilde{\partial}}\right)=$$
$$=\left(Y_a+\tilde{g}_{ac}\circ h\circ\overset{*}{\pi}\frac{\partial\left(G_b-\frac{1}{4}F_b\right)}{\partial p_a}\tilde{Z}^c-\tfrac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\tilde{Z}^cL_{dc}^f\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{af}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\ a}{\tilde{\partial}}$$

Since

$$\left(\rho,\eta\right)\Gamma\left(\tilde{Z}^a\overset{*}{\tilde{\partial}}_a+Y_a\overset{\cdot\ a}{\tilde{\partial}}\right)=\left(Y_c-\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}\tilde{Z}^b\right)\overset{\cdot\ c}{\tilde{\partial}}$$

it results that relations (4.11.1.4) are satisfied. In addition, since

$$(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}=(\rho,\eta)\overset{*}{\Gamma}_{bc}-\frac{1}{4}\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\frac{\partial F_c}{\partial p_d}$$

and

$$\begin{array}{l}(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{b'c'}=(\rho,\eta)\overset{*}{\Gamma}_{b'c'}-\frac{1}{4}\tilde{g}_{e'c'}\circ h\circ\overset{*}{\pi}\frac{\partial F_{b'}}{\partial p_{e'}}\\ =M_{b'}^{b}\circ\overset{*}{\pi}\left(-\rho_c^i\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_b^{a'}}{\partial x^i}p_{a'}+(\rho,\eta)\overset{*}{\Gamma}_{bc}\right)M_{c'}^{c}\circ h\circ\overset{*}{\pi}\\ +M_{b'}^{b}\circ\overset{*}{\pi}\left(\frac{1}{4}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\cdot\frac{\partial F_b}{\partial p_e}\right)M_{c'}^{c}\circ h\circ\overset{*}{\pi}\\ =M_{b'}^{b}\circ\overset{*}{\pi}\left(-\rho_c^i\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_b^{a'}}{\partial x^i}p_{a'}+\left((\rho,\eta)\overset{*}{\Gamma}_{bc}-\frac{1}{4}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\cdot\frac{\partial F_b}{\partial p_e}\right)\right)M_{c'}^{c}\circ h\circ\overset{*}{\pi}\\ =M_{b'}^{b}\circ\overset{*}{\pi}\left(-\rho_c^i\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_b^{a'}}{\partial x^i}p_{a'}+(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}\right)M_{c'}^{c}\circ h\circ\overset{*}{\pi}\end{array}$$

it results the conclusion of the theorem. *q.e.d.*

Theorem 6.11.1.2 *The following properties hold:*

1° *Since*

$$\overset{*}{\mathring{\tilde{\delta}}}_c=\overset{*}{\tilde{\partial}}_c+(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}\overset{\cdot\, b}{\tilde{\partial}},\ c\in\overline{1,r},$$

it results that

$$\overset{*}{\mathring{\tilde{\delta}}}_c=\overset{*}{\tilde{\delta}}_c-\frac{1}{4}\tilde{g}_{ec}\circ h\circ\pi\cdot\frac{\partial F_b}{\partial p_e}\overset{\cdot\, b}{\tilde{\partial}},\ c\in\overline{1,r}.\tag{6.11.1.6}$$

2° *Since*

$$\mathring{\delta}\tilde{p}_b=-(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}d\tilde{z}^c+d\tilde{p}_b,$$

it results that

$$\mathring{\delta}\tilde{p}_b=\delta\tilde{p}_b+\frac{1}{4}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial\tilde{p}_e}d\tilde{z}^c,\ b\in\overline{1,r}.\tag{6.11.1.7}$$

Theorem 6.11.1.3 *The real local functions*

$$\left(\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{bc}}{\partial p_a},\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{bc}}{\partial p_a},0,\ 0\right),\ a,b,c\in\overline{1,r}\tag{6.11.1.8}$$

and

$$\left(\frac{\partial(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}}{\partial p_a},\frac{\partial(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}}{\partial p_a},0,\ 0\right),\ a,b,c\in\overline{1,r}\tag{6.11.1.8$'$}$$

respectively are the coefficients to a normal Berwald linear (ρ,η)*-connection for the generalized tangent bundle* $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Theorem 6.11.1.4 *The tensor of integrability of the* (ρ,η)*-connection* $(\rho,\eta)\overset{*}{\mathring{\Gamma}}$ *is as follows:*

$$\begin{array}{c}(\rho,\eta,h)\overset{*}{\mathring{\mathbb{R}}}_{b\ cd}=(\rho,\eta,h)\overset{*}{\mathbb{R}}_{b\ cd}+\frac{1}{4}\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}_{|c}-\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}_{|d}\right)+\\ +\frac{1}{16}\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_l}{\partial p_e}\tilde{g}_{fc}\circ h\circ\overset{*}{\pi}\frac{\partial^2F_b}{\partial p_l\partial p_f}-\tilde{g}_{fc}\circ h\circ\overset{*}{\pi}\frac{\partial F_l}{\partial p_f}\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial^2F_b}{\partial p_l\partial p_e}\right)+\\ +\frac{1}{4}\left(L_{cd}^f\circ h\circ\overset{*}{\pi}\right)\left(\tilde{g}_{ef}\circ h\circ\overset{*}{\pi}\right)\frac{\partial F_b}{\partial p_e},\end{array}\tag{6.11.1.9}$$

where $_{|c}$ *is the* h*-covariant derivation with respect to the normal Berwald linear* ρ*-connection* (6.11.1.8).

Proof. Since

$$\begin{aligned}(\rho,\eta,h)\overset{*}{\mathring{\mathbb{R}}}_{b\ cd}=\ &\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\mathring{\tilde{\delta}}}_c\right)\left((\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bd}\right)-\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\mathring{\tilde{\delta}}}_d\right)\left((\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}\right)\\ &-L_{cd}^e\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{be},\end{aligned}$$

and

$$\begin{aligned}\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\mathring{\tilde{\delta}}}_c\right)\left((\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bd}\right)\ &=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_c\right)\left((\rho,\eta)\overset{*}{\Gamma}_{bd}\right)\\ &+\frac{1}{4}\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_c\right)\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}\right)\\ &-\frac{1}{4}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_f}{\partial p_e}\frac{\partial}{\partial p_f}\left((\rho,\eta)\overset{*}{\Gamma}_{bd}\right)\\ &-\frac{1}{16}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_f}{\partial p_e}\frac{\partial}{\partial p_f}\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}\right),\end{aligned}$$

$$\begin{aligned}\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\mathring{\tilde{\delta}}}_d\right)\left((\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}\right)\ &=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_d\right)\left((\rho,\eta)\overset{*}{\Gamma}_{bc}\right)\\ &+\frac{1}{4}\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_d\right)\left(\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}\right)\\ &-\frac{1}{4}\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_f}{\partial p_e}\frac{\partial}{\partial p_f}\left((\rho,\eta)\overset{*}{\Gamma}_{bc}\right)\\ &-\frac{1}{16}\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_f}{\partial p_e}\frac{\partial}{\partial p_f}\left(\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}\right),\end{aligned}$$

$$\begin{aligned}L_{cd}^e\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{be}\ &=L_{cd}^e\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{\Gamma}_{be}\\ &+L_{cd}^e\circ h\circ\overset{*}{\pi}\cdot\left(\tilde{g}_{fe}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}\right)\end{aligned}$$

it results the conclusion of the theorem. *q.e.d.*

Theorem 6.11.1.5 *Let*

$$\overset{*}{\mathbb{T}}{}^a_{bc}\delta_a\otimes d\tilde{z}^b\otimes d\tilde{z}^c\in\mathcal{T}_{20}^{10}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and

$$\overset{*}{\mathbb{S}}{}_{a}^{bc}\,\tilde{\partial}_a \otimes \delta\tilde{y}^b \otimes \delta\tilde{y}^c \in \mathcal{T}^{02}_{01}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

such that they verify the following conditions:

$$\overset{*}{\mathbb{T}}{}^{a}_{bc} = -\overset{*}{\mathbb{T}}{}^{a}_{cb},\ \overset{*}{\mathbb{S}}{}^{bc}_{a} = -\overset{*}{\mathbb{S}}{}^{bc}_{a},\ \forall b,c\in\overline{1,r}.$$

If $\left((\rho,\eta)\,\overset{*}{\tilde{H}},(\rho,\eta)\,\overset{*}{\tilde{V}}\right)$ *is the distinguished linear* (ρ,η)*-connection presented in the Theorem 6.9.2, then the local real functions:*

$$\begin{aligned}(\rho,\eta)\,\overset{*}{\overset{\circ}{\tilde{H}}}{}^{a}_{bc} &= (\rho,\eta)\,\overset{*}{\tilde{H}}{}^{a}_{bc}+\tfrac{1}{8}g^{ae}\left(-\tilde{g}_{fc}\circ h\circ\overset{*}{\pi}\tfrac{\partial F_d}{\partial p_f}\tfrac{\partial\tilde{g}_{be}}{\partial p_d}\right.\\ &\quad\left.+\tilde{g}_{fe}\circ h\circ\overset{*}{\pi}\tfrac{\partial F_d}{\partial p_f}\tfrac{\partial\tilde{g}_{bc}}{\partial p_d}-\tilde{g}_{fb}\circ h\circ\overset{*}{\pi}\tfrac{\partial F_d}{\partial p_f}\tfrac{\partial\tilde{g}_{ec}}{\partial p_d}\right),\\ (\rho,\eta)\,\overset{*}{\overset{\circ}{\tilde{V}}}{}^{a}_{bc} &= (\rho,\eta)\,\overset{*}{\tilde{V}}{}^{a}_{bc}\end{aligned}\tag{6.11.1.10}$$

are the components of a normal distinguished linear (ρ,η)*-connection with* (ρ,η)*-*$\overset{*}{\mathcal{H}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right)$ *and* (ρ,η)*-*$\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{V}}\overset{*}{\mathcal{V}}\right)$ *torsions a priori given such that the generalized tangent bundle* $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ *derives generalized Hamilton* (ρ,η)*-space.*

In addition, we have:

$$\begin{aligned}(\rho,\eta,h)\,\overset{*}{\overset{\circ}{\tilde{\mathbb{T}}}}{}^{a}_{bc} &= \overset{*}{\mathbb{T}}{}^{a}_{bc}\\ (\rho,\eta,h)\,\overset{*}{\overset{\circ}{\tilde{\mathbb{S}}}}{}^{bc}_{a} &= \overset{*}{\mathbb{S}}{}^{bc}_{a}.\end{aligned}\tag{6.11.1.11}$$

The local functions $\tilde{g}_{fc},\ \tilde{g}_{fe},\ \tilde{g}_{fb}$ *are the local functions associated to the locally invertible* $\mathbf{B}^{\mathbf{v}}$*-morphism* (g,h).

Proposition 6.11.1.1 *If* $\overset{*}{S}$ *is the canonical* (ρ,η)*-semispray associated to the mechanical* (ρ,η)*-system* $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\,\overset{*}{\Gamma}\right)$ *and from locally invertible* $\mathbf{B}^{\mathbf{v}}$*-morphism* (g,h)*, then*

$$2G_{b'} = 2G_b\cdot M^{b}_{b'}\circ h\circ\overset{*}{\pi}-\left(g^{ae}\circ h\circ\overset{*}{\pi}\right)p_e\left(\rho^i_a\circ h\circ\overset{*}{\pi}\right)\frac{\partial p_{b'}}{\partial x^i}.\tag{6.11.1.12}$$

Proof. Since the Jacobian matrix of coordinates transformation is

$$\left\Vert\begin{array}{cc} M^{a'}_{a}\circ h\circ\overset{*}{\pi} & 0\\ \rho^i_a\circ h\circ\overset{*}{\pi}\frac{\partial M^a_{b'}\circ\overset{*}{\pi}}{\partial x^i}p_a & M^b_{b'}\circ\overset{*}{\pi}\end{array}\right\Vert=\left\Vert\begin{array}{cc} M^{a'}_{a}\circ h\circ\overset{*}{\pi} & 0\\ \rho^i_a\circ h\circ\overset{*}{\pi}\frac{\partial p_{b'}}{\partial x^i} & M^b_{b'}\circ\overset{*}{\pi}\end{array}\right\Vert$$

and

$$\begin{aligned}&\left\Vert\begin{array}{cc} M^{a'}_{a}\circ h\circ\overset{*}{\pi} & 0\\ \rho^i_a\circ h\circ\overset{*}{\pi}\frac{\partial p_{b'}}{\partial x^i} & M^b_{b'}\circ\overset{*}{\pi}\end{array}\right\Vert\cdot\left(\begin{array}{c}\left(g^{ae}\circ h\circ\overset{*}{\pi}\right)p_e\\ -2\left(G_b-\frac{1}{4}F_b\right)\end{array}\right)=\\ &=\left(\begin{array}{c}\left(g^{a'e'}\circ h\circ\overset{*}{\pi}\right)p_{e'}\\ -2\left(G_{b'}-\frac{1}{4}F_{b'}\right)\end{array}\right),\end{aligned}$$

the conclusion results immediately. *q.e.d.*

In the following we consider a differentiable curve $I \overset{c}{\to} M$ and its (g,h)-lift $\dot{c}$.

Definition 6.11.1.3 The curve $\dot{c}$ is an integral curve of the (ρ,η)-semispray $\overset{*}{S}$ of the dual mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$, if it is verify the following equality:

$$\frac{d\dot{c}(t)}{dt}=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\overset{*}{S}\left(\dot{c}\left(t\right)\right). \tag{6.11.1.13}$$

Theorem 6.11.1.6 *The integral curves of the canonical (ρ,η)-semispray associated to the mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible* $\mathbf{B^v}$*-morphism (g,h), are the (g,h)-lifts solutions of the equations:*

$$\frac{dp_b(t)}{dt}+2G_b\circ\overset{*}{u}\left(c,\dot{c}\right)\left(x\left(t\right)\right)=\frac{1}{2}F_b\circ\overset{*}{u}\left(c,\dot{c}\right)\left(x\left(t\right)\right),\ b\in\overline{1,r}, \tag{6.11.1.14}$$

where $x\left(t\right)=\left(\eta\circ h\circ c\right)\left(t\right).$

Proof. Since the equality

$$\frac{d\dot{c}(t)}{dt}=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\overset{*}{S}\left(\dot{c}\left(t\right)\right)$$

is equivalent with

$$\begin{array}{c}\frac{d}{dt}((\eta\circ h\circ c)^i(t),p_b(t))=\\ =\left(\rho_a^i\circ\eta\circ h\circ c(t)g^{ae}\circ h\circ c(t)p_e(t),-2\left(G_b-\frac{1}{4}F_b\right)((\eta\circ h\circ c)(t),p\left(t\right))\right),\end{array}$$

it results

$$\begin{array}{c}\frac{dp_b(t)}{dt}+2G_b(x\left(t\right),p\left(t\right))=\frac{1}{2}F_b(x\left(t\right),p\left(t\right)),\ b\in\overline{1,r},\\ \frac{dx^i(t)}{dt}=\rho_a^i\circ\eta\circ h\circ c\left(t\right)g^{ae}\circ h\circ c\left(t\right)p_e\left(t\right),\end{array}$$

where $x^i\left(t\right)=\left(\eta\circ h\circ c\right)^i\left(t\right).$ *q.e.d.*

Definition 6.11.1.4 If $\overset{*}{S}$ is a (ρ,η)-semispray, then the vector field

$$\left[\overset{*}{\mathbb{C}},\overset{*}{S}\right]_{(\rho,\eta)T\overset{*}{E}}-\overset{*}{S} \tag{6.11.1.15}$$

will be called the *derivation of (ρ,η)-semispray* $\overset{*}{S}$.

The (ρ,η)-semispray $\overset{*}{S}$ will be called *(ρ,η)-spray* if there are verified the following conditions:

1. $\overset{*}{S}\circ 0\in C^1$, where 0 is the null section;
2. Its derivation is the null vector field.

The (ρ,η)-semispray $\overset{*}{S}$ will be called *quadratic (ρ,η)-spray* if there are verified the following conditions:

1. $\overset{*}{S}\circ 0\in C^{2}$, where 0 is the null section;
2. Its derivation is the null vector field.

In particular, if $(\rho,\eta)=(id_{TM},Id_M)$ and $(g,h)=(Id_E,Id_M)$, then we obtain the *spray* and the *quadratic spray* which is similar with the classical spray and quadratic spray.

Theorem 6.11.1.7 *If $\overset{*}{S}$ is the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h), then*

$$\begin{array}{ll}(6.11.1.16) & \begin{array}{rl}2\left(G_b-\frac{1}{4}F_b\right) & =(\rho,\eta)\overset{*}{\Gamma}_{bc}\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_f\right)\\ & +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)L_{dc}^{a}\circ h\circ\overset{*}{\pi}\\ & \cdot\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_f\right),\ b\in\overline{1,r}.\end{array}\end{array}$$

Then we obtain the spray

$$\begin{array}{ll}(6.11.1.17) & \begin{array}{l}\overset{*}{S}=\left(g^{ae}\circ h\circ\overset{*}{\pi}\right)p_e\frac{\partial}{\partial\tilde{z}^{a}}+(\rho,\eta)\overset{*}{\Gamma}_{bc}\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_f\right)\frac{\partial}{\partial\tilde{p}_b}\\ +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)L_{dc}^{a}\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_f\right)\frac{\partial}{\partial\tilde{p}_b}.\end{array}\end{array}$$

This (ρ,η)-spray will be called the canonical (ρ,η)-spray associated to mechanical system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).

In particular, if $(\rho,\eta,g,h)=(Id_{TM},Id_M,Id_M,Id_M)$, then we get the canonical spray associated to connection $\overset{*}{\Gamma}$ which is similar with the classical canonical spray associated to connection $\overset{*}{\Gamma}$.

Proof. Since

$$\begin{array}{rl}\left[\overset{*}{\mathbb{C}},\overset{*}{S}\right]_{(\rho,\eta)T\overset{*}{E}} & =\left[p_a\overset{\cdot\,a}{\tilde{\partial}},\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_b\right]_{(\rho,\eta)T\overset{*}{E}}\\ & -2\left[p_a\overset{\cdot\,a}{\tilde{\partial}},\left(G_b-\frac{1}{4}F_b\right)\overset{\cdot\,b}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}},\end{array}$$

$$\begin{array}{rl}\left[p_a\overset{\cdot\,a}{\tilde{\partial}},\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_b\right]_{(\rho,\eta)T\overset{*}{E}} & =p_a\frac{\partial\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_a}\overset{*}{\tilde{\partial}}_b\\ & -\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\frac{\partial p_a}{\partial x^{i}}\overset{\cdot\,a}{\tilde{\partial}}\\ & =\left(p_a\cdot g^{be}\circ h\circ\overset{*}{\pi}\cdot\delta_a^{e}\right)\overset{*}{\tilde{\partial}}_b-0\\ & =\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_b\end{array}$$

and
$$\left[p_a\overset{\cdot\, a}{\tilde{\partial}}, \left(G_b-\tfrac{1}{4}F_b\right)\overset{\cdot\, b}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}} \begin{array}{l} = p_a\frac{\partial\left(G_b-\frac{1}{4}F_b\right)}{\partial p_a}\overset{\cdot\, b}{\tilde{\partial}} \\ -\left(G_b-\frac{1}{4}F_b\right)\delta_a^b\overset{\cdot\, a}{\tilde{\partial}} \\ = p_a\frac{\partial\left(G_b-\frac{1}{4}F_b\right)}{\partial p_a}\overset{\cdot\, b}{\tilde{\partial}}-\left(G_b-\frac{1}{4}F_b\right)\overset{\cdot\, b}{\tilde{\partial}}\end{array}$$

it results that

$$(S_1)\qquad \left[\overset{*}{\mathbb{C}}, \overset{*}{S}\right]_{(\rho,\eta)T\overset{*}{E}}-\overset{*}{S} = 2\left(-p_f\frac{\partial\left(G_b-\frac{1}{4}F_b\right)}{p_f}+2\left(G_b-\tfrac{1}{4}F_b\right)\right)\overset{\cdot\, b}{\tilde{\partial}}$$

Using the equality (6.11.1.4) it results that

$$(S_2)\qquad \frac{\partial\left(G_b-\frac{1}{4}F_b\right)}{p_f} \begin{array}{l} =-(\rho,\eta)\overset{*}{\Gamma}_{bc}\circ\overset{*}{u}(c,\dot{c})\circ(\eta\circ h\circ c)\cdot g^{cf}\circ h\circ\overset{*}{\pi} \\ +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\cdot L_{dc}^a\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\cdot g^{cf}\circ h\circ\overset{*}{\pi}.\end{array}$$

Using the equalities (S_1) and (S_2) it results the conclusion of the theorem. *q.e.d.*

Remark 6.11.1.2. If $(\rho,\eta,h)=(id_{TM}, Id_M, Id_M)$, then we get the canonical spray associated to connection $\overset{*}{\Gamma}$.

Theorem 6.11.1.8 *The integral curves of canonical (ρ,η)-spray associated to mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h) are the (g,h)-lifts solutions of the following system of equations:*

$$(6.11.1.17)\qquad \begin{array}{l} \frac{dp_b}{dt}-(\rho,\eta)\overset{*}{\Gamma}_{bc}\circ\overset{*}{u}(c,\dot{c})\circ(\eta\circ h\circ c)\cdot\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_f\right) \\ +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\cdot L_{dc}^a\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\cdot\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_f\right)=0,\end{array}$$

where $x(t)=\eta\circ h\circ c(t)$.

6.11.2 The Hamiltonian formalism for Hamilton mechanical (ρ,η)-systems

Let $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right)$ be an arbitrary Hamilton mechanical (ρ,η)-system.

The *natural dual (ρ,η)-base* $(d\tilde{z}^\alpha, d\tilde{p}_a)$ of natural (ρ,η)-base $\left(\frac{\partial}{\partial\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{p}_a}\right)$ is determined by the equations

$$\begin{cases} \left\langle d\tilde{z}^\alpha,\frac{\partial}{\partial\tilde{z}^\beta}\right\rangle=\delta_\beta^\alpha, & \left\langle d\tilde{z}^\alpha,\frac{\partial}{\partial\tilde{p}_b}\right\rangle=0, \\ \left\langle d\tilde{p}_a,\frac{\partial}{\partial\tilde{z}^\beta}\right\rangle=0, & \left\langle d\tilde{p}_a,\frac{\partial}{\partial\tilde{p}_b}\right\rangle=\delta_a^b.\end{cases}$$

It is very important to remark that the 1-forms $d\tilde{z}^\alpha,\ \alpha\in\overline{1,p}$ and $d\tilde{p}_a,\ a\in\overline{1,r}$ are not the differentials of coordinates functions as in the classical case, but we will use the same notations.

In this case

$$(d\tilde{z}^\alpha)\neq d^{(\rho,\eta)T\overset{*}{E}}(\tilde{z}^\alpha)=0,$$

where $d^{(\rho,\eta)T\overset{*}{E}}$ is the exterior differentiation operator associated to exterior differential $\mathcal{F}\left(\overset{*}{E}\right)$-algebra

$$\left(\Lambda\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot,\wedge\right).$$

Let H be a regular Hamiltonian and let (g,h) be a $\mathbf{B^v}$-morphism locally invertible of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ source and (E,π,M) target.

Definition 6.11.2.1 The 1-form

$$\theta_H=\left(\tilde{g}_{ea}\circ h\circ\overset{*}{\pi}\cdot H^e\right)d\tilde{z}^a \tag{6.11.2.1}$$

will be called the 1-*form of Poincaré-Cartan type associated to the regular Hamiltonian* H *and to the locally invertible* $\mathbf{B^v}$*-morphism* (g,h).

We obtain easily:

$$\theta_H\left(\frac{\partial}{\partial\tilde{z}^b}\right)=\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\cdot H^e,\quad\theta_H\left(\frac{\partial}{\partial\tilde{p}_b}\right)=0. \tag{6.11.2.2}$$

Definition 6.11.2.2 The 2-form

$$\omega_H=d^{(\rho,\eta)T\overset{*}{E}}\theta_L$$

will be called the 2-*form of Poincaré-Cartan type associated to the regular Hamiltonian* H *and to the locally invertible* $\mathbf{B^v}$*-morphism* (g,h).

By the definition of $d^{(\rho,\eta)T\overset{*}{E}}$, we obtain:

$$\begin{array}{c}\omega_H\left(U,V\right)=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(U\right)\left(\theta_H\left(V\right)\right)-\\ -\Gamma\left(\tilde{\rho},Id_E\right)\left(V\right)\left(\theta_H\left(U\right)\right)-\theta_H\left(\left[U,V\right]_{(\rho,\eta)T\overset{*}{E}}\right),\end{array} \tag{6.11.2.3}$$

for any $U,V\in\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

It follows:
(6.11.2.4)

$$\begin{cases}\omega_L\left(\frac{\partial}{\partial\tilde{z}^a},\frac{\partial}{\partial\tilde{z}^b}\right)&=\left(\rho_a^i\circ h\circ\overset{*}{\pi}\right)\cdot\frac{\partial\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i}\\ &-\left(\rho_b^i\circ h\circ\overset{*}{\pi}\right)\cdot\frac{\partial\left(\tilde{g}_{ea}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i}-L_{ab}^c\circ h\circ\overset{*}{\pi}\cdot\left(\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\cdot H^e\right);\\ \omega_L\left(\frac{\partial}{\partial\tilde{z}^a},\frac{\partial}{\partial\tilde{p}_b}\right)&=\tilde{g}_{ea}\circ h\circ\overset{*}{\pi}\cdot H^{eb};\\ \omega_L\left(\frac{\partial}{\partial\tilde{p}_a},\frac{\partial}{\partial\tilde{p}_b}\right)&=0.\end{cases}$$

Definition 6.11.2.3 The real function

$$\mathcal{E}_H=p_a\cdot H^a-H \tag{6.11.2.5}$$

will be called the *energy of regular Hamiltonian* H.

Theorem 6.11.2.1 *The equation*

$$(6.11.2.6)\quad i_{\overset{*}{S}}(\omega_H) = -d^{(\rho,\eta)T\overset{*}{E}}(\mathcal{E}_H),\ \overset{*}{S} \in \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

has an unique solution $\overset{*}{S}_H(g,h)$ *of the type:*

$$(6.11.2.7)\qquad \left(g^{ae}\circ h\circ\overset{*}{\pi}\right)p_e\tfrac{\partial}{\partial\tilde{z}^a} - 2\left(G_a - \tfrac{1}{4}F_a\right)\tfrac{\partial}{\partial\tilde{p}_a},$$

where

$$(6.11.2.8)\qquad 2G_a = \left(g^{be}\circ h\circ\overset{*}{\pi}\cdot\tilde{H}_{ea}\right)\cdot E_b(H,g,h) + \tfrac{1}{2}F_a$$

and

$$(6.11.2.9)\qquad \begin{array}{c} E_b(H,g,h)=\rho_b^i\circ h\circ\overset{*}{\pi}\cdot H_i - g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\cdot\rho_a^i\circ h\circ\overset{*}{\pi}\cdot\frac{\partial\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i} \\ +g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\cdot L_{ab}^d\circ h\circ\overset{*}{\pi}\cdot\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\cdot H^e\right). \end{array}$$

$\overset{*}{S}_H(g,h)$ will be called *the canonical* (ρ,η)*-semispray associated to Hamilton mechanical* (ρ,η)*-system* $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right)$ *and from locally invertible* $\mathbf{B^v}$*-morphism* (g,h).

Proof. We obtain

$$\begin{array}{c} i_{\overset{*}{S}}(\omega_H) = -d^{(\rho,\eta)T\overset{(}{E}}(\mathcal{E}_H) \Longleftrightarrow \\ \Longleftrightarrow \omega_H\left(\overset{*}{S},X\right) = -\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)(X)(\mathcal{E}_H), \\ \forall X\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) \end{array}$$

Particularly, we obtain:

$$\omega_H\left(\overset{*}{S},\tfrac{\partial}{\partial\tilde{z}^b}\right) = -\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tfrac{\partial}{\partial\tilde{z}^b}\right)(\mathcal{E}_H).$$

If we expand this equality using (6.11.2.2) and (6.11.2.4), we obtain

$$\begin{array}{c} g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\cdot\left[\rho_a^i\circ h\circ\overset{*}{\pi}\cdot\frac{\partial\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i} - \rho_b^i\circ h\circ\overset{*}{\pi}\cdot\frac{\partial\left(\tilde{g}_{ea}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i}\right. \\ \left.- L_{ab}^d\circ h\circ\overset{*}{\pi}\cdot\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\cdot H^e\right)\right] + 2\left(G_a - \tfrac{1}{4}F_a\right)\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\right)\cdot H^{ea} \\ = -\rho_b^i\circ h\circ\overset{*}{\pi}\cdot\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\cdot\frac{\partial\left(\tilde{g}_{ea}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i} + \rho_b^i\circ h\circ\overset{*}{\pi}\cdot H_i. \end{array}$$

After some calculations, we obtain

$$2\left(G_a - \frac{1}{4}F_a\right) = \left(g^{be}\circ h\circ\pi\cdot\tilde{H}_{ea}\right)\cdot E_b(H,g,h),$$

where

$$E_b\left(H,g,h\right)=\rho_b^i\circ h\circ\overset{*}{\pi}\cdot H_i-g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\cdot\rho_a^i\circ h\circ\overset{*}{\pi}\cdot\frac{\partial\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i}$$
$$+g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\cdot L_{ab}^d\circ h\circ\overset{*}{\pi}\cdot\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\cdot H^e\right).$$

q.e.d.

Theorem 6.11.2.2 *The real local functions*

(6.11.2.10)
$$\begin{array}{rl}(\rho,\eta)\overset{*}{\Gamma}_{bc} & =\frac{1}{2}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial\left(\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot H_{eb}\right)E_a(H,g,h)\right)}{\partial p_e}\\ & -\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)L_{dc}^a\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ab}\circ h\circ\overset{*}{\pi},\ b,c\in\overline{1,r}.\end{array}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$ for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ which will be called the (ρ,η)-connection associated to Hamilton mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right)$ and from locally invertible $\mathbf{B}^{\mathbf{v}}$-morphism (g,h).

Corollary 6.11.2.1 *The real local functions*

(6.11.2.11)
$$\begin{array}{rl}(\rho,\eta)\overset{\overset{*}{\circ}}{\Gamma}_{bc} & =\left(\tilde{g}_{ec}\circ h\circ\pi\right)\frac{\partial G_b}{\partial p_e}\\ & -\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)L_{dc}^a\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ab}\circ h\circ\overset{*}{\pi},\ b,c\in\overline{1,r}\end{array}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\overset{\overset{*}{\circ}}{\Gamma}$ for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

In addition, we have

(6.11.2.12)
$$(\rho,\eta)\overset{\overset{*}{\circ}}{\Gamma}_{bc}=(\rho,\eta)\overset{*}{\Gamma}_{bc}+\tfrac{1}{4}(\tilde{g}_{ec}\circ h\circ\pi)\cdot\tfrac{\partial F_b}{\partial p_e},\ \forall a,c\in\overline{1,r}.$$

Theorem 6.11.2.3 *The integral curves of the canonical (ρ,η)-semispray associated to $\left(\overset{*}{E},\overset{*}{F}_e,H\right)$ mechanical (ρ,η)-system and from locally invertible $\mathbf{B}^{\mathbf{v}}$-morphism (g,h) are the autoparallel lifts with respect to (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$.*

Definition 6.11.2.4 The equations

(6.11.2.13)
$$\frac{dp_b\left(t\right)}{dt}+\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot\tilde{H}_{eb}\cdot E_a\left(H,g,h\right)\right)\circ\overset{*}{u}\left(c,\dot{c}\right)\circ\left(\eta\circ h\circ c\left(t\right)\right)=0,$$

will be called the *equations of Hamilton-Jacobi type associated to Hamilton mechanical (ρ,η)-system $\left(\overset{*}{E},\overset{*}{F}_e,H\right)$ and from locally invertible $\mathbf{B}^{\mathbf{v}}$-morphism (g,h).*

Remark 6.11.2.1 The integral curves of the canonical (ρ,η)-semispray associated to dual mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right)$ and from locally invertible $\mathbf{B}^{\mathbf{v}}$-morphism (g,h) are the (g,h)-lifts solutions for the equations of Hamilton-Jacobi type (6.11.2.13).

7 The (horizontal) Legendre (ρ, η, h)-equivalence

Let (E, π, M) be a vector bundle.

We take (x^i, y^a) as canonical local coordinates on (E, π, M), where $i \in \overline{1, m}$ and $a \in \overline{1, r}$.

Consider

$$\left(x^i, y^a\right) \longrightarrow \left(x^{\acute{\imath}}\left(x^i\right), y^{\acute{a}}\left(x^i, y^a\right)\right)$$

a change of coordinates on (E, π, M). Then the coordinates y^a change to $y^{\acute{a}}$ by the rule:

$$y^{\acute{a}} = M_a^{\acute{a}} y^a. \tag{7.1}$$

Let $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ be the dual vector bundle of (E, π, M).

We take (x^i, p_a) as canonical local coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, where $i \in \overline{1, m}$ and $a \in \overline{1, r}$.

Consider

$$\left(x^i, p_a\right) \longrightarrow \left(x^{\acute{\imath}}\left(x^i\right), p_{\acute{a}}\left(x^i, p_a\right)\right)$$

a change of coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$. Then the coordinates p_a change to $p_{\acute{a}}$ by the rule:

$$p_{\acute{a}} = M_{\acute{a}}^a p_a. \tag{7.1'}$$

If (U, s_U) and $\left(U, \overset{*}{s}_U\right)$ are vector local $(m+r)$-charts then

$$M_{\acute{a}}^a(x) \cdot M_b^{\acute{a}}(x) = \delta_b^a, \ \forall x \in U.$$

Let L be a differentiable Lagrangian defined on the total space of the vector bundle (E, π, M).

If (U, s_U) is a vector local $(m+r)$-chart for (E, π, M), then we obtain the following real functions defined on $\pi^{-1}(U)$:

$$\begin{array}{ll} L_i \overset{put}{=} \frac{\partial L}{\partial x^i} & L_{ib} \overset{put}{=} \frac{\partial^2 L}{\partial x^i \partial y^b} \\ L_a \overset{put}{=} \frac{\partial L}{\partial y^a} & L_{ab} \overset{put}{=} \frac{\partial^2 L}{\partial y^a \partial y^b} \end{array}. \tag{7.3}$$

We build the fiber bundle morphism

$$\begin{array}{ccc} E & \xrightarrow{\varphi_L} & \overset{*}{E} \\ \pi \downarrow & & \downarrow \overset{*}{\pi} \\ M & \xrightarrow{Id_M} & M \end{array},$$

where φ_L is locally defined

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_L} & \overset{*}{\pi}^{-1}(U) \\ u_x & \longmapsto & L_b(u_x) s^a(x) \end{array}, \tag{7.4}$$

for any vector local $(m+r)$-chart (U,s_U) of (E,π,M) and for any vector local $(m+r)$-chart $\left(U,\overset{*}{s}_U\right)$ of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

Using the differentiable Lagrangian L, we build the differentiable Hamiltonian H, locally defined by

$$(7.2') \qquad \begin{array}{ccc} \overset{*}{\pi}^{-1}(U) & \xrightarrow{H} & \mathbb{R} \\ \overset{*}{u}_x = p_a s^a & \longmapsto & p_a y^a - L(u_x) \end{array},$$

for any vector local $(m+r)$-chart $\left(U,\overset{*}{s}_U\right)$ of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, where $\left(y^a,\ a\in\overline{1,r}\right)$ are the components solutions of the differentiable equations

$$p_b = L_b(u_x),\ u_x\in\pi^{-1}(U).$$

If $\left(U,\overset{*}{s}_U\right)$ is a vector local $(m+r)$-chart for $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then we obtain the following real functions defined on $\overset{*}{\pi}^{-1}(U)$:

$$(7.3') \qquad \begin{array}{cc} H_i = \frac{\partial H}{\partial x^i} & H_i^b = \frac{\partial^2 H}{\partial x^i \partial p_b} \\ H^a = \frac{\partial H}{\partial p_a} & H^{ab} = \frac{\partial^2 H}{\partial p_a \partial p_b} \end{array}.$$

Using this Hamiltonian, we build the fiber bundle morphism

$$\begin{array}{ccc} \overset{*}{E} & \xrightarrow{\varphi_H} & E \\ \overset{*}{\pi}\downarrow & & \downarrow\pi \\ M & \xrightarrow{Id_M} & M \end{array},$$

where φ_H is locally defined

$$(7.4') \qquad \begin{array}{ccc} \overset{*}{\pi}^{-1}(U) & \xrightarrow{\varphi_H} & \pi^{-1}(U) \\ \overset{*}{u}_x & \longmapsto & H^a\left(\overset{*}{u}_x\right)s_a(x) \end{array},$$

for any vector local $(m+r)$-chart (U,s_U) of (E,π,M) and for any vector local $(m+r)$-chart $\left(U,\overset{*}{s}_U\right)$ of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

Using the **B**-morphism (φ_L, Id_M), we build the $\mathbf{B^v}$-morphism $((\rho,\eta)T\varphi_L,\varphi_L)$ given by the diagram

$$(7.5) \qquad \begin{array}{ccc} (\rho,\eta)TE & \xrightarrow{(\rho,\eta)T\varphi_L} & (\rho,\eta)T\overset{*}{E} \\ (\rho,\eta)\tau_E\downarrow & & \downarrow(\rho,\eta)\tau_{\overset{*}{E}} \\ E & \xrightarrow{\varphi_L} & \overset{*}{E} \end{array},$$

such that

$$(7.6) \qquad \begin{array}{ll} \Gamma((\rho,\eta)T\varphi_L,\varphi_L)\left(\tilde{Z}^\alpha\tilde{\partial}_\alpha\right) & = \left(\tilde{Z}^\alpha\circ\varphi_H\right)\overset{*}{\tilde{\partial}}_\alpha + \left[\left(\rho_\alpha^i\circ h\circ\pi\right)\tilde{Z}^\alpha L_{ib}\right]\circ\varphi_H\overset{\cdot}{\tilde{\partial}}{}^b, \\ \Gamma((\rho,\eta)T\varphi_L,\varphi_L)\left(Y^a\overset{\cdot}{\tilde{\partial}}_a\right) & = (Y^a L_{ab})\circ\varphi_H\overset{\cdot}{\tilde{\partial}}{}^b, \end{array}$$

for any $\tilde{Z}^{\alpha}\tilde{\partial}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a}\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right).$

The $\mathbf{B^{v}}$-morphism $\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)$ will be called the (ρ,η)*-tangent application of the Legendre bundle morphism associated to the Lagrangian* L.

Using the $\mathbf{B}$-morphism $\left(\varphi_{H},Id_{M}\right)$, we build the $\mathbf{B^{v}}$-morphism $\left(\left(\rho,\eta\right)T\varphi_{H},\varphi_{H}\right)$ given by the diagram

$$(7.5')\qquad \begin{array}{ccc}\left(\rho,\eta\right)T\overset{*}{E} & \xrightarrow{(\rho,\eta)T\varphi_{H}} & \left(\rho,\eta\right)TE\\ \left(\rho,\eta\right)\tau_{\overset{*}{E}}\downarrow & & \downarrow\left(\rho,\eta\right)\tau_{E}\\ E^{*} & \xrightarrow{\varphi_{H}} & E,\end{array}$$

such that

$$(7.6')\qquad \begin{array}{ll}\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\left(\tilde{Z}^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}\right) & =\left(\tilde{Z}^{\alpha}\circ\varphi_{L}\right)\tilde{\partial}_{\alpha}+\left[\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^{\alpha}H_{i}^{b}\right]\circ\varphi_{L}\dot{\tilde{\partial}}_{b},\\ \Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\left(Y_{a}\dot{\tilde{\partial}}^{a}\right) & =\left(Y_{a}H^{ab}\right)\circ\varphi_{L}\dot{\tilde{\partial}}_{b},\end{array}$$

for any $\tilde{Z}^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}+Y_{a}\dot{\tilde{\partial}}^{a}\in\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right).$

The $\mathbf{B^{v}}$-morphism $\left(\left(\rho,\eta\right)T\varphi_{H},\varphi_{H}\right)$ will be called the (ρ,η)*-tangent application of the Legendre bundle morphism associated to the Hamiltonian* H.

Let

$$(7.7)\qquad \left(\frac{\partial}{\partial x^{i}},\frac{\partial}{\partial y^{a}}\right)\overset{put}{=}\left(\partial_{i},\dot{\partial}_{a}\right)$$

be the natural base for sections Lie algebra $\left(\Gamma\left(TE,\tau_{E},E\right),+,\cdot,\left[,\right]_{TE}\right).$

Let

$$(7.7')\qquad \left(\frac{\partial}{\partial x^{i}},\frac{\partial}{\partial p_{a}}\right)\overset{put}{=}\left(\frac{\overset{*}{\partial}}{\partial x^{i}},\frac{\partial}{\partial p_{a}}\right)\overset{put}{=}\left(\overset{*}{\partial}_{i},\dot{\partial}^{a}\right)$$

be the natural base for sections Lie algebra $\left(\Gamma\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot,\left[,\right]_{T\overset{*}{E}}\right).$

Using the diagram:

$$(7.8)\qquad \begin{array}{ccc}E & & \left(F,\left[,\right]_{F,h},\left(\rho,\eta\right)\right)\\ \pi\downarrow & & \downarrow\nu\\ M & \xrightarrow{h} & N\end{array},$$

where $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$ is a generalized Lie algebroid, we build the generalized tangent bundle

$$(7.9)\qquad \left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right).$$

The natural (ρ,η)-base of sections is denoted

$$(7.10)\qquad \left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^{a}}\right)\overset{put}{=}\left(\tilde{\partial}_{\alpha},\dot{\tilde{\partial}}_{a}\right).$$

Using the diagram:

$$(7.8') \qquad \begin{array}{ccc} \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \overset{*}{\pi}\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array},$$

where $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid, we build the generalized tangent bundle

$$(7.9') \qquad \left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right).$$

The natural (ρ,η)-base of sections is denoted

$$(7.10') \qquad \left(\frac{\overset{*}{\partial}}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{p}_a}\right)\overset{put}{=}\left(\overset{*}{\tilde{\partial}}_{\alpha},\dot{\tilde{\partial}}^{a}\right).$$

7.1 The duality between mechanical systems

Let $((E,\pi,M),F_e,(\rho,\eta)\Gamma)$ be a mechanical (ρ,η)-system.

Let $g\in\mathbf{Man}\,(E,E)$ such that (g,h) is a $\mathbf{B^v}$-morphism locally invertible of (E,π,M) source and (E,π,M) target, on components g_b^a.

The **Mod**-endomorphism

$$(7.1.1) \qquad \begin{array}{ccc} \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right) & \xrightarrow{\mathcal{J}_{(g,h)}} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right) \\ \tilde{Z}^a\tilde{\partial}_a+Y^b\dot{\tilde{\partial}}_b & \longmapsto & \left(\tilde{g}_a^b\circ h\circ\pi\right)\tilde{Z}^a\dot{\tilde{\partial}}_b \end{array}$$

is the almost tangent structure associated to $\mathbf{B^v}$-morphism (g,h).

The vertical section

$$(7.1.2) \qquad \mathbb{C}=y^a\dot{\tilde{\partial}}_a$$

is the Liouville section.

Let $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ be a dual mechanical (ρ,η)-system.

Let $g\in\mathbf{Man}\left(\overset{*}{E},E\right)$ be such that (g,h) is a $\mathbf{B^v}$-morphism locally invertible of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ source and (E,π,M) target, on components g^{ab}.

The **Mod**-endomorphism

$$(7.1.1)' \qquad \begin{array}{ccc} \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{J}}_{(g,h)}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) \\ \tilde{Z}^a\overset{*}{\tilde{\partial}}_a+Y_b\dot{\tilde{\partial}}^{b} & \longmapsto & \left(\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^a\dot{\tilde{\partial}}^{b} \end{array}$$

is the almost tangent structure associated to $\mathbf{B^v}$-morphism (g,h).

The vertical section

$$(7.1.2)' \qquad \overset{*}{\mathbb{C}}=p_b\dot{\tilde{\partial}}^{b}$$

is the Liouville section.

Let

$$S=y^{b}\left(g_{b}^{a}\circ h\circ\pi\right)\frac{\partial}{\partial\tilde{z}^{a}}-2\left(G^{a}-\frac{1}{4}F^{a}\right)\frac{\partial}{\partial\tilde{y}^{a}}\tag{7.1.3}$$

be the (ρ,η)-semispray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,(\rho,\eta)\Gamma)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h) and let

$$\overset{*}{S}=p_{b}\left(g^{ab}\circ h\circ\overset{*}{\pi}\right)\frac{\overset{*}{\partial}}{\partial\tilde{z}^{a}}-2\left(G_{a}-\frac{1}{4}F_{a}\right)\frac{\partial}{\partial\tilde{p}_{a}}\tag{7.1.3$'$}$$

be the (ρ,η)-semispray associated to the mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),F_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).

Theorem 7.1.1 *If*

$$\Gamma\left((\rho,\eta)T\varphi_{L},\varphi_{L}\right)(S)=\overset{*}{S},\tag{7.1.4}$$

then we obtain:

$$y^{b}\left(g_{b}^{a}\circ h\circ\pi\right)\circ\varphi_{H}\quad=p_{b}\left(g^{ab}\circ h\circ\overset{*}{\pi}\right)\tag{7.1.5}$$

and

$$\begin{array}{cl}2\left(G_{b}-\frac{1}{4}F_{b}\right)&=2\left[\left(G^{a}-\frac{1}{4}F^{a}\right)\cdot L_{ab}\right]\circ\varphi_{H}\\&-y^{c}\left\{\left[\left(g_{c}^{a}\cdot\rho_{a}^{i}\right)\circ h\circ\pi\right]\cdot L_{ib}\right\}\circ\varphi_{H}.\end{array}\tag{7.1.6}$$

Theorem 7.1.2 *Dual, if*

$$\Gamma\left((\rho,\eta)T\varphi_{H},\varphi_{H}\right)\left(\overset{*}{S}\right)=S,\tag{7.1.4$'$}$$

then we obtain:

$$p_{b}\left(g^{ba}\circ h\circ\overset{*}{\pi}\right)\circ\varphi_{L}\quad=y^{b}\left(g_{b}^{a}\circ h\circ\pi\right)\tag{7.1.5$'$}$$

and

$$\begin{array}{cl}2\left(G^{a}-\frac{1}{4}F^{a}\right)&=2\left[\left(G_{b}-\frac{1}{4}F_{b}\right)\cdot H^{ab}\right]\circ\varphi_{L}\\&-p_{c}\left\{\left[\left(g^{ac}\cdot\rho_{a}^{i}\right)\circ h\circ\overset{*}{\pi}\right]\cdot H_{i}^{b}\right\}\circ\varphi_{L}.\end{array}\tag{7.1.6$'$}$$

7.2 The duality between Lie algebroids structures

The generalized tangent bundle

$$((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$$

can be endowed with a Lie algebroid structure

$$\left([,]_{(\rho,\eta)TE},(\tilde{\rho},Id_E)\right).$$

The Lie bracket $[,]_{(\rho,\eta)TE}$ is defined by

$$(7.2.1)\quad \begin{array}{l}\left[\left(\tilde{Z}_1^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y_1^a \frac{\partial}{\partial \tilde{y}^a}\right), \left(\tilde{Z}_2^\beta \frac{\partial}{\partial \tilde{z}^\beta} + Y_2^b \frac{\partial}{\partial \tilde{y}^b}\right)\right]_{(\rho,\eta)TE} = \\ = \left[\tilde{Z}_1^\alpha T_a, \tilde{Z}_2^\beta T_\beta\right]_{\pi^*(h^*F)} \oplus \left[\left(\rho_\alpha^i \circ h \circ \pi\right) \tilde{Z}_1^\alpha \frac{\partial}{\partial x^i} + Y_1^a \frac{\partial}{\partial \tilde{y}^a},\right. \\ \qquad\qquad \left.\left(\rho_\beta^i \circ h \circ \pi\right) \tilde{Z}_2^\beta \frac{\partial}{\partial x^i} + Y_2^b \frac{\partial}{\partial \tilde{y}^b}\right]_{TE},\end{array}$$

for any sections $\left(\tilde{Z}_1^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y_1^a \frac{\partial}{\partial \tilde{y}^a}\right)$ and $\left(\tilde{Z}_2^\beta \frac{\partial}{\partial \tilde{z}^\beta} + Y_2^b \frac{\partial}{\partial \tilde{y}^b}\right)$.

The anchor map $(\tilde{\rho}, Id_E)$ is a $\mathbf{B^v}$-morphism of $((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E)$ source and (TE, τ_E, E) target, where

$$(5.2.2)\quad \begin{array}{c}(\rho,\eta)\, TE \overset{\tilde{\rho}}{\to} TE \\ \left(\tilde{Z}^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y^a \frac{\partial}{\partial \tilde{y}^a}\right)(u_x) \longmapsto \left(\left(\rho_\alpha^i \circ h \circ \pi\right) \tilde{Z}^\alpha \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial \tilde{y}^a}\right)(u_x)\end{array}$$

The generalized tangent bundle

$$\left((\rho,\eta)\, T\overset{*}{E}, (\rho,\eta)\, \tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

can be endowed with a Lie algebroid structure

$$\left([,]_{(\rho,\eta)T\overset{*}{E}}, \left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\right).$$

The Lie bracket $[,]_{(\rho,\eta)T\overset{*}{E}}$ is defined by

$$(7.2.1)'\quad \begin{array}{l}\left[\left(\tilde{Z}_1^\alpha \frac{\overset{*}{\partial}}{\partial \tilde{z}^\alpha} + Y_{1a} \frac{\partial}{\partial \tilde{p}_a}\right), \left(\tilde{Z}_2^\beta \frac{\overset{*}{\partial}}{\partial \tilde{z}^\beta} + Y_{2b} \frac{\partial}{\partial \tilde{p}_b}\right)\right]_{(\rho,\eta)T\overset{*}{E}} = \\ = \left[\tilde{Z}_1^\alpha T_a, \tilde{Z}_2^\beta T_\beta\right]_{\overset{*}{\pi}{}^*(h^*F)} \oplus \left[\left(\rho_\alpha^i \circ h \circ \overset{*}{\pi}\right) \tilde{Z}_1^\alpha \frac{\partial}{\partial x^i} + Y_{1a} \frac{\partial}{\partial \tilde{p}_a},\right. \\ \qquad\qquad \left.\left(\rho_\beta^i \circ h \circ \overset{*}{\pi}\right) \tilde{Z}_2^\beta \frac{\partial}{\partial x^i} + Y_{2b} \frac{\partial}{\partial \tilde{p}_b}\right]_{T\overset{*}{E}},\end{array}$$

for any sections $\left(\tilde{Z}_1^\alpha \frac{\overset{*}{\partial}}{\partial \tilde{z}^\alpha} + Y_{1a} \frac{\partial}{\partial \tilde{p}_a}\right)$ and $\left(\tilde{Z}_2^\beta \frac{\overset{*}{\partial}}{\partial \tilde{z}^\beta} + Y_{2b} \frac{\partial}{\partial \tilde{p}_b}\right)$.

The anchor map $\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)$ is a $\mathbf{B^v}$-morphism of $\left((\rho,\eta)\, T\overset{*}{E}, (\rho,\eta)\, \tau_{\overset{*}{E}}, \overset{*}{E}\right)$ source and $\left(T\overset{*}{E}, \tau_{\overset{*}{E}}, \overset{*}{E}\right)$ target, where

$$(5.2.2)'\quad \begin{array}{c}(\rho,\eta)\, T\overset{*}{E} \overset{\overset{*}{\tilde{\rho}}}{\to} T\overset{*}{E} \\ \left(\tilde{Z}^\alpha \frac{\overset{*}{\partial}}{\partial \tilde{z}^\alpha} + Y_a \frac{\partial}{\partial \tilde{p}_a}\right)(\overset{*}{u}_x) \longmapsto \left(\left(\rho_\alpha^i \circ h \circ \overset{*}{\pi}\right) \tilde{Z}^\alpha \frac{\overset{*}{\partial}}{\partial x^i} + Y_a \frac{\partial}{\partial \tilde{p}_a}\right)(\overset{*}{u}_x)\end{array}$$

Theorem 7.2.1 *If the $\mathbf{B^v}$-morphism $((\rho,\eta)\, T\varphi_L, \varphi_L)$ is morphism of Lie algebroids, then we obtain:*

$$(7.2.3)\quad \left(L_{\alpha\beta}^\gamma \circ h \circ \pi\right) \circ \varphi_H = L_{\alpha\beta}^\gamma \circ h \circ \overset{*}{\pi},$$

$$(7.2.4)\quad \begin{array}{rl} \left[\left(L_{\alpha\beta}^{\gamma}\rho_{\gamma}^{k}\right)\circ h\circ\pi\cdot L_{kb}\right]\circ\varphi_H & =\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot\frac{\overset{*}{\partial}}{\partial x^i}\left[\left(\rho_{\beta}^{j}\circ h\circ\pi\cdot L_{jb}\right)\circ\varphi_H\right] \\ & -\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\cdot\frac{\overset{*}{\partial}}{\partial x^j}\left[\left(\rho_{\alpha}^{i}\circ h\circ\pi\cdot L_{ib}\right)\circ\varphi_H\right] \\ & +\left(\rho_{\alpha}^{i}\circ h\circ\pi\cdot L_{ia}\right)\circ\varphi_H\cdot\frac{\partial}{\partial p_a}\left[\left(\rho_{\beta}^{j}\circ h\circ\pi\cdot L_{jb}\right)\circ\varphi_H\right] \\ & -\left(\rho_{\beta}^{j}\circ h\circ\pi\cdot L_{ja}\right)\circ\varphi_H\cdot\frac{\partial}{\partial p_a}\left[\left(\rho_{\alpha}^{i}\circ h\circ\pi\cdot L_{ib}\right)\circ\varphi_H\right], \end{array}$$

$$(7.2.5)\quad \begin{array}{rl} 0 & =\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot\frac{\overset{*}{\partial}}{\partial x^i}\left(L_{ba}\circ\varphi_H\right) \\ & +\left(\rho_{\alpha}^{i}\circ h\circ\pi\cdot L_{bc}\right)\circ\varphi_H\frac{\partial}{\partial p_c}\left(L_{ba}\circ\varphi_H\right) \\ & -L_{bc}\circ\varphi_H\cdot\frac{\partial}{\partial p_c}\left[\left(\rho_{\alpha}^{i}\circ h\circ\pi\cdot L_{ia}\right)\circ\varphi_H\right] \end{array}$$

and

$$(7.2.6)\quad \begin{array}{rl} 0 & =L_{ac}\circ\varphi_H\cdot\frac{\partial}{\partial p_c}\left(L_{bd}\circ\varphi_H\right) \\ & -L_{bc}\circ\varphi_H\cdot\frac{\partial}{\partial p_c}\left(L_{ad}\circ\varphi_H\right). \end{array}$$

Proof. Developing the following equalities

$$\begin{array}{c} \Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\left[\tilde{\partial}_{\alpha},\tilde{\partial}_{\beta}\right]_{(\rho,\eta)TE} \\ =\left[\Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\tilde{\partial}_{\alpha},\Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\tilde{\partial}_{\beta}\right]_{(\rho,\eta)T\overset{*}{E}}, \end{array}$$

$$\begin{array}{c} \Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\left[\tilde{\partial}_{\alpha},\dot{\tilde{\partial}}_{b}\right]_{(\rho,\eta)TE} \\ =\left[\Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\tilde{\partial}_{\alpha},\Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\dot{\tilde{\partial}}_{b}\right]_{(\rho,\eta)T\overset{*}{E}} \end{array}$$

and

$$\begin{array}{c} \Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\left[\dot{\tilde{\partial}}_{a},\dot{\tilde{\partial}}_{b}\right]_{(\rho,\eta)TE} \\ =\left[\Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\dot{\tilde{\partial}}_{a},\Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\dot{\tilde{\partial}}_{b}\right]_{(\rho,\eta)T\overset{*}{E}} \end{array}$$

it results the conclusion of the theorem. *q.e.d.*

Corollary 7.2.1 *In particular, if* $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$*, then we obtain:*

$$(7.2.4)'\quad \begin{array}{rl} 0 & =\frac{\overset{*}{\partial}}{\partial x^i}\left(L_{jb}\circ\varphi_H\right)-\frac{\overset{*}{\partial}}{\partial x^j}\left(L_{ib}\circ\varphi_H\right) \\ & +L_{ia}\circ\varphi_H\cdot\frac{\partial}{\partial p_a}\left(L_{jb}\circ\varphi_H\right)-L_{ja}\circ\varphi_H\cdot\frac{\partial}{\partial p_a}\left(L_{ib}\circ\varphi_H\right) \end{array}$$

$$(5.2.5)'\quad \begin{array}{rl} 0 & =\frac{\overset{*}{\partial}}{\partial x^i}\left(L_{ba}\circ\varphi_H\right)+L_{bc}\circ\varphi_H\cdot\frac{\partial}{\partial p_c}\left(L_{ba}\circ\varphi_H\right) \\ & -L_{bc}\circ\varphi_H\cdot\frac{\partial}{\partial p_c}\left(L_{ia}\circ\varphi_H\right) \end{array}$$

and

$$(7.2.6)'\quad \begin{array}{rl} 0 & =L_{ac}\circ\varphi_H\cdot\frac{\partial}{\partial p_c}\left(L_{bd}\circ\varphi_H\right) \\ & -L_{bc}\circ\varphi_H\cdot\frac{\partial}{\partial p_c}\left(L_{ad}\circ\varphi_H\right). \end{array}$$

Theorem 7.2.2 *Dual, if the the* $\mathbf{B}^{\mathbf{v}}$*-morphism* $((\rho,\eta)\,T\varphi_H,\varphi_H)$ *is morphism of Lie algebroids, then we obtain:*

$$\left(L_{\alpha\beta}^{\gamma}\circ h\circ\overset{*}{\pi}\right)\circ\varphi_L=L_{\alpha\beta}^{\gamma}\circ h\circ\pi, \tag{7.2.7}$$

$$\begin{array}{ll}\left[\left(L_{\alpha\beta}^{\gamma}\rho_{\gamma}^{k}\right)\circ h\circ\overset{*}{\pi}\cdot H_{k}^{b}\right]\circ\varphi_L & =\rho_{\alpha}^{i}\circ h\circ\pi\cdot\frac{\partial}{\partial x^{i}}\left[\left(\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\cdot H_{j}^{b}\right)\circ\varphi_L\right]\\ & -\rho_{\beta}^{j}\circ h\circ\pi\cdot\frac{\partial}{\partial x^{j}}\left[\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot H_{i}^{b}\right)\circ\varphi_L\right]\\ & +\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot H_{i}^{c}\right)\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left[\left(\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\cdot H_{j}^{b}\right)\circ\varphi_L\right]\\ & -\left(\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\cdot H_{j}^{c}\right)\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left[\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot H_{i}^{b}\right)\circ\varphi_L\right],\end{array} \tag{7.2.8}$$

$$\begin{array}{ll}0 & =\rho_{\alpha}^{i}\circ h\circ\pi\cdot\frac{\partial}{\partial x^{i}}\left(H^{ba}\circ\varphi_L\right)\\ & +\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot H^{bc}\right)\circ\varphi_L\frac{\partial}{\partial y^{c}}\left(H^{ba}\circ\varphi_L\right)\\ & -H^{bc}\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left[\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot H_{i}^{a}\right)\circ\varphi_L\right]\end{array} \tag{7.2.9}$$

and

$$\begin{array}{ll}0 & =H^{ac}\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left(H^{bd}\circ\varphi_L\right)\\ & -H^{bc}\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left(H^{ad}\circ\varphi_L\right).\end{array} \tag{7.2.10}$$

Proof. Developing the following equalities

$$\begin{array}{c}\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\left[\overset{*}{\tilde{\partial}}_{\alpha},\overset{*}{\tilde{\partial}}_{\beta}\right]_{(\rho,\eta)T\overset{*}{E}}\\ =\left[\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\overset{*}{\tilde{\partial}}_{\alpha},\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\overset{*}{\tilde{\partial}}_{\beta}\right]_{(\rho,\eta)TE},\end{array}$$

$$\begin{array}{c}\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\left[\overset{*}{\tilde{\partial}}_{\alpha},\overset{\cdot\, b}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}}\\ =\left[\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\overset{*}{\tilde{\partial}}_{\alpha},\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\overset{\cdot\, b}{\tilde{\partial}}\right]_{(\rho,\eta)TE}\end{array}$$

and

$$\begin{array}{c}\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\left[\overset{\cdot\, a}{\tilde{\partial}},\overset{\cdot\, b}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}}\\ =\left[\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\overset{\cdot\, a}{\tilde{\partial}},\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\overset{\cdot\, b}{\tilde{\partial}}\right]_{(\rho,\eta)TE}\end{array}$$

it results the conclusion of the theorem. *q.e.d.*

Corollary 7.2.2 *In particular, if* $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$*, then we obtain:*

$$\begin{array}{ll}0 & =\frac{\partial}{\partial x^{i}}\left(H_{j}^{b}\circ\varphi_L\right)-\frac{\partial}{\partial x^{j}}\left(H_{i}^{b}\circ\varphi_L\right)\\ & +H_{i}^{c}\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left(H_{j}^{b}\circ\varphi_L\right)-H_{j}^{c}\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left(H_{i}^{b}\circ\varphi_L\right)\end{array} \tag{7.2.8$'$}$$

$$
(7.2.9)' \qquad \begin{array}{rl} 0 & = \frac{\partial}{\partial x^i}\left(H^{ba}\circ\varphi_L\right)+H^{bc}\circ\varphi_L\cdot\frac{\partial}{\partial y^c}\left(H^{ba}\circ\varphi_L\right) \\ & -H^{bc}\circ\varphi_L\cdot\frac{\partial}{\partial y^c}\left(H_i^a\circ\varphi_L\right) \end{array}
$$

and

$$
(7.2.10)' \qquad \begin{array}{rl} 0 & = H^{ac}\circ\varphi_L\cdot\frac{\partial}{\partial y^c}\left(H^{bd}\circ\varphi_L\right) \\ & -H^{bc}\circ\varphi_L\cdot\frac{\partial}{\partial y^c}\left(H^{ad}\circ\varphi_L\right). \end{array}
$$

Definition 7.2.1 If $((\rho,\eta)\,T\varphi_L,\varphi_L)$ and $((\rho,\eta)\,T\varphi_H,\varphi_H)$ are Lie algebroids morphisms, then we will say that (E,π,M) *and* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ *are Legendre* (ρ,η,h)*-equivalent.*

We will write

$$
(E,\pi,M)\underset{(\rho,\eta,h)}{\overset{\mathcal{L}}{\frown}}\left(\overset{*}{E},\overset{*}{\pi},M\right).
$$

Theorem 7.2.3 *If*

$$
(E,\pi,M)\underset{(\rho,\eta,h)}{\overset{\mathcal{L}}{\frown}}\left(\overset{*}{E},\overset{*}{\pi},M\right),
$$

then, using the equalities (7.2.3) *and* (7.2.7) *it results that for any vcetor local* $(m+r)$*-chart* (U,s_U) *of* (E,π,M) *and for any vector local* $(m+r)$*-chart* $\left(U,\overset{*}{s}_U\right)$ *of* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ *we obtain:*

$$
(7.2.11) \qquad \varphi_H\circ\varphi_L=Id_{\pi^{-1}(U)}
$$

and

$$
(7.2.12) \qquad \varphi_L\circ\varphi_H=Id_{\overset{*}{\pi}^{-1}(U)}.
$$

Therefore, locally, φ_L *is diffeomorphism and* $\varphi_L^{-1}=\varphi_H$.

7.3 The duality betwen adapted (ρ,η)-basis

If $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M), then the adapted (ρ,η)-base of sections is

$$
(7.3.1) \qquad \left(\frac{\partial}{\partial\tilde{z}^\alpha}-(\rho,\eta)\,\Gamma_\alpha^a\frac{\partial}{\partial\tilde{y}^a},\frac{\partial}{\partial\tilde{y}^a}\right)\overset{put}{=}\left(\frac{\delta}{\delta\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{y}^a}\right)\overset{put}{=}\left(\tilde{\delta}_\alpha,\dot{\tilde{\partial}}_a\right).
$$

If $(\rho,\eta)\,\overset{*}{\Gamma}$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then the adapted (ρ,η)-base of sections is

$$
(7.3.1)' \qquad \left(\frac{\overset{*}{\partial}}{\partial\tilde{z}^\alpha}+(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\frac{\partial}{\partial\tilde{p}_b},\frac{\partial}{\partial\tilde{p}_a}\right)\overset{put}{=}\left(\frac{\overset{*}{\delta}}{\delta\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{p}_a}\right)\overset{put}{=}\left(\overset{*}{\tilde{\delta}}_\alpha,\dot{\tilde{\partial}}^a\right).
$$

Theorem 7.3.1 *If*

$$
\Gamma\left((\rho,\eta)\,T\varphi_L,\varphi_L\right)\left(\tilde{\delta}_\alpha\right)=\overset{*}{\tilde{\delta}}_\alpha,
$$

then

$$
(7.3.2) \qquad (\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}=\left[\left(\rho_\alpha^i\circ h\circ\pi\right)\cdot L_{ib}-(\rho,\eta)\,\Gamma_\alpha^a\cdot L_{ab}\right]\circ\varphi_H.
$$

Proof. After some calculations, it results the conclusion of the theorem. *q.e.d.*

Corollary 7.3.1 *In particular, if* $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$*, then the equality*

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\left(\tilde{\delta}_k\right)=\overset{*}{\tilde{\delta}}_k,$$

implies the equality

$$\overset{*}{\Gamma}_{bk}=\left[L_{kb}-\Gamma_k^a\cdot L_{ab}\right]\circ\varphi_H. \tag{7.3.2$'$}$$

Theorem 7.3.2 *Dual, if*

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\left(\overset{*}{\tilde{\delta}}_\alpha\right)=\tilde{\delta}_\alpha,$$

then

$$-\left(\rho,\eta\right)\Gamma_\alpha^a=\left[\left(\rho_\alpha^i\circ h\circ\overset{*}{\pi}\right)\cdot H_i^a+\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\cdot H^{ba}\right]\circ\varphi_L. \tag{7.3.3}$$

Proof. After some calculations, it results the conclusion of the theorem. *q.e.d.*

Corollary 7.3.2 *In particular, if* $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$*, then the equality*

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\left(\overset{*}{\tilde{\delta}}_k\right)=\tilde{\delta}_k,$$

implies the equality

$$-\Gamma_k^a=\left[H_k^a+\overset{*}{\Gamma}_{bk}\cdot H^{ba}\right]\circ\varphi_L. \tag{7.3.3$'$}$$

Definition 7.3.2 If

$$\left(E,\pi,M\right)\underset{(\rho,\eta,h)}{\overset{\mathcal{L}}{\frown}}\left(\overset{*}{E},\overset{*}{\pi},M\right).$$

and

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\left(\tilde{\delta}_\alpha\right)=\overset{*}{\tilde{\delta}}_\alpha, \tag{7.3.4}$$

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\left(\overset{*}{\tilde{\delta}}_\alpha\right)=\tilde{\delta}_\alpha, \tag{7.3.4$'$}$$

then we will say that (E,π,M) *and* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ *are horizontal Legendre* (ρ,η,h)*-equivalent.*

We will write

$$\left(E,\pi,M\right)\underset{(\rho,\eta,h)}{\overset{\mathcal{HL}}{\frown}}\left(\overset{*}{E},\overset{*}{\pi},M\right).$$

The dual natural (ρ,η)-base of the natural (ρ,η)-base $\left(\tilde{\partial}_\alpha, \dot{\tilde{\partial}}_a\right)$ is denoted $(d\tilde{z}^\alpha, d\tilde{y}^a)$ and the dual adapted (ρ,η)-base of the adapted (ρ,η)-base $\left(\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_a\right)$ is denoted

$$(d\tilde{z}^\alpha, \delta\tilde{y}^a) \overset{put}{=} (d\tilde{z}^\alpha, d\tilde{y}^a + (\rho,\eta)\,\Gamma^a_\alpha \cdot d\tilde{z}^\alpha)\,. \tag{7.3.5}$$

The dual natural (ρ,η)-base of the natural (ρ,η)-base $\left(\overset{*}{\tilde{\partial}}_\alpha, \dot{\tilde{\partial}}^a\right)$ is denoted $(d\tilde{z}^\alpha, d\tilde{p}_a)$ and the dual adapted (ρ,η)-base of the adapted (ρ,η)-base $\left(\overset{*}{\tilde{\delta}}_\alpha, \dot{\tilde{\partial}}_a\right)$ is denoted

$$(d\tilde{z}^\alpha, \delta\tilde{p}_a) \overset{put}{=} \left(d\tilde{z}^\alpha, d\tilde{p}_a - (\rho,\eta)\,\overset{*}{\Gamma}_{a\alpha} \cdot d\tilde{z}^\alpha\right). \tag{7.3.5$'$}$$

Theorem 7.3.3 *The equality* (7.3.4) *is equivalent with the equality:*

$$\Gamma\left((\rho,\eta)\,T\varphi_L, \varphi_L\right)^*(\delta\tilde{p}_a) = L_{ab}\cdot\delta\tilde{y}^b \tag{7.3.6}$$

and the equality (7.3.4)$'$ *is equivalent with the equality:*

$$\Gamma\left((\rho,\eta)\,T\varphi_H, \varphi_H\right)^*(\delta\tilde{y}^a) = H^{ab}\cdot\delta\tilde{p}_b \tag{7.3.6$'$}$$

Theorem 7.3.4 *If*

$$(E,\pi,M)\underset{(\rho,\eta,h)}{\overset{\mathcal{HL}}{\frown}}\left(\overset{*}{E},\overset{*}{\pi},M\right),$$

then we obtain:

$$(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha} = \left[\left(\rho^i_\alpha\circ h\circ\pi\right)\cdot L_{ib} - (\rho,\eta)\,\Gamma^a_\alpha\cdot L_{ab}\right]\circ\varphi_H \tag{7.3.7}$$

and

$$-(\rho,\eta)\,\Gamma^a_\alpha = \left[\left(\rho^i_\alpha\circ h\circ\overset{*}{\pi}\right)\cdot H^a_i + (\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\cdot H^{ba}\right]\circ\varphi_L. \tag{7.3.7$'$}$$

If the Lagrangian L is regular, then we will define the real local functions $\tilde{L}^{ab}$ such that

$$\left\|\tilde{L}^{ab}(u_x)\right\| = \left\|L_{ab}(u_x)\right\|^{-1},\ \forall u_x\in\pi^{-1}(U)\,.$$

If the Hamiltonian H is regular, then we will define the real local functions $\tilde{H}_{ab}$ such that

$$\left\|\tilde{H}_{ab}\left(\overset{*}{u}_x\right)\right\| = \left\|H^{ab}\left(\overset{*}{u}_x\right)\right\|^{-1},\ \forall\overset{*}{u}_x\in\overset{*}{\pi}^{-1}(U)\,.$$

Remark 7.3.1 If the Lagrangian L is regular and

$$(E,\pi,M)\underset{(\rho,\eta,h)}{\overset{\mathcal{HL}}{\frown}}\left(\overset{*}{E},\overset{*}{\pi},M\right)$$

then, using the equalities (7.3.7) and (7.3.7)$'$, we obtain:

$$\left(\rho^i_\alpha\circ h\circ\pi\right)\cdot L_{ib}\cdot\tilde{L}^{ab} = -\left[\left(\rho^i_\alpha\circ h\circ\overset{*}{\pi}\right)\cdot H^b_i\right]\circ\varphi_L \tag{7.3.8}$$

and

$$\tilde{L}^{ab} = H^{ab} \circ \varphi_L. \tag{7.3.9}$$

Therefore, the Hamiltonian H is regular and

$$\tilde{H}_{ab} = L_{ab} \circ \varphi_H. \tag{7.3.10}$$

It is known that the following equalities hold good

$$\left[\tilde{\delta}_\alpha, \tilde{\delta}_\beta\right]_{(\rho,\eta)TE} = \left(L_{\alpha\beta}^\gamma \circ h \circ \pi\right) \tilde{\delta}_\gamma + (\rho, \eta, h)\, \mathbb{R}^a{}_{\alpha\beta} \dot{\tilde{\partial}}_a, \tag{7.3.11}$$

and

$$\left[\overset{*}{\tilde{\delta}}_\alpha, \overset{*}{\tilde{\delta}}_\beta\right]_{(\rho,\eta)T\overset{*}{E}} = \left(L_{\alpha\beta}^\gamma \circ h \circ \overset{*}{\pi}\right) \overset{*}{\tilde{\delta}}_\gamma + (\rho, \eta, h)\, \mathbb{R}_{b\ \alpha\beta} \dot{\tilde{\partial}}^b, \tag{7.3.11$'$}$$

Theorem 7.3.5 *If*

$$(E, \pi, M) \overset{\mathcal{HL}}{\widetilde{\ (\rho,\eta,h)\ }} \left(\overset{*}{E}, \overset{*}{\pi}, M\right),$$

then, we obtain:

$$(\rho, \eta, h)\, \mathbb{R}_{b\ \alpha\beta} = \left[(\rho, \eta, h)\, \mathbb{R}^a{}_{\alpha\beta} \cdot L_{ab}\right] \circ \varphi_H \tag{7.3.12}$$

and

$$(\rho, \eta, h)\, \mathbb{R}^a{}_{\alpha\beta} = \left[(\rho, \eta, h)\, \mathbb{R}_{b\ \alpha\beta} \cdot H^{ba}\right] \circ \varphi_L. \tag{7.3.12$'$}$$

Theorem 7.3.6 *If*

$$(E, \pi, M) \overset{\mathcal{HL}}{\widetilde{\ (\rho,\eta,h)\ }} \left(\overset{*}{E}, \overset{*}{\pi}, M\right),$$

then we obtain

$$\begin{aligned} \left(\tfrac{\partial(\rho,\eta)\Gamma_\alpha^a}{\partial y^b} \cdot L_{ac}\right) \circ \varphi_H &= L_{ba} \circ \varphi_H \cdot \tfrac{\partial(\rho,\eta)\Gamma_{c\alpha}}{\partial p_a} \\ &+ \left(\rho_\alpha^i \circ h \circ \overset{*}{\pi}\right) \cdot \tfrac{\overset{*}{\partial}}{\partial x^i} \left(L_{bc} \circ \varphi_H\right) \\ &+ (\rho, \eta)\, \Gamma_{a\alpha} \cdot \tfrac{\partial}{\partial p_a} \left(L_{bc} \circ \varphi_H\right) \end{aligned} \tag{7.3.13}$$

and

$$\begin{aligned} -\left(\tfrac{\partial(\rho,\eta)\Gamma_\alpha^a}{\partial y^b} \cdot L_{ac}\right) \circ \varphi_H &= L_{ba} \circ \varphi_H \cdot \tfrac{\partial(\rho,\eta)\Gamma_{c\alpha}}{\partial p_a} \\ &+ \left(\rho_\alpha^i \circ h \circ \overset{*}{\pi}\right) \cdot \tfrac{\overset{*}{\partial}}{\partial x^i} \left(L_{bc} \circ \varphi_H\right) \\ &+ (\rho, \eta)\, \Gamma_{a\alpha} \cdot \tfrac{\partial}{\partial p_a} \left(L_{bc} \circ \varphi_H\right) \end{aligned} \tag{7.3.13$'$}$$

Proof. Developing the following equalities

$$\begin{aligned} &\Gamma\left((\rho, \eta)\, T\varphi_L, \varphi_L\right) \left(\left[\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_a\right]_{(\rho,\eta)TE}\right) \\ &= \left[\Gamma\left((\rho, \eta)\, T\varphi_L, \varphi_L\right) \tilde{\delta}_\alpha, \Gamma\left((\rho, \eta)\, T\varphi_L, \varphi_L\right) \dot{\tilde{\partial}}_a\right]_{(\rho,\eta)T\overset{*}{E}} \end{aligned}$$

and

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\left(\left[\overset{*}{\tilde{\delta}}_\alpha,\overset{\cdot}{\tilde{\partial}}^{a}\right]_{(\rho,\eta)T\overset{*}{E}}\right)$$
$$=\left[\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\overset{*}{\tilde{\delta}}_\alpha,\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\overset{\cdot}{\tilde{\partial}}_a\right]_{(\rho,\eta)TE}$$

it results the conclusion of the theorem. *q.e.d.*

7.4 The duality between distinguished linear (ρ,η)-connections

Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M) and let

(7.4.1) $$(X,T)\xrightarrow{(\rho,\eta)D}(\rho,\eta)\,D_XT$$

be a covariant (ρ,η)-derivative for the tensor algebra of generalized tangent bundle

$$\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$$

which preserves the horizontal and vertical IDS by parallelism.

If (U,s_U) is a vector local $(m+r)$-chart for (E,π,M), then the real local functions

$$\left((\rho,\eta)\,H^\alpha_{\beta\gamma},(\rho,\eta)\,H^a_{b\gamma},(\rho,\eta)\,V^\alpha_{\beta c},(\rho,\eta)\,V^a_{bc}\right)$$

defined on $\pi^{-1}(U)$ and determined by the following equalities:

(7.4.2) $$\begin{array}{ll}(\rho,\eta)\,D_{\tilde{\delta}_\gamma}\tilde{\delta}_\beta=(\rho,\eta)\,H^\alpha_{\beta\gamma}\tilde{\delta}_\alpha, & (\rho,\eta)\,D_{\tilde{\delta}_\gamma}\overset{\cdot}{\tilde{\partial}}_b=(\rho,\eta)\,H^a_{b\gamma}\overset{\cdot}{\tilde{\partial}}_a\\ (\rho,\eta)\,D_{\overset{\cdot}{\tilde{\partial}}_c}\tilde{\delta}_\beta=(\rho,\eta)\,V^\alpha_{\beta_c}\tilde{\delta}_\alpha, & (\rho,\eta)\,D_{\overset{\cdot}{\tilde{\partial}}_c}\overset{\cdot}{\tilde{\partial}}_b=(\rho,\eta)\,V^a_{bc}\overset{\cdot}{\tilde{\partial}}_a\end{array}$$

are the components of a distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$.

Let $(\rho,\eta)\,\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and let

(7.4.1)′ $$(X,T)\xrightarrow{(\rho,\eta)\overset{*}{D}}(\rho,\eta)\,\overset{*}{D}_XT$$

be a covariant (ρ,η)-derivative for the tensor algebra of generalized tangent bundle

$$\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

which preserves the horizontal and vertical IDS by parallelism.

If $\left(U,\overset{*}{s}_U\right)$ is a vector local $(m+r)$-chart for $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then the real local functions

$$\left((\rho,\eta)\,\overset{*}{H}{}^\alpha_{\beta\gamma},(\rho,\eta)\,\overset{*}{H}{}^a_{b\gamma},(\rho,\eta)\,\overset{*}{V}{}^{\alpha c}_{\beta},(\rho,\eta)\,\overset{*}{V}{}^{ac}_{b}\right)$$

defined on $\overset{*}{\pi}{}^{-1}(U)$ and determined by the following equalities:

(7.4.2)′ $$\begin{array}{ll}(\rho,\eta)\,\overset{*}{D}_{\overset{*}{\tilde{\delta}}_\gamma}\overset{*}{\tilde{\delta}}_\beta=(\rho,\eta)\,\overset{*}{H}{}^\alpha_{\beta\gamma}\overset{*}{\tilde{\delta}}_\alpha, & (\rho,\eta)\,\overset{*}{D}_{\overset{*}{\tilde{\delta}}_\gamma}\overset{\cdot}{\tilde{\partial}}{}^a=(\rho,\eta)\,\overset{*}{H}{}^a_{b\gamma}\overset{\cdot}{\tilde{\partial}}{}^b\\ (\rho,\eta)\,\overset{*}{D}_{\overset{\cdot}{\tilde{\partial}}{}^c}\overset{*}{\tilde{\delta}}_\beta=(\rho,\eta)\,\overset{*}{V}{}^{\alpha c}_{\beta}\overset{*}{\tilde{\delta}}_\alpha, & (\rho,\eta)\,\overset{*}{D}_{\overset{\cdot}{\tilde{\partial}}{}^c}\overset{\cdot}{\tilde{\partial}}{}^b=(\rho,\eta)\,\overset{*}{V}{}^{bc}_{a}\overset{\cdot}{\tilde{\partial}}{}^a\end{array}$$

are the components of a distinguished linear (ρ, η)-connection

$$\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V}\right).$$

Theorem 7.4.1 *If*

$$(E, \pi, M) \underset{(\rho,\eta,h)}{\overset{\mathcal{HL}}{\frown}} \left(\overset{*}{E}, \overset{*}{\pi}, M\right)$$

and

$$\Gamma\left((\rho, \eta) T\varphi_L, \varphi_L\right)\left((\rho, \eta) D_X Y\right) = (\rho, \eta) \overset{*}{D}_{\Gamma((\rho,\eta)T\varphi_L,\varphi_L)X}\Gamma\left((\rho, \eta) T\varphi_L, \varphi_L\right) Y,$$

for any $X, Y \in \Gamma\left((\rho, \eta) TE, (\rho, \eta) \tau_E, E\right)$, then we obtain:

$$(\rho, \eta) H^{\alpha}_{\beta\gamma} \circ \varphi_H = (\rho, \eta) \overset{*}{H}{}^{\alpha}_{\beta\gamma}, \tag{7.4.3}$$

$$\begin{aligned} \left((\rho, \eta) H^{a}_{b\gamma} \cdot L_{ac}\right) \circ \varphi_H &= \left(\rho^k_\gamma \circ h \circ \overset{*}{\pi}\right) \cdot \frac{\overset{*}{\partial}}{\partial x^k}\left(L_{bc} \circ \varphi_H\right) \\ &+ (\rho, \eta) \overset{*}{\Gamma}_{b\gamma} \cdot \frac{\partial}{\partial p_b}\left(L_{bc} \circ \varphi_H\right) \\ &- (\rho, \eta) \overset{*}{H}{}^{a}_{b\gamma} \cdot \left(L_{ac} \circ \varphi_H\right), \end{aligned} \tag{7.4.4}$$

$$(\rho, \eta) V^{\alpha}_{\beta d} \circ \varphi_H = (\rho, \eta) \overset{*}{V}{}^{\alpha c}_{\beta} \cdot \left(L_{cd} \circ \varphi_H\right) \tag{7.4.5}$$

and

$$\begin{aligned} \left((\rho, \eta) V^{a}_{bc} \cdot L_{ad}\right) \circ \varphi_H &= \left(L_{ce} \circ \varphi_H\right) \cdot \frac{\partial}{\partial p_e}\left(L_{bd} \circ \varphi_H\right) \\ &- \left(L_{ce} \circ \varphi_H\right) \cdot (\rho, \eta) \overset{*}{V}{}^{ef}_{d} \cdot \left(L_{bf} \circ \varphi_H\right). \end{aligned} \tag{7.4.6}$$

Theorem 7.4.2 *Dual, if*

$$(E, \pi, M) \underset{(\rho,\eta,h)}{\overset{\mathcal{HL}}{\frown}} \left(\overset{*}{E}, \overset{*}{\pi}, M\right)$$

and

$$\Gamma\left((\rho, \eta) T\varphi_H, \varphi_H\right)\left((\rho, \eta) \overset{*}{D}_X Y\right) = (\rho, \eta) D_{\Gamma((\rho,\eta)T\varphi_H,\varphi_H)X}\Gamma\left((\rho, \eta) T\varphi_H, \varphi_H\right) Y,$$

for any $X, Y \in \Gamma\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$, then we obtain:

$$(\rho, \eta) \overset{*}{H}{}^{\alpha}_{\beta\gamma} \circ \varphi_L = (\rho, \eta) H^{\alpha}_{\beta\gamma}, \tag{7.4.3$'$}$$

$$\begin{aligned} \left((\rho, \eta) \overset{*}{H}{}^{a}_{b\gamma} \cdot H^{bc}\right) \circ \varphi_L &= \left(\rho^k_\gamma \circ h \circ \pi\right) \cdot \frac{\partial}{\partial x^k}\left(H^{ac} \circ \varphi_L\right) \\ &+ (\rho, \eta) \Gamma^{b}_{\gamma} \cdot \frac{\partial}{\partial y^b}\left(H^{ac} \circ \varphi_L\right) \\ &- (\rho, \eta) H^{a}_{b\gamma} \cdot \left(H^{bc} \circ \varphi_L\right), \end{aligned} \tag{7.4.4$'$}$$

$$(\rho, \eta) \overset{*}{V}{}^{\alpha c}_{\beta} \circ \varphi_L = (\rho, \eta) V^{\alpha}_{\beta c} \cdot \left(H^{cd} \circ \varphi_L\right) \tag{7.4.5$'$}$$

and

$$\begin{aligned} \left((\rho, \eta) \overset{*}{V}{}^{bc}_{a} \cdot H^{ad}\right) \circ \varphi_L &= \left(H^{ce} \circ \varphi_H\right) \cdot \frac{\partial}{\partial y^e}\left(H^{bd} \circ \varphi_L\right) \\ &- \left(H^{ce} \circ \varphi_L\right) \cdot (\rho, \eta) V^{d}_{ef} \cdot \left(H^{bf} \circ \varphi_L\right). \end{aligned} \tag{7.4.6$'$}$$

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THE GENERALIZED LIE ALGEBROIDS AND THEIR APPLICATIONS

CONSTANTIN M. ARCUŞ

In memory of my uncles
Prof. Dr. Gheorghe RADU and Acad. Dr. Doc. Cornelius RADU

Dedicated to Acad. Prof. Dr. Doc. Radu MIRON at his 83^{th} anniversary

Abstract

In this paper we introduce the notion of generalized Lie algebroid and we develop a new formalism necessary to obtain a new solution for the Weistein's Problem [61]. Many applications emphasize the importance and the utility of this new framework determined by the introduction of generalized Lie algebroids.

We introduce and develop the exterior differential calculus for generalized Lie algebroids and, in this general framework, we establish the structure equations of Maurer-Cartan type. In particular, we obtain a new point of view over the exterior differential calculus for Lie algebroids.

Using the (generalized) Lie algebroids theory, we build the Lie algebroid generalized tangent bundle and, using that, we obtain a new method by determining the (linear) connections for fiber bundles, in general, and for vector bundles, in particular.

Using the linear connections theory we develop the study of the geometry of vector bundles. Moreover, using the connections theory, we develop the geometry of total space of the generalized tangent bundle for a vector bundle.

We present a geometric description of metrizability for the total space of the Lie algebroid generalized tangent bundle, where we extend the notions of generalized Lagrange space, Lagrange space and Finsler space. Using the Lie algebroid generalized tangent bundle of a generalized Lie algebroid, we introduce and develop a mechanical systems theory and we present a Lagrangian formalism for these mechanical systems. In particular, using the Lie algebroid generalized tangent bundle of a Lie algebroid, we obtain a new solution for the Weinstein's Problem.

A geometric description of metrizability for the total space of the Lie algebroid generalized tangent bundle for dual vector bundle is presented. We extend the notions of generalized Hamilton space, Hamilton space and Cartan space. Using the Lie algebroid generalized tangent bundle of dual of a generalized Lie algebroid, we introduce and develop the dual mechanical systems theory and we present a Hamiltonian formalism for dual mechanical systems.

Finally, we introduce and develop the concept of (horizontal) Legendre equivalence between a vector bundle and its dual vector bundle.

We remark that, if the morphisms used are identities morphisms, then we obtain similar results to the classical results, but which are not classical results though.

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Contents

1 Introduction

The motivation for our researches was the

> **Weinstein's Problem:**
>
> *Develop a Lagrangian formalism directly on the given Lie algebroid similar to Klein's formalism for ordinary Lagrangian Mechanics* [25].

This problem was formulated by A. Weinstein in [61], where the author gave the theory of Lagrangians on Lie algebroids and obtained the Euler-Lagrange equations using the dual of a Lie algebroid and the Legendre transformation defined by a regular Lagrangian.

In [28], P. Liberman showed that such a formalism is not possible if one consider the tangent bundle of a Lie algebroid as space for developing the theory. Using the prolongation of a Lie algebroid over a smooth map introduced by P.J. Higgins and K. Mackenzie in [15], E. Martinez solved the *Weinstein's Problem* in [61] (see also [13], [27]).

Finding an other space for developing the theory, we discovered the generalized Lie algebroids which are presented in Subsection 3.3.

Since any Lie algebroid can be regarded as a generalized Lie algebroid, we proposed to obtain a new solution for the *Weinstein's Problem* using the new notion of generalized Lie algebroid.

To solve this problem it was necessary to introduce and develop a new formalism. In order to develop our researches, new and interesting notions and results appeared, which determined the apparition of new theories which are naturally integrated in our paper. In particular, using identity morphisms, we obtain similar results with S. Vacaru [59] (see also [56], [57], [58]) and L. Popescu (see: [45]-[49]).

So, in Subsection 3.2 we introduce and develop the exterior differential calculus for generalized Lie algebroids and, using that, we establish the structure equations of Maurer-Cartan type for generalized Lie algebroids. In particular, we obtain a new point of view over the exterior differential calculus for Lie algebroids. We introduced the notion of *interior differential system* of a generalized Lie algebroid in Paragraph 3.1.3 and we obtain a theorem of Cartan type. In addition, we introduced the notion of *exterior differential system* of a generalized Lie algebroid and we characterized the involutivity of an interior differential system in Subsection 3.3.

Inspired by the general framework of Yang-Mills theory, presented synthetically in the following diagram:

$$\begin{array}{ccc} (E, \langle , \rangle_E) & & (TM, [,]_{TM}, (Id_{TM}, Id_M), g) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tau_M} \\ M & \xrightarrow{Id_M} & M \end{array}$$

where:

1. (E, π, M) is a vector bundle,

2. $\langle , \rangle_E$ is an inner product for the module of sections $\Gamma(E, \pi, M)$,

3. $((Id_{TM}, Id_M), [,]_{TM})$ is the usual Lie algebroid structure for the tangent vector bundle (TM, τ_M, M) and

4. $g \in \Gamma\left((T^*M, \tau_M^*, M) \otimes (T^*M, \tau_M^*, M)\right)$ such that (M, g) is a Riemannian manifold,

we build the *Lie algebroid generalized tangent bundle* in Subsection 3.3.

Using this in Subsection 3.4, we introduce and develop a (linear) connections theory for fiber bundles, in general, and for vector bundles, in particular.

We can define the covariant derivatives with respect to sections of the generalized Lie algebroid
$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right).$$

In particular, if we use the generalized Lie algebroid structure
$$\left([,]_{TM,Id_M}, (Id_{TM}, Id_M)\right)$$
for the tangent bundle (TM, τ_M, M) in our theory, then the linear connections obtained are similar with the classical linear connections for the vector bundle (E, π, M), but not classical linear connections.

It is known that in Yang-Mills theory the set
$$Cov^0_{(E,\pi,M)}$$
of covariant derivatives for the vector bundle (E, π, M) such that
$$X\left(\langle u, v\rangle_E\right) = \langle D_X(u), v\rangle_E + \langle u, D_X(v)\rangle_E,$$
for any $X \in \mathcal{X}(M)$ and $u, v \in \Gamma(E, \pi, M)$, is very important, because the Yang-Mills theory is a variational theory which use (cf. [6]) the Yang-Mills functional
$$\begin{array}{ccc} Cov^0_{(E,\pi,M)} & \stackrel{\mathcal{YM}}{\longrightarrow} & \mathbb{R} \\ D_X & \longmapsto & \frac{1}{2}\int\limits_M \left\|\mathbb{R}^{D_X}\right\|^2 v_g \end{array}$$
where $\mathbb{R}^{D_X}$ is the curvature.

Using the linear connections theory, we succeed to extend at maximum the set $Cov^0_{(E,\pi,M)}$ of Yang-Mills theory, because using all generalized Lie algebroid structures for the tangent bundle (TM, τ_M, M), we obtain all possible linear connections for the vector bundle (E, π, M).

We emphasize the importance and the utility of linear connections theory for vector bundles in Chapter IV of our paper, where we present many applications. In particular, we obtain similar results to the classical results, but which are not classical results though.

After that we study the geometry of total space of the Lie algebroid generalized tangent bundle for a vector bundle in Section 5 of our paper, where we emphasize the importance and the utility of connections theory presented in Subsection 3.4.

The geometry of Lagrange spaces, introduced and studied in [24] and [35], was extensively examined in the last two decades by geometers and physicists from Romania, Japan, Hungary, Canada, Germany, Italy, Russia and USA. Many international conferences devoted to debate this subject, proceedings and monographs where published [3], [4], [41], [42]. A large area of applicability of this geometry is suggested by the

connections to Biology, Mechanics and Physics and also by its general setting as a generalization of Finsler and Riemann geometries.

As the (generalized) Lagrange space has been certified as an excellent model for some important problems in Relativity, Gauge Theory and Electromagnetism, in Subsections 5.8 and 5.9 we continue and we present a geometric description of metrizability for the total space of the Lie algebroid generalized tangent bundle for a vector bundle. We extend the notions of generalized Lagrange space, Lagrange space and Finsler space and we define the Einstein equations in this general framework.

Subsection 5.11 is devoted to introduce and study of a new class of mechanical systems called by us *mechanical* (ρ,η)*-systems, generalized Lagrange mechanical* (ρ,η)*-systems*, *Lagrange mechanical* (ρ,η)*-systems and Finsler mechanical* (ρ,η)*-systems.*

For these mechanical systems we develop a theory of semisprays and sprays. We develop a Lagrangian formalism for Lagrange mechanical systems.

We determine and we study the (ρ,η)-semispray associated to a regular Lagrangian L and external force F_e which are applied on the total space of a generalized Lie algebroid and we derive the equations of Euler-Lagrange type.

In particular, using the Lie algebroid generalized tangent bundle of a Lie algebroid, we obtain a new solution for the *Weinstein's Problem* different by the Martinez's solution [61].

Moreover, if the Lie algebroid used is

$$((TM,\tau_M,M),[,]_{TM},(Id_{TM},Id_M)),$$

then we obtain similar results to those presented by I. Bucataru and R. Miron in [7].

It is known that in 1918, immediately after the birth of general relativity, Weyl proposed the first unified theory of gravitation and electromagnetism, by generalizing the Riemannian space.

We are interested in finding the answer to the following question:

- *Could we to extend the study of the Riemannian geometry from the usual Lie algebroid*

  $$((TM,\tau_M,M),[,]_{TM},(Id_{TM},Id_M)),$$

  to an arbitrary (generalized) Lie algebroid and can we obtain a general framework necessary to unify the theory of gravitation with the theory of electromagnetism?

The future will show how far our theory can be used in this direction.

Our researches continue in Section 6, where we study the geometry of total space of the Lie algebroid generalized tangent bundle of a dual vector bundle and so, we emphasize the importance and the utility of the generalized connections theory presented in paragraph 3.4.1.

We present the adapted (ρ,η)-basis and adapted dual (ρ,η)-basis and remarkable endomorphisms of $(\Gamma((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}),+,\cdot)$ module (projectors, almost product structure, almost tangent structure, almost complex structure, (ρ,η)-tension endomorphism) and we present the (ρ,η)-torsion and the (ρ,η)-curvature of a (ρ,η)-connection $(\rho,\eta)\,\Gamma$. We introduce and studied distinguished linear (ρ,η)-connections and we build the (g,h)-lift of accelerations for a differentiable curve. Using the distinguished linear (ρ,η)-connections theory, we introduced and study the (ρ,η)-torsion, the (ρ,η)-curvature and we present the formulas of Ricci type and the identities of Cartan and Bianchi type.

The concept of Hamilton space, introduced in [36], [40], was intensively studied in [19], [20], [21], and it has been successful, as a geometric theory of the Hamiltonian function. The modern formulation of the geometry of Cartan spaces was given by R. Miron ([36], [38]) although some results where obtained by É. Cartan [9] and A. Kawaguchi [23] . Since the fundamental entity in Mechanics and Physics is the (generalized) Hamilton space, in Subsections 4.8 and 4.9 we continue to present a geometric description of metrizability for the total space of the Lie algebroid generalized tangent bundle of dual vector bundle. We extend the notions of generalized Hamilton space, Hamilton space and Cartan space and we define the Einstein equations in this general framework.

Subsection 4.11 is devoted to the introduction and the study of a new class of mechanical systems, called by us *dual mechanical* (ρ, η)*-systems, generalized Hamilton mechanical* (ρ, η)*-systems, Hamilton mechanical* (ρ, η)*-systems and Cartan mechanical* (ρ, η)*-systems*. For dual mechanical systems we develop a theory of semisprays and sprays. For Hamilton mechanical systems we develop a Hamiltonian formalism. We determine and study the (ρ, η)-semispray associated to a regular Hamiltonian H and external force $\overset{*}{F}_e$, which are applied on the total space of the dual of a generalized Lie algebroid and we derive the equations of Hamilton-Jacobi type. One remarks that, if the morphisms used are identities, then similar results can be obtained by classical results, but not classical ones.

The classical Legendre's duality makes possible a natural connection between Lagrange and Hamilton spaces. It reveals new concepts and geometrical objects of Hamilton spaces that are dual to those which are similar in Lagrange spaces. The geometrical theory of Hamilton (Cartan) spaces was investigated from the Legendre duality point of view in the papers [36], [38], [20], [21].

In our paper, we propose a new point of view over the Legendre duality. We introduce and develop the notion of *(horizontal) Legendre* (ρ, η, h)*-equivalence* between an arbitrary vector bundle and its dual. For this new theory it was necessary to build the (ρ, η)*-tangent application* of the Legendre bundle morphism associated to a Lagrangian or a Hamiltonian.

We consider that this new theory can be used in the develop of the Poisson Geometry and Symplectic Geometry.

2 Preliminaries

In general, if $\mathcal{C}$ is a category, then we denoted by $|\mathcal{C}|$ the class of objects and we denoted by $\overrightarrow{\mathcal{C}}$ the class of arrows (morphisms). For any $A, B \in |\mathcal{C}|$, we denote by $\mathcal{C}(A, B)$ the morphisms set of A source and B target.

Let **Vect**, **Liealg**, **Mod**, **Man**, **B** and $\mathbf{B^v}$ be the category of real vector spaces, Lie algebras, modules, manifolds, fiber bundles and vector bundles respectively.

2.1 The category of Lie algebroids

We assume that $N \in |\mathbf{Man}|$ and let $[,]_{TN}$ be the usual Lie bracket such that

$$(\Gamma(TN, \tau_N, N), +, \cdot, [,]_{TN}) \in |\mathbf{LieAlg}|.$$

Definition 2.1.1 If $(F, \nu, N) \in |\mathbf{B^v}|$ such that there exists

$$(\rho, Id_N) \in \mathbf{B^v}((F, \nu, N), (TN, \tau_N, N))$$

and an operation

$$\begin{array}{ccc} \Gamma(F,\nu,N)\times\Gamma(F,\nu,N) & \xrightarrow{[,]_F} & \Gamma(F,\nu,N) \\ (u,v) & \longmapsto & [u,v]_F \end{array}$$

with the following properties:

LA_1. the equality holds good

$$[u,f\cdot v]_F = f\,[u,v]_F + \Gamma(\rho, Id_N)(u)\,f\cdot v,$$

for all $u,v\in\Gamma(F,\nu,N)$ and $f\in\mathcal{F}(N)$,

LA_2. the 4-tuple

$$(\Gamma(F,\nu,N),+,\cdot,[,]_F)$$

is a Lie $\mathcal{F}(N)$-algebra,

LA_3. the **Mod**-morphism $\Gamma(\rho, Id_N)$ is a **LieAlg**-morphism of

$$(\Gamma(F,\nu,N),+,\cdot,[,]_F)$$

source and

$$(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$$

target,

then we will say that *the triple*

$$((F,\nu,N),[,]_F,(\rho,Id_N)) \tag{2.1.1}$$

is a Lie algebroid.

The couple

$$([,]_F,(\rho,Id_N))$$

is called *Lie algebroid structure.*

Definition 2.1.2 We define the morphisms set of

$$((F,\nu,N),[,]_F,(\rho,Id_N))$$

source and

$$\left((F',\nu',N'),[,]_{F'},(\rho',Id_{N'})\right)$$

target as being the set

$$\{(\varphi,\varphi_0)\in\mathbf{B^v}((F,\nu,N),(F',\nu',N'))\}$$

such that the **Mod**-morphism $\Gamma(\varphi,\varphi_0)$ is a **LieAlg**-morphism of

$$(\Gamma(F,\nu,N),+,\cdot,[,]_F)$$

source and

$$\left(\Gamma(F',\nu',N'),+,\cdot,[,]_{F'}\right)$$

target.

Remark 2.1.1 Note that we can discuss about *the category of Lie algebroids.* This category is denoted by **LA**.

If

$$((F,\nu,N),[,]_F,(\rho,Id_N))$$

is a Lie algebroid, then we assume that (F,ν,N) is a vector bundle with type fibre the real vector space $(\mathbb{R}^p,+,\cdot)$ and structure group a Lie subgroup of $(\mathbf{GL}(p,\mathbb{R}),\cdot)$.

We take $(\varkappa^{\tilde{\imath}},z^\alpha)$ as canonical local coordinates on (F,ν,N), where $\tilde{\imath}\in\overline{1,n}$, $\alpha\in\overline{1,p}$.

Consider

$$(\varkappa^{\tilde{\imath}},z^\alpha)\longrightarrow\left(\varkappa^{\tilde{\imath}},z^{\acute{\alpha}}\right)$$

a change of coordinates on (F,ν,N). Then the coordinates z^α change to $z^{\acute{\alpha}}$ by the rule:

$$z^{\acute{\alpha}}=\Lambda_{\alpha}^{\acute{\alpha}}z^\alpha. \tag{2.1.2}$$

The coefficients $\rho_\alpha^{\tilde{\imath}}$ change to $\rho_{\acute{\alpha}}^{\tilde{\imath}}$ by the rule:

$$\rho_{\acute{\alpha}}^{\tilde{\imath}}=\Lambda_{\acute{\alpha}}^{\alpha}\rho_\alpha^{\tilde{\imath}}\frac{\partial\varkappa^{\tilde{\imath}}}{\partial\varkappa^{\tilde{\imath}}}, \tag{2.1.3}$$

where

$$\|\Lambda_{\acute{\alpha}}^{\alpha}\|=\left\|\Lambda_{\alpha}^{\acute{\alpha}}\right\|^{-1}.$$

Locally, we obtain

$$[t_\alpha,t_\beta]_F\overset{put}{=}L_{\alpha\beta}^{\gamma}t_\gamma. \tag{2.1.4}$$

The real local functions

$$L_{\alpha\beta}^{\gamma},\ \alpha,\beta,\gamma\in\overline{1,p}$$

will be called *structure functions of the Lie algebroid*

$$((F,\nu,N),[,]_F,(\rho,Id_N)).$$

It is easy to prove that

$$L_{\alpha\beta}^{\gamma}=-L_{\beta\alpha}^{\gamma},\ \forall\alpha,\beta,\gamma\in\overline{1,p}.$$

2.2 The pull-back Lie algebroid of a Lie algebroid

We consider the following diagram:

$$\begin{array}{ccc} & & (F,[,]_F,(\rho,Id_N)) \\ & & \downarrow\nu \\ E & \xrightarrow{\quad\pi\quad} & N \end{array} \tag{2.2.1}$$

where (E,π,M) is a fiber bundle and $((F,\nu,N),[,]_F,(\rho,Id_N))$ is a Lie algebroid.

We assume that (E,π,M) has the type fibre a manifold of dimension r and structure group a Lie group $(\mathbf{G},\cdot)$.

Proposition 2.2.1 *Using the tangent* $\mathbf{B^v}$*-morphism* $(T\pi,\pi)$ *of* (TE,τ_E,E) *source and* (TN,τ_N,N) *target, we obtain that*

$$\frac{\partial f\circ\pi}{\partial x^{\tilde{\imath}}}=\frac{\partial f}{\partial x^{\tilde{\imath}}}\circ\pi,\ \forall f\in\mathcal{F}(N) \tag{2.2.2}$$

and

$$\frac{\partial f\circ\pi}{\partial y^a}=0,\ \forall f\in\mathcal{F}(N)\,. \tag{2.2.3}$$

Let $\mathcal{AF}_F$ be a representative of vector fibred $(n+p)$-structure for the vector bundle (F,ν,N) and let $\mathcal{AF}_E$ be a representative of fibred $(n+r)$-structure for the fiber bundle (E,π,N). Let $(\pi^*F,\pi^*\nu,E)$ be the pull-back vector bundle through π.

If $(U,\xi_U)\in\mathcal{AF}_E$ and $(V,s_V)\in\mathcal{AF}_F$ such that $U\cap V\neq\phi$, then we define the application

$$\begin{array}{ccc}\pi^*\nu^{-1}(\pi^{-1}(U\cap V)) & \xrightarrow{\tilde{s}_{\pi^{-1}(U\cap V)}} & \pi^{-1}(U\cap V)\times\mathbb{R}^p\\ \left(u,\tilde{Z}(u)\right) & \longmapsto & \left(\varkappa,t_{V,\pi(u)}^{-1}\tilde{Z}(u)\right).\end{array}$$

Proposition 2.2.2 *The set*

$$\widetilde{\mathcal{AF}}_{\pi^*F}\overset{put}{=}\bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_E,\ (V,s_V)\in\mathcal{AF}_F\\ U\cap V\neq\phi}}\left\{\left(\pi^{-1}(U\cap V),\tilde{s}_{\pi^{-1}(U\cap V)}\right)\right\}$$

is a vector fibred $(m+r)+p$-atlas for the vector bundle $(\pi^*F,\pi^*\nu,E)$.

If

$$z=z^\alpha t_\alpha\in\Gamma(F,\nu,N)\,,$$

then, using the vector fibred $(m+r)+p$-structure $\left[\widetilde{\mathcal{AF}}_{\pi^*F}\right]$, we obtain the section

$$\tilde{Z}=(z^\alpha\circ\pi)\,\tilde{T}_\alpha\in\Gamma(\pi^*F,\pi^*\nu,E)$$

such that

$$\tilde{Z}(u_x)=z(x)\,,$$

for any $u_x\in\pi^{-1}(U\cap V)$.

The set $\left\{\tilde{T}_\alpha,\ \alpha\in\overline{1,p}\right\}$ is a base for the module of sections

$$(\Gamma(\pi^*F,\pi^*\nu,E),+,\cdot)\,.$$

Let $\left(\overset{\pi^*F}{\rho},Id_E\right)$ be the $\mathbf{B^v}$-morphism of

$$(\pi^*F,\pi^*\nu,E)$$

source and

$$(TE,\tau_E,E)$$

target, where

$$\begin{array}{ccc}\pi^*F & \xrightarrow{\overset{\pi^*F}{\rho}} & TE\\ \tilde{Z}^\alpha\tilde{T}_\alpha(u_x) & \longmapsto & \left(\tilde{Z}^\alpha\cdot\rho_\alpha^{\tilde{\imath}}\circ\pi\frac{\partial}{\partial x^{\tilde{\imath}}}\right)(u_x)\end{array} \tag{2.2.4}$$

We consider the operation

$$\Gamma(\pi^*F,\pi^*\nu,E)\times\Gamma(\pi^*F,\pi^*\nu,E)\xrightarrow{[,]_{\pi^*F}}\Gamma(\pi^*F,\pi^*\nu,E)$$

defined by

$$
(2.2.5)\qquad
\begin{aligned}
\left[\tilde{T}_\alpha, \tilde{T}_\beta\right]_{\pi^*F} &= \left(L_{\alpha\beta}^\gamma \circ \pi\right) \tilde{T}_\gamma,\\
\left[\tilde{T}_\alpha, f\tilde{T}_\beta\right]_{\pi^*F} &= f\left(L_{\alpha\beta}^\gamma \circ \pi\right) \tilde{T}_\gamma + \left(\rho_\alpha^{\tilde{i}} \circ \pi\right) \frac{\partial f}{\partial x^{\tilde{i}}} \tilde{T}_\beta,\\
\left[f\tilde{T}_\alpha, \tilde{T}_\beta\right]_{\pi^*F} &= -\left[\tilde{T}_\beta, f\tilde{T}_\alpha\right]_{\pi^*F},
\end{aligned}
$$

for any $f \in \mathcal{F}(E)$.

Lemma 2.2.1 *The following equality holds good*

$$
\left[\tilde{U}, f\tilde{V}\right]_{\pi^*F} = f\left[\tilde{U}, \tilde{V}\right]_{\pi^*F} + \Gamma\left(\overset{\pi^*F}{\rho}, Id_E\right)\left(\tilde{U}\right) f \cdot \tilde{V},
$$

for any $\tilde{U}, \tilde{V} \in \Gamma(\pi^*F, \pi^*\nu, E)$ *and for any* $f \in \mathcal{F}(E)$.

Proof. We observe that for any $\alpha, \beta \in \overline{1,p}$, we obtain

$$
\left[\tilde{T}_\alpha, f\tilde{T}_\beta\right]_{\pi^*F} = f\left[\tilde{T}_\alpha, \tilde{T}_\beta\right]_{\pi^*F} + \Gamma\left(\overset{h^*F}{\rho}, Id_E\right) \tilde{T}_\alpha(f),\ \forall f \in \mathcal{F}(E).
$$

Using this equality and the definition of the operation $[,]_{\pi^*F}$ it results the conclusion of the lemma. *q.e.d.*

Lemma 2.2.2 *The* $\mathcal{F}(E)$*-algebra*

$$
\left(\Gamma(\pi^*F, \pi^*\nu, E), +, \cdot, [,]_{\pi^*F}\right)
$$

is a Lie $\mathcal{F}(E)$*-algebra.*

Proof. Using the definition of the operation $[,]_{\pi^*F}$ it results that

$$
\left[\tilde{U}, \tilde{V}\right]_{\pi^*F} = -\left[\tilde{V}, \tilde{U}\right]_{\pi^*F},
$$

for any $\tilde{U}, \tilde{V} \in \Gamma(\pi^*F, \pi^*\nu, E)$. Therefore, we obtain

$$
(1)\qquad \left[\tilde{U}, \tilde{U}\right]_{\pi^*F} = 0,\ \forall \tilde{U} \in \Gamma(\pi^*F, \pi^*\nu, E).
$$

Since

$$
\left(\Gamma(F, \nu, N), +, \cdot, [,]_F\right)
$$

is a Lie $\mathcal{F}(N)$-algebra, we obtain the equality:

$$
\sum_{cyclic(\alpha,\beta,\gamma)} \left(L_{\beta\gamma}^\varepsilon L_{\alpha\varepsilon}^\delta + \rho_\alpha^{\tilde{i}} \frac{\partial L_{\beta\gamma}^\delta}{\partial \varkappa^{\tilde{i}}}\right) = 0.
$$

Using (2.2.2), we obtain the equality:

$$
\sum_{cyclic(\alpha,\beta,\gamma)} \left(\left(L_{\beta\gamma}^\varepsilon \circ \pi\right)\left(L_{\alpha\varepsilon}^\delta \circ \pi\right) + \rho_\alpha^{\tilde{i}} \circ \pi \frac{\partial\left(L_{\beta\gamma}^\delta \circ \pi\right)}{\partial \varkappa^{\tilde{i}}}\right) = 0.
$$

and multiplying with $\tilde{T}_\delta$, we obtain

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(\left(L_{\beta\gamma}^{\varepsilon}\circ\pi\right)\left(L_{\alpha\varepsilon}^{\delta}\circ\pi\right)\tilde{T}_\delta+\rho_\alpha^{\tilde{\imath}}\circ\pi\frac{\partial\left(L_{\beta\gamma}^{\delta}\circ\pi\right)}{\partial\varkappa^{\tilde{\imath}}}\tilde{T}_\delta\right)=0.$$

which is equivalent with

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(\left(L_{\beta\gamma}^{\varepsilon}\circ\pi\right)\left[\tilde{T}_\alpha,\tilde{T}_\varepsilon\right]_{\pi^*F}+\rho_\alpha^{\tilde{\imath}}\circ\pi\frac{\partial L_{\beta\gamma}^{\delta}\circ\pi}{\partial\varkappa^{\tilde{\imath}}}\tilde{T}_\delta\right)=0.$$

Since this equality implies

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left[\tilde{T}_\alpha,\left(L_{\beta\gamma}^{\varepsilon}\circ\pi\right)\tilde{T}_\varepsilon\right]_{\pi^*F}=0,$$

it results that the following Jacobi identity is satisfied

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left[\tilde{T}_\alpha,\left[\tilde{T}_\beta,\tilde{T}_\gamma\right]_{\pi^*F}\right]_{\pi^*F}=0.$$

In general, for any $\tilde{U},\tilde{V},\tilde{Z}\in\Gamma\left(\pi^*F,\pi^*\nu,E\right)$, we obtain the Jacobi identity:

$$\left[\tilde{U},\left[\tilde{V},\tilde{Z}\right]_{\pi^*F}\right]_{\pi^*F}+\left[\tilde{Z},\left[\tilde{U},\tilde{V}\right]_{\pi^*F}\right]_{\pi^*F}+\left[\tilde{V},\left[\tilde{Z},\tilde{U}\right]_{\pi^*F}\right]_{\pi^*F}=0. \tag{2}$$

Using the affirmations (1) and (2) it results the conclusion of the lemma. *q.e.d.*

Lemma 2.2.3 *The* **Mod**-*morphism*

$$\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)$$

is a **Liealg**-*morphism of*

$$\left(\Gamma\left(\pi^*F,\pi^*\nu,E\right),+,\cdot,[,]_{\pi^*F}\right)$$

source and

$$\left(\Gamma(TE,\tau_E,E),+,\cdot,[,]_{TE}\right)$$

target.

Proof. As the **Mod**-morphism $\Gamma(\rho,Id_N)$ is a **Liealg**-morphism of

$$\left(\Gamma\left(F,\nu,N\right),+,\cdot,[,]_F\right)$$

source and

$$\left(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN}\right)$$

target, then we obtain

$$L_{\alpha\beta}^{\gamma}\rho_\gamma^{\tilde{k}}=\rho_\alpha^{\tilde{\imath}}\frac{\partial\left(\rho_\beta^{\tilde{k}}\right)}{\partial\varkappa^{\tilde{\imath}}}-\rho_\beta^{\tilde{\jmath}}\frac{\partial\left(\rho_\alpha^{\tilde{k}}\right)}{\partial\varkappa^{\tilde{\jmath}}}$$

Using relations (2.2.2), we obtain:

$$\left(L_{\alpha\beta}^{\gamma}\circ\pi\right)\left(\rho_\gamma^{\tilde{k}}\circ\pi\right)=\rho_\alpha^{\tilde{\imath}}\circ\pi\frac{\partial\left(\rho_\beta^{\tilde{k}}\circ\pi\right)}{\partial\varkappa^{\tilde{\imath}}}-\rho_\beta^{\tilde{\jmath}}\circ\pi\frac{\partial\left(\rho_\alpha^{\tilde{k}}\circ\pi\right)}{\partial\varkappa^{\tilde{\jmath}}}$$

Multiplying with $\frac{\partial}{\partial\varkappa^{\tilde{k}}}$, we obtain the equality

$$\left(L_{\alpha\beta}^{\gamma}\circ\pi\right)\left(\rho_{\gamma}^{\tilde{k}}\circ\pi\right)\frac{\partial}{\partial\varkappa^{\tilde{k}}}=\rho_{\alpha}^{\tilde{i}}\circ\pi\frac{\partial\left(\rho_{\beta}^{\tilde{k}}\circ\pi\right)}{\partial\varkappa^{\tilde{i}}}\frac{\partial}{\partial\varkappa^{\tilde{k}}}-\rho_{\beta}^{\tilde{j}}\circ\pi\frac{\partial\left(\rho_{\alpha}^{\tilde{k}}\circ\pi\right)}{\partial\varkappa^{\tilde{j}}}\frac{\partial}{\partial\varkappa^{\tilde{k}}}$$

which is equivalent with the equality

$$\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\left[\tilde{T}_\alpha,\tilde{T}_\beta\right]_{\pi^*F}=\left[\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\tilde{T}_\alpha,\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\tilde{T}_\beta\right]_{TN}$$

for any base sections $\tilde{T}_\alpha,\tilde{T}_\beta$.

In general, we obtain the equality

$$\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\left[\tilde{U},\tilde{V}\right]_{\pi^*F}=\left[\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\tilde{U},\Gamma\left(\overset{\pi^*F}{\rho},Id_E\right)\tilde{V}\right]_{TN},$$

for any $\tilde{U},\tilde{V}\in\Gamma\left(h^*F,h^*\nu,M\right)$. *q.e.d.*

Using *Lemmas 2.2.1, 2.2.2* and *2.2.3*, we obtain the following

Theorem 2.2.1 *The couple*

$$\left([,]_{\pi^*F},\left(\overset{\pi^*F}{\rho},Id_E\right)\right)$$

is a Lie algebroid structure for the vector bundle $\left(\pi^*F,\pi^*\nu,E\right)$.

This Lie algebroid will be called *the pull-back Lie algebroid of the Lie algebroid*

$$\left(\left(F,\nu,N\right),[,]_F,\left(\rho,Id_N\right)\right).$$

3 Generalized Lie algebroids, exterior differential calculus and (linear) connections

3.1 The category of generalized Lie algebroids

We assume that $N\in|\mathbf{Man}|$ and let $[,]_{TN}$ be the usual Lie bracket such that

$$\left(\Gamma\left(TN,\tau_N,N\right),+,\cdot,[,]_{TN}\right)\in|\mathbf{LieAlg}|.$$

Let $h\in\mathbf{Man}\left(M,N\right)$ be a surjective application.

Definition 3.1.1 If $\left(F,\nu,N\right)\in|\mathbf{B^v}|$ such that there exists

$$\left(\rho,\eta\right)\in\mathbf{B^v}\left(\left(F,\nu,N\right),\left(TM,\tau_M,M\right)\right)$$

and an operation

$$\begin{array}{ccc}\Gamma\left(F,\nu,N\right)\times\Gamma\left(F,\nu,N\right) & \overset{[,]_{F,h}}{\longrightarrow} & \Gamma\left(F,\nu,N\right)\\ \left(u,v\right) & \longmapsto & [u,v]_{F,h}\end{array}$$

with the following properties:

GLA_1. the equality holds good

$$[u, f \cdot v]_{F,h} = f\,[u, v]_{F,h} + \Gamma\left(Th \circ \rho, h \circ \eta\right)(u)\, f \cdot v,$$

for all $u, v \in \Gamma\left(F, \nu, N\right)$ and $f \in \mathcal{F}\left(N\right)$.

GLA_2. the 4-tuple

$$\left(\Gamma\left(F, \nu, N\right), +, \cdot, [,]_{F,h}\right)$$

is a Lie $\mathcal{F}\left(N\right)$-algebra,

GLA_3. the **Mod**-morphism $\Gamma\left(Th \circ \rho, h \circ \eta\right)$ is a **LieAlg**-morphism of

$$\left(\Gamma\left(F, \nu, N\right), +, \cdot, [,]_{F,h}\right)$$

source and

$$\left(\Gamma\left(TN, \tau_N, N\right), +, \cdot, [,]_{TN}\right)$$

target,

then we will say that *the triple*

$$\left(\left(F, \nu, N\right), [,]_{F,h}, (\rho, \eta)\right) \tag{3.1.1}$$

is a generalized Lie algebroid.

The couple

$$\left([,]_{F,h}, (\rho, \eta)\right)$$

will be called *generalized Lie algebroid structure.*

Definition 3.1.2 We define the morphisms set of

$$\left(\left(F, \nu, N\right), [,]_{F,h}, (\rho, \eta)\right)$$

source and

$$\left(\left(F', \nu', N'\right), [,]_{F',h'}, (\rho', \eta')\right)$$

target as being the set

$$\left\{(\varphi, \varphi_0) \in \mathbf{B^v}\left(\left(F, \nu, N\right), \left(F', \nu', N'\right)\right)\right\}$$

such that the **Mod**-morphism $\Gamma\left(\varphi, \varphi_0\right)$ is a **LieAlg**-morphism of

$$\left(\Gamma\left(F, \nu, N\right), +, \cdot, [,]_{F,h}\right)$$

source and

$$\left(\Gamma\left(F', \nu', N'\right), +, \cdot, [,]_{F',h'}\right)$$

target.

Remark 3.1.1 Note that we discuss about *the category of generalized Lie algebroids.* This category will be denoted by **GLA**.

In the following we will build some examples of generalized Lie algebroids.

We assume that $((F,\nu,N),[,]_F,(\rho,Id_N))$ is a Lie algebroid and let $h\in\mathbf{Man}(N,N)$ be a surjective application.

Let $\mathcal{AF}_F$ be a representative of vector fibred $(n+p)$-structure for the vector bundle (F,ν,N) and let $\mathcal{AF}_{TN}$ be a representative of vector fibred $(n+n)$-structure for the vector bundle (TN,τ_N,N).

If $(U,\xi_U)\in\mathcal{AF}_{TN}$ and $(V,s_V)\in\mathcal{AF}_F$ such that $U\cap h^{-1}(V)\neq\phi$, then we define the application

$$\begin{array}{ccc}\tau_N^{-1}(U\cap h^{-1}(V))) & \xrightarrow{\bar{\xi}_{U\cap h^{-1}(V)}} & \left(U\cap h^{-1}(V)\right)\times\mathbb{R}^n\\ (\varkappa,u(\varkappa)) & \longmapsto & \left(\varkappa,\xi_{U,\varkappa}^{-1}u(\varkappa)\right).\end{array}$$

Proposition 3.1.1 *The set*

$$\overline{\mathcal{AF}}_{TN}\overset{put}{=}\bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_{TN},\ (V,s_V)\in\mathcal{AF}_F\\ U\cap h^{-1}(V)\neq\phi}}\left\{\left(U\cap h^{-1}(V),\bar{\xi}_{U\cap h^{-1}(V)}\right)\right\}$$

is a vector fibred $n+n$-atlas for the vector bundle (TN,τ_N,N).

If

$$X=X^{\tilde{\imath}}\tfrac{\partial}{\partial\varkappa^{\tilde{\imath}}}\in\Gamma(TN,\tau_N,N)$$

then, using the vector fibred $n+n$-structure $\left[\overline{\mathcal{AF}}_{TN}\right]$, we obtain the section

$$\bar{X}=\bar{X}^{\tilde{\imath}}\circ h\tfrac{\partial}{\partial\bar{\varkappa}^{\tilde{\imath}}}\in\Gamma(TN,\tau_N,N),$$

such that

$$\bar{X}(\bar{\varkappa})=X(h(\bar{\varkappa})),$$

for any $\bar{\varkappa}\in U\cap h^{-1}(V)$.

The set $\left\{\frac{\partial}{\partial\bar{\varkappa}^{\tilde{\imath}}},\ \tilde{\imath}\in\overline{1,n}\right\}$ is a base for the $\mathcal{F}(N)$-module $(\Gamma(TN,\tau_N,N),+,\cdot)$.

We consider the operation

$$\Gamma(F,\nu,N)\times\Gamma(F,\nu,N)\xrightarrow{[,]_{F,h}}\Gamma(F,\nu,N)$$

defined by

$$\begin{aligned}[][t_\alpha,t_\beta]_{F,h}&=\left(L_{\alpha\beta}^\gamma\circ h\right)t_\gamma,\\ [t_\alpha,ft_\beta]_{F,h}&=f\left(L_{\alpha\beta}^\gamma\circ h\right)t_\gamma+\rho_\alpha^{\tilde{\imath}}\circ h\frac{\partial f}{\partial\bar{\varkappa}^{\tilde{\imath}}}t_\beta,\\ [ft_\alpha,t_\beta]_{F,h}&=-[t_\beta,ft_\alpha]_{F,h},\end{aligned}$$

for any $f\in\mathcal{F}(N)$.

Lemma 3.1.1 *The following equality holds good*

$$[z,fv]_{F,h}=f[z,v]_{F,h}+\Gamma(Th\circ\rho,h)(z)f\cdot v,$$

for any $z,v\in\Gamma(F,\nu,N)$ and $f\in\mathcal{F}(N)$.

Proof. We obtain easily that

$$[t_\alpha, ft_\beta]_{F,h} = [t_\alpha, t_\beta]_{F,h} + \Gamma(Th\circ\rho, h)(t_\alpha) f\cdot t_\beta$$

for any $\forall f \in \mathcal{F}(N)$.

Using this equality and the definition of the operation $[,]_{F,h}$ it results the conclusion of the lemma. *q.e.d.*

Lemma 3.1.2 *The $\mathcal{F}(N)$-algebra*

$$\left(\Gamma(F,\nu,N), +, \cdot, [,]_{F,h}\right)$$

is a Lie $\mathcal{F}(N)$-algebra.

Proof. Using the definition of the operation $[,]_{F,h}$ it results that

$$[u,v]_{F,h} = -[v,u]_{F,h},$$

for any $u, v \in \Gamma(F,\nu,N)$. Therefore, we obtain

$$[u,u]_{F,h} = 0,\ \forall u \in \Gamma(F,\nu,N). \tag{1}$$

Since $(\Gamma(F,\nu,N), +, \cdot, [,]_F)$ is a Lie $\mathcal{F}(N)$-algebra, it results that

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(L_{\beta\gamma}^{\varepsilon} L_{\alpha\varepsilon}^{\delta} + \rho_\alpha^{\tilde{\imath}} \frac{\partial L_{\beta\gamma}^{\delta}}{\partial \varkappa^{\tilde{\imath}}}\right) = 0.$$

Therefore,

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(\left(L_{\beta\gamma}^{\varepsilon}\circ h\right)\left(L_{\alpha\varepsilon}^{\delta}\circ h\right) + \rho_\alpha^{\tilde{\imath}}\circ h \frac{\partial\left(L_{\beta\gamma}^{\delta}\circ h\right)}{\partial \bar{\varkappa}^{\tilde{\imath}}}\right) = 0.$$

Multiplying with t_δ, we obtain the equality

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(\left(L_{\beta\gamma}^{\varepsilon}\circ h\right)\left(L_{\alpha\varepsilon}^{\delta}\circ h\right) t_\delta + \rho_\alpha^{\tilde{\imath}}\circ h \frac{\partial\left(L_{\beta\gamma}^{\delta}\circ h\right)}{\partial \bar{\varkappa}^{\tilde{\imath}}} t_\delta\right) = 0$$

which is equivalent with the following equality:

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(\left(L_{\beta\gamma}^{\varepsilon}\circ h\right)[t_\alpha, t_\varepsilon]_{F,h} + \rho_\alpha^{\tilde{\imath}}\circ h \frac{\partial\left(L_{\beta\gamma}^{\delta}\circ h\right)}{\partial \bar{\varkappa}^{\tilde{\imath}}} t_\delta\right) = 0.$$

Therefore, we obtain the Jacobi identity

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left[t_\alpha, [t_\beta, t_\gamma]_{F,h}\right]_{F,h} = 0.$$

For any $u, v, w \in \Gamma(F,\nu,N)$, we obtain the Jacobi identity

$$\left[u, [v,w]_{F,h}\right]_{F,h} + \left[v, [w,u]_{F,h}\right]_{F,h} + \left[w, [u,v]_{F,h}\right]_{F,h} = 0, \tag{2}$$

Using (1) and (2) it results the conclusion of lemma. *q.e.d.*

Lemma 3.1.3 *The* **Mod**-*morphism* $\Gamma(Th\circ\rho,h)$ *is a* **Liealg**-*morphism of*

$$\left(\Gamma(F,\nu,N),+,\cdot,[,]_{F,h}\right)$$

source and

$$(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$$

target.

Proof. As the **Mod**-morphism $\Gamma(\rho,Id_N)$ is a **LieAlg**-morphisms of

$$(\Gamma(F,\nu,N),+,\cdot,[,]_F)$$

source and

$$(\Gamma(TN,\tau_N,N),+,\cdot,[,]_{TN})$$

target, then we obtain

$$L_{\alpha\beta}^{\gamma}\rho_{\gamma}^{\tilde{k}}=\rho_{\alpha}^{\tilde{\imath}}\frac{\partial\rho_{\beta}^{\tilde{k}}}{\partial\varkappa^{\tilde{\imath}}}-\rho_{\beta}^{\tilde{j}}\frac{\partial\rho_{\alpha}^{\tilde{k}}}{\partial\varkappa^{\tilde{j}}}.$$

Therefore, we obtain

$$\left(L_{\alpha\beta}^{\gamma}\circ h\right)\left(\rho_{\gamma}^{\tilde{k}}\circ h\right)=\rho_{\alpha}^{\tilde{\imath}}\circ h\frac{\partial\rho_{\beta}^{\tilde{k}}\circ h}{\partial\bar{\varkappa}^{\tilde{\imath}}}-\rho_{\beta}^{\tilde{j}}\circ h\frac{\partial\rho_{\alpha}^{\tilde{k}}\circ h}{\partial\bar{\varkappa}^{\tilde{j}}}.$$

Moreover, we obtain

$$\left(L_{\alpha\beta}^{\gamma}\circ h\right)\left(\rho_{\gamma}^{\tilde{k}}\circ h\right)\frac{\partial}{\partial\bar{\varkappa}^{\tilde{k}}}=\rho_{\alpha}^{\tilde{\imath}}\circ h\frac{\partial\rho_{\beta}^{\tilde{k}}\circ h}{\partial\bar{\varkappa}^{\tilde{\imath}}}\frac{\partial}{\partial\bar{\varkappa}^{\tilde{k}}}-\rho_{\beta}^{\tilde{j}}\circ h\frac{\partial\rho_{\alpha}^{\tilde{k}}\circ h}{\partial\bar{\varkappa}^{\tilde{j}}}\frac{\partial}{\partial\bar{\varkappa}^{\tilde{k}}}.$$

After some calculations, we obtain that

$$\Gamma(Th\circ\rho,h)[t_\alpha,t_\beta]_{F,h}=[\Gamma(Th\circ\rho,h)t_\alpha,\Gamma(Th\circ\rho,h)t_\beta]_{TN}.$$

We obtain easily that

$$\Gamma(Th\circ\rho,h)[u,v]_{F,h}=[\Gamma(Th\circ\rho,h)u,\Gamma(Th\circ\rho,h)v]_{TN},$$

for any $u,v\in\Gamma(F,\nu,N)$. *q.e.d.*

Using *Lemmas 3.1.1, 3.1.2* and *3.1.3* we obtain the following

Theorem 3.1.1 (example of generalized Lie algebroid) *The couple*

$$\left([,]_{F,h},(\rho,Id_N)\right)$$

is a generalized Lie algebroid structure for the vector bundle (F,ν,N).

Definition 3.1.3 The generalized Lie algebroid

$$\left((F,\nu,N),[,]_{F,h},(\rho,Id_N)\right)$$

given by the previous theorem, will be called *the generalized Lie algebroid associated to the Lie algebroid*

$$((F,\nu,N),[,]_F,(\rho,Id_N))$$

and to the surjective application

$$h \in \mathbf{Man}(N, N).$$

In particular, if $h = Id_N$, then the generalized Lie algebroid

$$\left((F, \nu, N), [,]_{F, Id_N}, (\rho, Id_N)\right)$$

will be called *the generalized Lie algebroid associated to the Lie algebroid*

$$((F, \nu, N), [,]_F, (\rho, Id_N)).$$

Note that any Lie algebroid can be regarded as a generalized Lie algebroid.

Theorem 3.1.2 (example of generalized Lie algebroid) *Let $M \in |\mathbf{Man}_m|$ and $g, h \in Iso_{\mathbf{Man}}(M)$.*

Let $[,]_{TM}$ be the usual Lie bracket such that

$$(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM}) \in |\mathbf{LieAlg}|.$$

Using the tangent $\mathbf{B^v}$-morphism (Tg, g) and the operation

$$\begin{array}{ccc} \Gamma(TM, \tau_M, M) \times \Gamma(TM, \tau_M, M) & \xrightarrow{[,]_{TM,h}} & \Gamma(TM, \tau_M, M) \\ (u, v) & \longmapsto & [u, v]_{TM,h} \end{array}$$

where

$$[u, v]_{TM,h} = \Gamma\left(T(h \circ g)^{-1}, (h \circ g)^{-1}\right)([\Gamma(T(h \circ g), h \circ g) u, \Gamma(T(h \circ g), h \circ g) v]_{TM}),$$

for any $u, v \in \Gamma(TM, \tau_M, M)$, then we obtain that

$$\left((TM, \tau_M, M), [u, v]_{TM,h}, (Tg, g)\right)$$

is a generalized Lie algebroid.

Proof: As the operation $[,]_{TM,h}$ is biadditive, then we obtain that

$$\left(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM,h}\right) \in |\mathbf{Alg}|.$$

Using the definition of the operation $[,]_{TM,h}$ we obtain that

$$\Gamma(T(h \circ g), h \circ g)\left([u, v]_{TM,h}\right) = [\Gamma(T(h \circ g), h \circ g) u, \Gamma(T(h \circ g), h \circ g) v]_{TM}$$

for any $u, v \in \Gamma(TM, \tau_M, M)$.

1) Therefore, $\Gamma(T(h \circ g), h \circ g)$ is a **Alg**-morphism of

$$\left(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM,h}\right)$$

source and

$$(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM})$$

target.

For any $u, v \in \Gamma(TM, \tau_M, M)$ and $f \in \mathcal{F}(M)$ we obtain that

$$\begin{aligned}
[u, fv]_{TM,h} &= \Gamma\left(T(h\circ g)^{-1}, (h\circ g)^{-1}\right)\left([\Gamma(T(h\circ g), h\circ g)u, \Gamma(T(h\circ g), h\circ g)fv]_{TM}\right) \\
&= \Gamma\left(T(h\circ g)^{-1}, (h\circ g)^{-1}\right)\left(f\cdot[\Gamma(T(h\circ g), h\circ g)u, \Gamma(T(h\circ g), h\circ g)v]_{TM}\right) \\
&+\Gamma\left(T(h\circ g)^{-1}, (h\circ g)^{-1}\right)\left(\Gamma(T(h\circ g), h\circ g)u\right)(f)\cdot\Gamma(T(h\circ g), h\circ g)v \\
&= f\cdot\Gamma\left(T(h\circ g)^{-1}, (h\circ g)^{-1}\right)[\Gamma(T(h\circ g), h\circ g)u, \Gamma(T(h\circ g), h\circ g)v]_{TM} \\
&+\left(\Gamma(T(h\circ g), h\circ g)u\right)(f)\cdot v
\end{aligned}$$

2) Therefore, we obtain that

$$[u, fv]_{TM,h} = f\cdot[u, v]_{TM,h} + \left(\Gamma(T(h\circ g), h\circ g)u\right)(f)\cdot v$$

for any $u, v \in \Gamma(TM, \tau_M, M)$ and $f \in \mathcal{F}(M)$.

We remark that

$$\Gamma\left(T(h\circ g)^{-1}, (h\circ g)^{-1}\right)(0) = 0.$$

As

$$\left(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM}\right) \in |\mathbf{LieAlg}|$$

and

$$\begin{aligned}
[u,[v,z]_{TM,h}]_{TM,h} &= \Gamma\left(T(h\circ g)^{-1},(h\circ g)^{-1}\right)\left[\Gamma(T(h\circ g),h\circ g)u,\Gamma(T(h\circ g),h\circ g)[v,z]_{TM,h}\right]_{TM} \\
&= \Gamma\left(T(h\circ g)^{-1},(h\circ g)^{-1}\right)\left[\Gamma(T(h\circ g),h\circ g)u,[\Gamma(T(h\circ g),h\circ g)v,\Gamma(T(h\circ g),h\circ g)z]_{TM}\right]_{TM}
\end{aligned}$$

for any $u, v, z \in \Gamma(TM, \tau_M, M)$, it results that

$$[u, u]_{TM,h} = 0$$

for any $u \in \Gamma(TM, \tau_M, M)$ and

$$\left[u, [v, z]_{TM,h}\right]_{TM,h} + \left[z, [u, v]_{TM,h}\right]_{TM,h} + \left[v, [z, u]_{TM,h}\right]_{TM,h} = 0,$$

for any $u, v, z \in \Gamma(TM, \tau_M, M)$.

3) Therefore, we have that

$$\left(\Gamma(TM, \tau_M, M), +, \cdot, [,]_{TM,h}\right) \in |\mathbf{LieAlg}|.$$

Using the affirmations 1), 2) and 3) it results the conclusion of the theorem.

Remark 3.1.2 For any **Man**-isomorphisms g and h we obtain new and interesting generalized Lie algebroid structures for the tangent vector bundle (TM, τ_M, M).

For any base $\left\{t_\alpha,\ \alpha \in \overline{1, m}\right\}$ of the module of sections $(\Gamma(TM, \tau_M, M), +, \cdot)$ we obtain the structure functions

$$L_{\alpha\beta}^{\gamma} = \left(\theta_\alpha^i \frac{\partial \theta_\beta^j}{\partial x^i} - \theta_\beta^i \frac{\partial \theta_\alpha^j}{\partial x^i}\right)\tilde{\theta}_j^{\gamma},\ \alpha, \beta, \gamma \in \overline{1, m}$$

where

$$\theta_\alpha^i,\ i, \alpha \in \overline{1, m}$$

are real local functions such that

$$\Gamma\left(T\left(h\circ g\right),h\circ g\right)\left(t_{\alpha}\right)=\theta_{\alpha}^{i}\frac{\partial}{\partial x^{i}}$$

and

$$\tilde{\theta}_{j}^{\gamma},\ i,\gamma\in\overline{1,m}$$

are real local functions such that

$$\Gamma\left(T\left(h\circ g\right)^{-1},\left(h\circ g\right)^{-1}\right)\left(\frac{\partial}{\partial x^{j}}\right)=\tilde{\theta}_{j}^{\gamma}t_{\gamma}.$$

In particular, using arbitrary basis for the module of sections and arbitrary isometries (symmetries, translations, rotations,...) we obtain a lot of generalized Lie algebroid structures for the tangent vector bundle $(T\Sigma,\tau_{\Sigma},\Sigma)$ and we can study its geometry using our theory which is develop in the next.

3.1.1 Structure functions for generalized Lie algebroids

Let

$$\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$$

be a generalized Lie algebroid given by the diagram:

(3.1.1.1)
$$\begin{array}{ccc} & & \left(F,\left[,\right]_{F,h},\left(\rho,\eta\right)\right)\\ & & \downarrow\nu\\ M & \xrightarrow{\quad h\quad} & N\end{array}$$

We assume that (F,ν,N) is a vector bundle with type fibre the real vector space $(\mathbb{R}^{p},+,\cdot)$ and structure group a Lie subgroup of $(\mathbf{GL}\left(p,\mathbb{R}\right),\cdot)$.

We take $\left(x^{i},y^{i}\right)$ as canonical local coordinates on (TM,τ_{M},M), where $i\in\overline{1,m}$.

Consider

$$\left(x^{i},y^{i}\right)\longrightarrow\left(x^{i'}\left(x^{i}\right),y^{i'}\left(x^{i},y^{i}\right)\right)$$

a change of coordinates on (TM,τ_{M},M). Then the coordinates y^{i} change to $y^{i'}$ by the rule:

(3.1.1.2)
$$y^{i'}=\frac{\partial x^{i'}}{\partial x^{i}}y^{i}.$$

We take $(\varkappa^{\tilde{i}},z^{\alpha})$ as canonical local coordinates on (F,ν,N), where $\tilde{i}\in\overline{1,n}$, $\alpha\in\overline{1,p}$.

Consider

$$\left(\varkappa^{\tilde{i}},z^{\alpha}\right)\longrightarrow\left(\varkappa^{\tilde{i}},z^{\alpha'}\right)$$

a change of coordinates on (F,ν,N). Then the coordinates z^{α} change to $z^{\alpha'}$ by the rule:

(3.1.3)
$$z^{\alpha'}=\Lambda_{\alpha}^{\alpha'}z^{\alpha}.$$

We assume that $(\theta,\mu)\overset{put}{=}(Th\circ\rho,h\circ\eta)$.

If $z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right)$ is arbitrary, then

(3.1.1.4)
$$\begin{array}{c}\Gamma\left(Th\circ\rho,h\circ\eta\right)\left(z^{\alpha}t_{\alpha}\right)f\left(h\circ\eta\left(\varkappa\right)\right)=\\ =\left(\theta_{\alpha}^{\tilde{i}}z^{\alpha}\frac{\partial f}{\partial\varkappa^{\tilde{i}}}\right)\left(h\circ\eta\left(\varkappa\right)\right)=\left(\left(\rho_{\alpha}^{i}\circ h\right)\left(z^{\alpha}\circ h\right)\frac{\partial f\circ h}{\partial x^{i}}\right)\left(\eta\left(\varkappa\right)\right),\end{array}$$

for any $f \in \mathcal{F}(N)$ and $\varkappa \in N$.

The coefficients ρ_α^i respectively $\theta_\alpha^{\tilde{i}}$ change to $\rho_{\acute{\alpha}}^{\acute{i}}$ respectively $\theta_{\acute{\alpha}}^{\tilde{\acute{i}}}$ by the rule:

$$\rho_{\acute{\alpha}}^{\acute{i}} = \Lambda_{\acute{\alpha}}^{\alpha} \rho_\alpha^i \frac{\partial x^{\acute{i}}}{\partial x^i}, \tag{3.1.1.5}$$

respectively

$$\theta_{\acute{\alpha}}^{\tilde{\acute{i}}} = \Lambda_{\acute{\alpha}}^{\alpha} \theta_\alpha^{\tilde{i}} \frac{\partial \varkappa^{\tilde{\acute{i}}}}{\partial \varkappa^{\tilde{i}}}, \tag{3.1.1.6}$$

where

$$\left\| \Lambda_{\acute{\alpha}}^{\alpha} \right\| = \left\| \Lambda_{\alpha}^{\acute{\alpha}} \right\|^{-1}.$$

Locally, we set

$$[t_\alpha, t_\beta]_F \overset{put}{=} L_{\alpha\beta}^\gamma t_\gamma. \tag{3.1.1.7}$$

We easily obtain that

$$L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma, \ \forall \alpha, \beta, \gamma \in \overline{1,p}.$$

The real local functions

$$\left\{ L_{\alpha\beta}^\gamma, \ \alpha, \beta, \gamma \in \overline{1,p} \right\}$$

will be called the *structure functions of the generalized Lie algebroid*

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta) \right).$$

Theorem 3.1.1.1 *The following equalities hold good:*

$$\rho_\alpha^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left(\theta_\alpha^{\tilde{i}} \frac{\partial f}{\partial \varkappa^{\tilde{i}}} \right) \circ h, \forall f \in \mathcal{F}(N). \tag{3.1.1.8}$$

and

$$\left(L_{\alpha\beta}^\gamma \circ h \right) \left(\rho_\gamma^k \circ h \right) = \left(\rho_\alpha^i \circ h \right) \frac{\partial \left(\rho_\beta^k \circ h \right)}{\partial x^i} - \left(\rho_\beta^j \circ h \right) \frac{\partial \left(\rho_\alpha^k \circ h \right)}{\partial x^j}. \tag{3.1.1.9}$$

Proof. Using the relation (3.1.1.4), we obtain the equality (3.1.1.8). Since

$$\begin{array}{l} \Gamma(Th \circ \rho, h \circ \eta) [t_\alpha, t_\beta]_F (f) \\ = [\Gamma((Th, h) \circ (\rho, \eta)) t_\alpha, \Gamma((Th, h) \circ (\rho, \eta)) t_\beta]_F (f) \\ = \Gamma(Th, h) \left([\Gamma(\rho, \eta) t_\alpha, \Gamma(\rho, \eta) t_\beta]_{TM} \right)(f), \ \forall f \in \mathcal{F}(N), \end{array}$$

it results that

$$\left(L_{\alpha\beta}^\gamma \circ h \right) \left(\rho_\gamma^k \circ h \right) \tfrac{\partial f \circ h}{\partial x^k} = \left(\left(\rho_\alpha^i \circ h \right) \tfrac{\partial \left(\rho_\beta^k \circ h \right)}{\partial x^i} - \left(\rho_\beta^j \circ h \right) \tfrac{\partial \left(\rho_\alpha^k \circ h \right)}{\partial x^j} \right) \tfrac{\partial f \circ h}{\partial x^k},$$

for any $f \in \mathcal{F}(N)$. *q.e.d.*

3.1.2 The pull-back Lie algebroid of a generalized Lie algebroid

Let

$$\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$$

be a generalized Lie algebroid given by the diagram (3.1.1.1).

Let $\mathcal{AF}_F$ be a representative of vector fibred $(n+p)$-structure for the vector bundle (F,ν,N) and let $\mathcal{AF}_{TM}$ be a representative of vector fibred $(m+m)$-structure for the vector bundle (TM,τ_M,M).

Let $(h^*F,h^*\nu,M)$ be the pull-back vector bundle through h.

If $(U,\xi_U)\in\mathcal{AF}_{TM}$ and $(V,s_V)\in\mathcal{AF}_F$ such that $U\cap h^{-1}(V)\neq\phi$, then we define the application

$$\begin{array}{ccc} h^*\nu^{-1}(U\cap h^{-1}(V))) & \xrightarrow{\bar{s}_{U\cap h^{-1}(V)}} & \left(U\cap h^{-1}(V)\right)\times\mathbb{R}^p \\ (\varkappa,z\left(h\left(\varkappa\right)\right)) & \longmapsto & \left(\varkappa,t_{V,h(\varkappa)}^{-1}z\left(h\left(\varkappa\right)\right)\right). \end{array}$$

Proposition 3.1.2.1 *The set*

$$\overline{\mathcal{AF}}_F\overset{put}{=}\bigcup_{\substack{(U,\xi_U)\in\mathcal{AF}_{TM},\ (V,s_V)\in\mathcal{AF}_F\\ U\cap h^{-1}(V)\neq\phi}}\left\{\left(U\cap h^{-1}\left(V\right),\bar{s}_{U\cap h^{-1}(V)}\right)\right\}$$

is a vector fibred $m+p$-atlas for the vector bundle $\left(h^*F,h^*\nu,M\right)$.

If

$$z=z^\alpha t_\alpha\in\Gamma\left(F,\nu,N\right),$$

then, using the vector fibred $m+p$-structure $\left[\overline{\mathcal{AF}}_F\right]$, we obtain the section

$$Z=\left(z^\alpha\circ h\right)T_\alpha\in\Gamma\left(h^*F,h^*\nu,M\right)$$

such that

$$Z\left(x\right)=z\left(h\left(x\right)\right),$$

for any $x\in U\cap h^{-1}\left(V\right)$.

The set $\left\{T_\alpha,\ \alpha\in\overline{1,p}\right\}$ is a base for the module of sections

$$\left(\Gamma\left(h^*F,h^*\nu,M\right),+,\cdot\right).$$

Let $\left(\overset{h^*F}{\rho},Id_M\right)$ be the $\mathbf{B^v}$-morphism of

$$(h^*F,h^*\nu,M)$$

source and

$$(TM,\tau_M,M)$$

target, where

$$\begin{array}{ccc} h^*F & \overset{\overset{h^*F}{\rho}}{\longrightarrow} & TM \\ Z^\alpha T_\alpha\left(x\right) & \longmapsto & \left(Z^\alpha\cdot\rho_\alpha^i\circ h\right)\dfrac{\partial}{\partial x^i}\left(x\right) \end{array}\tag{3.1.2.1}$$

We consider the operation

$$\Gamma\left(h^{*}F,h^{*}\nu,M\right)\times\Gamma\left(h^{*}F,h^{*}\nu,M\right)\quad\xrightarrow{[,]_{h^{*}F}}\quad\Gamma\left(h^{*}F,h^{*}\nu,M\right)$$

defined by

$$\begin{aligned}
\left[T_{\alpha},T_{\beta}\right]_{h^{*}F} &= \left(L_{\alpha\beta}^{\gamma}\circ h\right)T_{\gamma},\\
\left[T_{\alpha},fT_{\beta}\right]_{h^{*}F} &= f\left(L_{\alpha\beta}^{\gamma}\circ h\right)T_{\gamma}+\left(\rho_{\alpha}^{i}\circ h\right)\frac{\partial f}{\partial x^{i}}T_{\beta},\\
\left[fT_{\alpha},T_{\beta}\right]_{h^{*}F} &= -\left[T_{\beta},fT_{\alpha}\right]_{h^{*}F},
\end{aligned}\tag{3.1.2.2}$$

for any $f\in\mathcal{F}\left(M\right)$.

Lemma 3.1.2.1 *The following equality holds good*

$$\left[U,fV\right]_{h^{*}F}=f\left[U,V\right]_{h^{*}F}+\Gamma\left(\overset{h^{*}F}{\rho},Id_{M}\right)\left(U\right)f\cdot V,$$

for any $U,V\in\Gamma\left(h^{*}F,h^{*}\nu,M\right)$ *and for any* $f\in\mathcal{F}\left(M\right)$.

Proof. We observe that for any $\alpha,\beta\in\overline{1,p}$, we obtain

$$\left[T_{\alpha},fT_{\beta}\right]_{h^{*}F}=f\left[T_{\alpha},T_{\beta}\right]_{h^{*}F}+\Gamma\left(\overset{h^{*}F}{\rho},Id_{M}\right)\left(T_{\alpha}\right)f\cdot T_{\beta},\ \forall f\in\mathcal{F}\left(N\right).$$

Using this equality and the definition of the operation $[,]_{h^{*}F}$ it results the conclusion of the lemma. *q.e.d.*

Lemma 3.1.2.1 *The* $\mathcal{F}\left(M\right)$*-algebra*

$$\left(\Gamma\left(h^{*}F,h^{*}\nu,M\right),+,\cdot,[,]_{h^{*}F}\right)$$

is a Lie $\mathcal{F}\left(M\right)$*-algebra.*

Proof. Using the definition of the operation $[,]_{h^{*}F}$ it results that

$$\left[U,V\right]_{h^{*}F}=-\left[V,U\right]_{h^{*}F},$$

for any $U,V\in\Gamma\left(h^{*}F,h^{*}\nu,M\right)$. Therefore, we obtain

$$\left[U,U\right]_{h^{*}F}=0,\ \forall U\in\Gamma\left(h^{*}F,h^{*}\nu,M\right).\tag{1}$$

Since $\left(\Gamma\left(F,\nu,M\right),+,\cdot,[,]_{F}\right)$ is a Lie $\mathcal{F}\left(M\right)$-algebra, we obtain the equality:

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(L_{\beta\gamma}^{\varepsilon}L_{\alpha\varepsilon}^{\delta}+\theta_{\alpha}^{\tilde{i}}\frac{\partial L_{\beta\gamma}^{\delta}}{\partial\varkappa^{\tilde{i}}}\right)=0.$$

Using (3.1.1.9), we obtain the equality:

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(\left(L_{\beta\gamma}^{\varepsilon}\circ h\right)\left(L_{\alpha\varepsilon}^{\delta}\circ h\right)+\rho_{\alpha}^{i}\circ h\frac{\partial\left(L_{\beta\gamma}^{\delta}\circ h\right)}{\partial x^{i}}\right)=0.$$

and multiplying with T_δ, we obtain

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left(\left(L_{\beta\gamma}^{\varepsilon}\circ h\right)\left(L_{\alpha\varepsilon}^{\delta}\circ h\right)T_\delta\ +\rho_\alpha^i\circ h\frac{\partial\left(L_{\beta\gamma}^{\delta}\circ h\right)}{\partial x^i}T_\delta\right)\ =0.$$

which is equivalent with

$$(2)\qquad \sum_{cyclic(\alpha,\beta,\gamma)}\left(\left(L_{\beta\gamma}^{\varepsilon}\circ h\right)[T_\alpha,T_\varepsilon]_{h^*F}+\rho_\alpha^i\circ h\frac{\partial L_{\beta\gamma}^{\delta}\circ h}{\partial x^i}T_\delta\right)=0.$$

Since this equality implies

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left[T_\alpha,\left(L_{\beta\gamma}^{\varepsilon}\circ h\right)T_\varepsilon\right]_{h^*F}=0,$$

it results that it is satisfied the Jacobi identity

$$\sum_{cyclic(\alpha,\beta,\gamma)}\left[T_\alpha,[T_\beta,T_\gamma]_{h^*F}\right]_{h^*F}=0.$$

In general, for any $U,V,Z\in\Gamma\left(h^*F,h^*\nu,M\right)$, we obtain the Jacobi identity:

$$(2)\qquad [U,[V,Z]_{h^*F}]_{h^*F}+[Z,[U,V]_{h^*F}]_{h^*F}+[V,[Z,U]_{h^*F}]_{h^*F}=0.$$

Using affirmations (1) and (2), we get the conclusion of the lemma. *q.e.d.*

Lemma 3.1.2.3 *The* **Mod**-*morphism*

$$\Gamma\left(\overset{h^*F}{\rho},Id_M\right)$$

is a **Liealg**-*morphism of*

$$\left(\Gamma\left(h^*F,h^*\nu,M\right),+,\cdot,[,]_{h^*F}\right)$$

source and

$$(\Gamma(TM,\tau_M,M),+,\cdot,[,]_{TM})$$

target.

Proof. Using relations (3.1.1.9), we obtain:

$$\left(L_{\alpha\beta}^{\gamma}\circ h\right)\left(\rho_\gamma^k\circ h\right)\frac{\partial}{\partial x^k}=\left(\rho_\alpha^i\circ h\right)\frac{\partial\left(\rho_\beta^k\circ h\right)}{\partial x^i}\frac{\partial}{\partial x^k}-\left(\rho_\beta^j\circ h\right)\frac{\partial\left(\rho_\alpha^k\circ h\right)}{\partial x^j}\frac{\partial}{\partial x^k},$$

Therefore,

$$\Gamma\left(\overset{h^*F}{\rho},Id_M\right)[T_\alpha,T_\beta]_{h^*F}=\left[\Gamma\left(\overset{h^*F}{\rho},Id_M\right)T_\alpha,\Gamma\left(\overset{h^*F}{\rho},Id_M\right)T_\beta\right]_{TM},$$

for any base sections T_α,T_β.

In general, we obtain the equality

$$\Gamma\left(\overset{h^*F}{\rho},Id_M\right)[U,V]_{h^*F}=\left[\Gamma\left(\overset{h^*F}{\rho},Id_M\right)U,\Gamma\left(\overset{h^*F}{\rho},Id_M\right)U\right]_{TM},$$

for any $U,V\in\Gamma\left(h^{*}F,h^{*}\nu,M\right).$ *q.e.d.*

Using Lemmas 3.1.2.1, 3.1.2.2 and 3.1.2.3, we obtain the following

Theorem 3.1.2.1 *The couple*

$$\left(\left[,\right]_{h^{*}F},\left(\overset{h^{*}F}{\rho},Id_{M}\right)\right)$$

is a Lie algebroid structure for the vector bundle $\left(h^{*}F,h^{*}\nu,M\right).$

This Lie algebroid will be called *the pull-back Lie algebroid of the generalized Lie algebroid*

$$\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right).$$

3.1.3 Interior Differential Systems

We consider a generalized Lie algebroid $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$ given by the diagrams:

$$\begin{array}{ccccccc} & & F & \stackrel{\rho}{\longrightarrow} & TM & \stackrel{Th}{\longrightarrow} & TN \\ & & \downarrow\nu & & \downarrow\tau_{M} & & \downarrow\tau_{N} \\ M & \stackrel{h}{\longrightarrow} & N & \stackrel{\eta}{\longrightarrow} & M & \stackrel{h}{\longrightarrow} & N \end{array}$$

Let $\left(\left(h^{*}F,h^{*}\nu,M\right),\left[,\right]_{h^{*}F},\left(\overset{h^{*}F}{\rho},Id_{M}\right)\right)$ be the pull-back Lie algebroid.

Definition 3.1.3.1 Any vector subbundle $\left(E,\pi,M\right)$ of the vector bundle $\left(h^{*}F,h^{*}\nu,M\right)$ will be called *interior differential system (IDS) of the generalized Lie algebroid* $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right).$

In particular, if $h=Id_{N}=\eta$, then any vector subbundle $\left(E,\pi,N\right)$ of the vector bundle $\left(F,\nu,N\right)$ will be called *interior differential system of the Lie algebroid* $\left(\left(F,\nu,N\right),\left[,\right]_{F},\left(\rho,Id_{N}\right)\right).$

Remark 3.1.3.1 If $\left(E,\pi,M\right)$ is an IDS of the generalized Lie algebroid $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right),$ then $\left(\Gamma\left(E,\pi,M\right),+,\cdot\right)$ is a $\mathcal{F}\left(M\right)$-submodule of the $\mathcal{F}\left(M\right)$-module $\left(\Gamma\left(h^{*}F,h^{*}\nu,M\right),+,\cdot\right).$

In addition, if

$$\Gamma\left(E^{\perp},\pi^{\perp},M\right)\overset{put}{=}\left\{\Omega\in\Gamma\left(\overset{*}{h^{*}F},\overset{*}{h^{*}\nu},M\right):\Omega\left(S\right)=0,\ \forall S\in\Gamma\left(E,\pi,M\right)\right\},$$

then $\left(\Gamma\left(E^{\perp},\pi^{\perp},M\right),+,\cdot\right)$ is $\mathcal{F}\left(M\right)$-submodule of the $\mathcal{F}\left(M\right)$-module $\left(\Gamma\left(\overset{*}{h^{*}F},\overset{*}{h^{*}\nu},M\right),+,\cdot\right).$

We obtain a vector subbundle $\left(E^{\perp},\pi^{\perp},M\right)$ of the vector bundle $\left(\overset{*}{h^{*}F},\overset{*}{h^{*}\nu},M\right)$ which will be called *the annihilator vector subbundle for the IDS* $\left(E,\pi,M\right).$

Proposition 3.1.3.1 *Let* $\left(E,\pi,M\right)$ *be an IDS of the generalized Lie algebroid* $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right).$

If $dim_{\mathcal{F}(M)}\Gamma\left(E,\pi,M\right)=r\leq p=dim_{\mathcal{F}(M)}\Gamma\left(h^{*}F,h^{*}\nu,M\right)$*, then* $dim_{\mathcal{F}(M)}\Gamma\left(E^{\perp},\pi^{\perp},M\right)=p-r.$

Therefore, if $\Gamma\left(E,\pi,M\right)=\left\langle S_{1},...,S_{r}\right\rangle$*, then it exists* $\Theta^{r+1},...,\Theta^{p}\in\Gamma\left(\overset{*}{h^{*}F},\overset{*}{h^{*}\nu},M\right)$ *linearly independent such that* $\Gamma\left(E^{\perp},\pi^{\perp},M\right)=\left\langle\Theta^{r+1},...,\Theta^{p}\right\rangle.$

Conversely, if $\Gamma\left(E^{\perp},\pi^{\perp},M\right)=\left\langle\Theta^{r+1},...,\Theta^{p}\right\rangle$*, then it exists* $S_1,...,S_r\in\Gamma(E,\pi,M)$ *linearly independent such that* $\Gamma(E,\pi,M)=\langle S_1,...,S_r\rangle$ *.*

Definition 3.1.3.2 The IDS (E,π,M) of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ will be called *involutive* if

$$[S,T]_{h^*F}\in\Gamma(E,\pi,M)\,,\ \forall S,T\in\Gamma(E,\pi,M)\,.$$

Proposition 3.1.3.2 *Let* (E,π,M) *be an IDS of the generalized Lie algebroid* $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$*.*

If $\{S_1,...,S_r\}$ *is a base for the* $\mathcal{F}(M)$*-submodule* $(\Gamma(E,\pi,M),+,\cdot)$ *then* (E,π,M) *is involutive if and only if*

$$[S_a,S_b]_{h^*F}\in\Gamma(E,\pi,M)\,,\ \forall a,b\in\overline{1,r}.$$

3.2 Exterior differential calculus for generalized Lie algebroids

We propose an exterior differential calculus in the general framework of generalized Lie algebroids. As any Lie algebroid can be regarded as a generalized Lie algebroid, in particular, we obtain a new point of view over the exterior differential calculus for Lie algebroids.

Let

$$\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$$

be a generalized Lie algebroid given by the diagram $(3.1.1.1)$.

Definition 3.2.1 For any $q\in\mathbb{N}$ we denote by $(\Sigma_q,\circ)$ the permutations group of the set $\{1,2,...,q\}$.

Definition 3.2.2 We denoted by $\Lambda^q(F,\nu,N)$ the set of q-linear applications

$$\begin{array}{ccc}\Gamma(F,\nu,N)^q & \xrightarrow{\omega} & \mathcal{F}(N)\\ (z_1,...,z_q) & \longmapsto & \omega(z_1,...,z_q)\end{array}$$

such that

$$\omega\left(z_{\sigma(1)},...,z_{\sigma(q)}\right)=sgn(\sigma)\cdot\omega(z_1,...,z_q)$$

for any $z_1,...,z_q\in\Gamma(F,\nu,N)$ and for any $\sigma\in\Sigma_q$.

The elements of $\Lambda^q(F,\nu,N)$ will be called *differential forms of degree* q or *differential* q*-forms*.

Remark 3.2.1 If $\omega\in\Lambda^q(F,\nu,N)$, then $\omega(z_1,...,z,...,z,...z_q)=0$. Therefore, if $\omega\in\Lambda^q(F,\nu,N)$, then

$$\omega(z_1,...,z_i,...,z_j,...z_q)=-\omega(z_1,...,z_j,...,z_i,...z_q)\,.$$

Theorem 3.2.1 *If* $q\in N$*, then* $(\Lambda^q(F,\nu,N),+,\cdot)$ *is a* $\mathcal{F}(N)$*-module.*

Definition 3.2.3 If $\omega \in \Lambda^q(F,\nu,N)$ and $\theta \in \Lambda^r(F,\nu,N)$, then the $(q+r)$-form $\omega \wedge \theta$ defined by

$$\begin{aligned} \omega \wedge \theta (z_1, ..., z_{q+r}) &= \sum_{\substack{\sigma(1)<...<\sigma(q) \\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma)\, \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right) \theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right) \\ &= \frac{1}{q!r!} \sum_{\sigma \in \Sigma_{q+r}} sgn(\sigma)\, \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right) \theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right), \end{aligned}$$

for any $z_1, ..., z_{q+r} \in \Gamma(F,\nu,N)$, will be called *the exterior product of the forms ω and θ.*

Using the previous definition, we obtain

Theorem 3.2.2 *The following affirmations hold good:*

1. *If $\omega \in \Lambda^q(F,\nu,N)$ and $\theta \in \Lambda^r(F,\nu,N)$, then*

$$\omega \wedge \theta = (-1)^{q \cdot r}\, \theta \wedge \omega. \tag{3.2.1}$$

2. *For any $\omega \in \Lambda^q(F,\nu,N)$, $\theta \in \Lambda^r(F,\nu,N)$ and $\eta \in \Lambda^s(F,\nu,N)$ we obtain*

$$(\omega \wedge \theta) \wedge \eta = \omega \wedge (\theta \wedge \eta). \tag{3.2.2}$$

3. *For any $\omega, \theta \in \Lambda^q(F,\nu,N)$ and $\eta \in \Lambda^s(F,\nu,N)$ we obtain*

$$(\omega + \theta) \wedge \eta = \omega \wedge \eta + \theta \wedge \eta. \tag{3.2.3}$$

4. *For any $\omega \in \Lambda^q(F,\nu,N)$ and $\theta, \eta \in \Lambda^s(F,\nu,N)$ we obtain*

$$\omega \wedge (\theta + \eta) = \omega \wedge \theta + \omega \wedge \eta. \tag{3.2.4}$$

5. *For any $f \in \mathcal{F}(N)$, $\omega \in \Lambda^q(F,\nu,N)$ and $\theta \in \Lambda^s(F,\nu,N)$ we obtain*

$$(f \cdot \omega) \wedge \theta = f \cdot (\omega \wedge \theta) = \omega \wedge (f \cdot \theta). \tag{3.2.5}$$

Theorem 3.2.3 *If*

$$\Lambda(F,\nu,N) = \underset{q \geq 0}{\oplus} \Lambda^q(F,\nu,N),$$

then

$$(\Lambda(F,\nu,N), +, \cdot, \wedge)$$

is a $\mathcal{F}(N)$-algebra. This algebra will be called *the exterior differential algebra of the vector bundle (F,ν,N).*

Remark 3.2.2 If $\left\{t^\alpha,\ \alpha \in \overline{1,p}\right\}$ is the coframe associated to the frame $\left\{t_\alpha,\ \alpha \in \overline{1,p}\right\}$ of the vector bundle (F,ν,N) in the vector local $(n+p)$-chart U, then

$$t^{\alpha_1} \wedge ... \wedge t^{\alpha_q}\left(z_1^\alpha t_\alpha, ..., z_q^\alpha t_\alpha\right) = \tfrac{1}{q!} \det \left\Vert \begin{array}{ccc} z_1^{\alpha_1} & ... & z_1^{\alpha_q} \\ ... & ... & ... \\ z_q^{\alpha_1} & ... & z_q^{\alpha_q} \end{array} \right\Vert,$$

for any $q \in \overline{1,p}$.

Remark 3.2.3 If $\left\{t^{\alpha},\ \alpha\in\overline{1,p}\right\}$ is the coframe associated to the frame $\left\{t_{\alpha},\ \alpha\in\overline{1,p}\right\}$ of the vector bundle (F,ν,N) in the vector local $(n+p)$-chart U, then, for any $q\in\overline{1,p}$ we define C_p^q exterior differential forms of the type

$$t^{\alpha_1}\wedge...\wedge t^{\alpha_q}$$

such that $1\leq\alpha_1<...<\alpha_q\leq p$.

The set

$$\{t^{\alpha_1}\wedge...\wedge t^{\alpha_q},1\leq\alpha_1<...<\alpha_q\leq p\}$$

is a base for the $\mathcal{F}(N)$-module

$$\left(\Lambda^q\left(F,\nu,N\right),+,\cdot\right).$$

Therefore, if $\omega\in\Lambda^q\left(F,\nu,N\right)$, then

$$\omega=\omega_{\alpha_1...\alpha_q}t^{\alpha_1}\wedge...\wedge t^{\alpha_q}.$$

In particular, if ω is an exterior differential p-form ω, then we can written

$$\omega=a\cdot t^1\wedge...\wedge t^p,$$

where $a\in\mathcal{F}(N)$.

Definition 3.2.4 If

$$\omega=\omega_{\alpha_1...\alpha_q}t^{\alpha_1}\wedge...\wedge t^{\alpha_q}\in\Lambda^q\left(F,\nu,N\right)$$

such that

$$\omega_{\alpha_1...\alpha_q}\in C^r\left(N\right),$$

for any $1\leq\alpha_1<...<\alpha_q\leq p$, then we will say that *the q-form ω is differentiable of C^r-class.*

Definition 3.2.5 For any $z\in\Gamma\left(F,\nu,N\right)$, the $\mathcal{F}(N)$-multilinear application

$$\Lambda\left(F,\nu,N\right)\ \xrightarrow{L_z}\ \Lambda\left(F,\nu,N\right)\ ,$$

defined by

$$L_z\left(f\right)=\Gamma\left(Th\circ\rho,h\circ\eta\right)z\left(f\right),\ \forall f\in\mathcal{F}\left(N\right)$$

and

$$\begin{array}{ll}L_z\omega\left(z_1,...,z_q\right) & =\Gamma\left(Th\circ\rho,h\circ\eta\right)z\left(\omega\left(\left(z_1,...,z_q\right)\right)\right)\\ & -\sum\limits_{i=1}^{q}\omega\left(\left(z_1,...,\left[z,z_i\right]_{F,h},...,z_q\right)\right),\end{array}$$

for any $\omega\in\Lambda^q\ \left(F,\nu,N\right)$ and $z_1,...,z_q\in\Gamma\left(F,\nu,N\right)$, will be called *the covariant Lie derivative with respect to the section z.*

Theorem 3.2.4 *If $z\in\Gamma\left(F,\nu,N\right),\omega\in\Lambda^q\left(F,\nu,N\right)$ and $\theta\in\Lambda^r\left(F,\nu,N\right)$, then*

$$L_z\left(\omega\wedge\theta\right)=L_z\omega\wedge\theta+\omega\wedge L_z\theta. \tag{3.2.6}$$

Proof. Let $z_1, ..., z_{q+r} \in \Gamma(F, \nu, N)$ be arbitrary. Since

$$L_z(\omega \wedge \theta)(z_1, ..., z_{q+r}) = \Gamma(Th \circ \rho, h \circ \eta) z((\omega \wedge \theta)(z_1, ..., z_{q+r}))$$
$$-\sum_{i=1}^{q+r}(\omega \wedge \theta)\left(\left(z_1, ..., [z, z_i]_{F,h}, ..., z_{q+r}\right)\right)$$
$$= \Gamma(Th \circ \rho, h \circ \eta) z\left(\sum_{\substack{\sigma(1)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \cdot \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right)\right.$$
$$\left.\cdot\theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right)\right) - \sum_{i=1}^{q+r}(\omega \wedge \theta)\left(\left(z_1, ..., [z, z_i]_{F,h}, ..., z_{q+r}\right)\right)$$
$$= \sum_{\substack{\sigma(1)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \cdot \Gamma(Th \circ \rho, h \circ \eta) z\left(\omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right)\right)$$
$$\cdot\theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right) + \sum_{\substack{\sigma(1)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \cdot \omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right)$$
$$\cdot\Gamma(Th \circ \rho, h \circ \eta) z\left(\theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right)\right) - \sum_{\substack{\sigma(1)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma)$$
$$\cdot\sum_{i=1}^{q}\omega\left(z_{\sigma(1)}, ..., \left[z, z_{\sigma(i)}\right]_{F,h}, ..., z_{\sigma(q+r)}\right) \cdot \theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right)$$
$$- \sum_{\substack{\sigma(1)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \sum_{i=q+1}^{q+r}\omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right)$$
$$\cdot\theta\left(z_{\sigma(q+1)}, ..., \left[z, z_{\sigma(i)}\right]_{F,h}, ..., z_{\sigma(q+r)}\right)$$
$$= \sum_{\substack{\sigma(1)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) L_z\omega\left(z_{\sigma(1)}, ..., \left[z, z_{\sigma(i)}\right]_{F,h}, ..., z_{\sigma(q+r)}\right)$$
$$\cdot\theta\left(z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}\right) + \sum_{\substack{\sigma(1)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma) \sum_{i=q+1}^{q+r}\omega\left(z_{\sigma(1)}, ..., z_{\sigma(q)}\right)$$
$$\cdot L_z\theta\left(z_{\sigma(q+1)}, ..., \left[z, z_{\sigma(i)}\right]_{F,h}, ..., z_{\sigma(q+r)}\right)$$
$$= (L_z\omega \wedge \theta + \omega \wedge L_z\theta)(z_1, ..., z_{q+r})$$

it results the conclusion of the theorem. *q.e.d.*

Definition 3.2.6 For any $z \in \Gamma(F, \nu, N)$, the $\mathcal{F}(N)$-multilinear application

$$\begin{array}{ccc} \Lambda(F, \nu, N) & \xrightarrow{i_z} & \Lambda(F, \nu, N) \\ \Lambda^q(F, \nu, N) \ni \omega & \longmapsto & i_z\omega \in \Lambda^{q-1}(F, \nu, N), \end{array}$$

where

$$i_z\omega(z_2, ..., z_q) = \omega(z, z_2, ..., z_q),$$

for any $z_2, ..., z_q \in \Gamma(F, \nu, N)$, will be called the *interior product associated to the section z.*

For any $f \in \mathcal{F}(N)$, we define $i_z f = 0$.

Remark 3.2.4 If $z \in \Gamma(F,\nu,N)$, $\omega \in \Lambda^p(F,\nu,N)$ and U is an open subset of N such that $z_{|U} = 0$ or $\omega_{|U} = 0$, then $(i_z\omega)_{|U} = 0$.

Theorem 3.2.5 *If* $z \in \Gamma(F,\nu,N)$, *then for any* $\omega \in \Lambda^q(F,\nu,N)$ *and* $\theta \in \Lambda^r(F,\nu,N)$ *we obtain*

$$i_z(\omega \wedge \theta) = i_z\omega \wedge \theta + (-1)^q\,\omega \wedge i_z\theta. \tag{3.2.7}$$

Proof. Let $z_1, ..., z_{q+r} \in \Gamma(F,\nu,N)$ be arbitrary. We observe that

$$\begin{aligned}
&i_{z_1}(\omega\wedge\theta)(z_2,...,z_{q+r}) = (\omega\wedge\theta)(z_1,z_2,...,z_{q+r})\\
&= \sum_{\substack{\sigma(1)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma)\cdot\omega\left(z_{\sigma(1)},...,z_{\sigma(q)}\right)\cdot\theta\left(z_{\sigma(q+1)},...,z_{\sigma(q+r)}\right)\\
&= \sum_{\substack{1=\sigma(1)<\sigma(2)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma)\cdot\omega\left(z_1,z_{\sigma(2)},...,z_{\sigma(q)}\right)\cdot\theta\left(z_{\sigma(q+1)},...,z_{\sigma(q+r)}\right)\\
&+ \sum_{\substack{\sigma(1)<...<\sigma(q)\\ 1=\sigma(q+1)<\sigma(q+2)<...<\sigma(q+r)}} sgn(\sigma)\cdot\omega\left(z_{\sigma(1)},...,z_{\sigma(q)}\right)\cdot\theta\left(z_1,z_{\sigma(q+2)},...,z_{\sigma(q+r)}\right)\\
&= \sum_{\substack{\sigma(2)<...<\sigma(q)\\ \sigma(q+1)<...<\sigma(q+r)}} sgn(\sigma)\cdot i_{z_1}\omega\left(z_{\sigma(2)},...,z_{\sigma(q)}\right)\cdot\theta\left(z_{\sigma(q+1)},...,z_{\sigma(q+r)}\right)\\
&+ \sum_{\substack{\sigma(1)<...<\sigma(q)\\ \sigma(q+2)<...<\sigma(q+r)}} sgn(\sigma)\cdot\omega\left(z_{\sigma(1)},...,z_{\sigma(q)}\right)\cdot i_{z_1}\theta\left(z_{\sigma(q+2)},...,z_{\sigma(q+r)}\right).
\end{aligned}$$

In the second sum, we have the permutation

$$\sigma = \begin{pmatrix} 1 & ... & q & q+1 & q+2 & ... & q+r \\ \sigma(1) & ... & \sigma(q) & 1 & \sigma(q+2) & ... & \sigma(q+r) \end{pmatrix}.$$

We observe that $\sigma = \tau\circ\tau'$, where

$$\tau = \begin{pmatrix} 1 & 2 & ... & q+1 & q+2 & ... & q+r \\ 1 & \sigma(1) & ... & \sigma(q) & \sigma(q+2) & ... & \sigma(q+r) \end{pmatrix}$$

and

$$\tau' = \begin{pmatrix} 1 & 2 & ... & q & q+1 & q+2 & ... & q+r \\ 2 & 3 & ... & q+1 & 1 & q+2 & ... & q+r \end{pmatrix}.$$

Since $\tau(2) < ... < \tau(q+1)$ and τ' has q inversions, it results that

$$sgn(\sigma) = (-1)^q\cdot sgn(\tau).$$

Therefore,

$$\begin{aligned}
&i_{z_1}(\omega\wedge\theta)(z_2,...,z_{q+r}) = (i_{z_1}\omega\wedge\theta)(z_2,...,z_{q+r})\\
&+(-1)^q \sum_{\substack{\tau(2)<...<\tau(q)\\ \tau(q+2)<...<\tau(q+r)}} sgn(\tau)\cdot\omega\left(z_{\tau(2)},...,z_{\tau(q)}\right)\cdot i_{z_1}\theta\left(z_{\tau(q+2)},...,z_{\tau(q+r)}\right)\\
&= (i_{z_1}\omega\wedge\theta)(z_2,...,z_{q+r}) + (-1)^q(\omega\wedge i_{z_1}\theta)(z_2,...,z_{q+r}).
\end{aligned}$$

q.e.d.

Theorem 3.2.6 *For any* $z, v \in \Gamma(F, \nu, N)$ *we obtain*

$$L_v \circ i_z - i_z \circ L_v = i_{[z,v]_{F,h}}. \tag{3.2.8}$$

Proof. Let $\omega \in \Lambda^q(F, \nu, N)$ be arbitrary. Since

$$\begin{aligned}
&i_z(L_v\omega)(z_2, ...z_q) = L_v\omega(z, z_2, ...z_q) \\
&= \Gamma(Th \circ \rho, h \circ \eta)\, v(\omega(z, z_2, ..., z_q)) - \omega\left([v, z]_{F,h}, z_2, ..., z_q\right) \\
&-\sum_{i=2}^{q}\omega\left(\left(z, z_2, ..., [v, z_i]_{F,h}, ..., z_q\right)\right) \\
&= \Gamma(Th \circ \rho, h \circ \eta)\, v(i_z\omega(z_2, ..., z_q)) - \sum_{i=2}^{q} i_z\omega\left(z_2, ..., [v, z_i]_{F,h}, ..., z_q\right) \\
&-i_{[v,z]_F}(z_2, ..., z_q) = \left(L_v(i_z\omega) - i_{[v,z]_{F,h}}\right)(z_2, ..., z_q),
\end{aligned}$$

for any $z_2, ..., z_q \in \Gamma(F, \nu, N)$ it result the conclusion of the theorem. *q.e.d.*

Definition 3.2.7 If $f \in \mathcal{F}(N)$ and $z \in \Gamma(F, \nu, N)$, then we define

$$d^F f(z) = \Gamma(Th \circ \rho, h \circ \eta)(z) f.$$

Theorem 3.2.7 *The* $\mathcal{F}(N)$*-multilinear application*

$$\begin{array}{ccc}
\Lambda^q\ (F, \nu, N) & \xrightarrow{d^F} & \Lambda^{q+1}\ (F, \nu, N) \\
\omega & \longmapsto & d\omega
\end{array}$$

defined by

$$\begin{aligned}
d^F\omega(z_0, z_1, ..., z_q) = &\sum_{i=0}^{q}(-1)^i\,\Gamma(Th \circ \rho, h \circ \eta)\, z_i(\omega((z_0, z_1, ..., \hat{z}_i, ..., z_q))) \\
&+ \sum_{i<j}(-1)^{i+j}\,\omega\left(\left([z_i, z_j]_{F,h}, z_0, z_1, ..., \hat{z}_i, ..., \hat{z}_j, ..., z_q\right)\right),
\end{aligned}$$

for any $z_0, z_1, ..., z_q \in \Gamma(F, \nu, N)$, *is unique with the following property:*

$$L_z = d^F \circ i_z + i_z \circ d^F,\ \forall z \in \Gamma(F, \nu, N). \tag{3.2.9}$$

This $\mathcal{F}(N)$-multilinear application will be called *the exterior differentiation operator for the exterior differential algebra of the generalized Lie algebroid* $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$.

Proof. We verify the property (3.2.9) Since

$$\begin{aligned}
&\left(i_{z_0}\circ d^F\right)\omega\left(z_1,...,z_q\right)=d\omega\left(z_0,z_1,...,z_q\right)\\
&=\sum_{i=0}^{q}\left(-1\right)^i\Gamma\left(Th\circ\rho,h\circ\eta\right)z_i\left(\omega\left(z_0,z_1,...,\hat{z}_i,...,z_q\right)\right)\\
&+\sum_{0\leq i<j}\left(-1\right)^{i+j}\omega\left(\left[z_i,z_j\right]_{F,h},z_0,z_1,...,\hat{z}_i,...,\hat{z}_j,...,z_q\right)\\
&=\Gamma\left(Th\circ\rho,h\circ\eta\right)z_0\left(\omega\left(z_1,...,z_q\right)\right)\\
&+\sum_{i=1}^{q}\left(-1\right)^i\Gamma\left(Th\circ\rho,h\circ\eta\right)z_i\left(\omega\left(z_0,z_1,...,\hat{z}_i,...,z_q\right)\right)\\
&+\sum_{i=1}^{q}\left(-1\right)^i\omega\left(\left[z_0,z_i\right]_{F,h},z_1,...,\hat{z}_i,...,z_q\right)\\
&+\sum_{1\leq i<j}\left(-1\right)^{i+j}\omega\left(\left[z_i,z_j\right]_{F,h},z_0,z_1,...,\hat{z}_i,...,\hat{z}_j,...,z_q\right)\\
&=\Gamma\left(Th\circ\rho,h\circ\eta\right)z_0\left(\omega\left(z_1,...,z_q\right)\right)\\
&-\sum_{i=1}^{q}\omega\left(z_1,...,\left[z_0,z_i\right]_{F,h},...,z_q\right)\\
&-\sum_{i=1}^{q}\left(-1\right)^{i-1}\Gamma\left(Th\circ\rho,h\circ\eta\right)z_i\left(i_{z_0}\omega\left(\left(z_1,...,\hat{z}_i,...,z_q\right)\right)\right)\\
&-\sum_{1\leq i<j}\left(-1\right)^{i+j-2}i_{z_0}\omega\left(\left(\left[z_i,z_j\right]_{F,h},z_1,...,\hat{z}_i,...,\hat{z}_j,...,z_q\right)\right)\\
&=\left(L_{z_0}-d^F\circ i_{z_0}\right)\omega\left(z_1,...,z_q\right),
\end{aligned}$$

for any $z_0,z_1,...,z_q\in\Gamma\left(F,\nu,N\right)$ it results that the property (3.2.9) is satisfied.

In the following, we verify the uniqueness of the operator d^F.

Let d'^F be an another exterior differentiation operator satisfying the property (3.2.9) .

Let $S=\left\{q\in\mathbb{N}:d^F\omega=d'^F\omega,\ \forall\omega\in\Lambda^q\left(F,\nu,N\right)\right\}$ be.

Let $z\in\Gamma\left(F,\nu,N\right)$ be arbitrary.

We observe that (3.2.9) is equivalent with

$$i_z\circ\left(d^F-d'^F\right)+\left(d^F-d'^F\right)\circ i_z=0. \tag{1}$$

Since $i_zf=0$, for any $f\in\mathcal{F}\left(N\right)$, it results that

$$\left(\left(d^F-d'^F\right)f\right)\left(z\right)=0,\ \forall f\in\mathcal{F}\left(N\right).$$

Therefore, we obtain that

$$0\in S. \tag{2}$$

In the following, we prove that

$$q\in S\Longrightarrow q+1\in S \tag{3}$$

Let $\omega\in\Lambda^{p+1}\left(F,\nu,N\right)$ be arbitrary. Since $i_z\omega\in\Lambda^q\left(F,\nu,N\right)$, using the equality (1), it results that

$$i_z\circ\left(d^F-d'^F\right)\omega=0.$$

We obtain that, $\left(\left(d^F-d'^F\right)\omega\right)\left(z_0,z_1,...,z_q\right)=0$, for any $z_1,...,z_q\in\Gamma\left(F,\nu,N\right)$. Therefore $d^F\omega=d'^F\omega$, namely $q+1\in S$.

Using the *Peano's Axiom* and the affirmations (2) and (3) it results that $S=\mathbb{N}$. Therefore, the uniqueness is verified. *q.e.d.*

Note that if $\omega = \omega_{\alpha_1 \dots \alpha_q} t^{\alpha_1} \wedge \dots \wedge t^{\alpha_q} \in \Lambda^q (F, \nu, N)$, then

$$\begin{aligned} d^F \omega \left(t_{\alpha_0}, t_{\alpha_1}, \dots, t_{\alpha_q}\right) &= \sum_{i=0}^{q} (-1)^i \theta_{\alpha_i}^{\tilde{k}} \frac{\partial \omega_{\alpha_0, \dots, \widehat{\alpha_i} \dots \alpha_q}}{\partial \varkappa^{\tilde{k}}} \\ &+ \sum_{i<j} (-1)^{i+j} L_{\alpha_i \alpha_j}^{\alpha} \cdot \omega_{\alpha, \alpha_0, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_q}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} d^F \omega = \Bigg(& \sum_{i=0}^{q} (-1)^i \theta_{\alpha_i}^{\tilde{k}} \frac{\partial \omega_{\alpha_0, \dots, \widehat{\alpha_i} \dots \alpha_q}}{\partial \varkappa^{\tilde{k}}} \\ & + \sum_{i<j} (-1)^{i+j} L_{\alpha_i \alpha_j}^{\alpha} \cdot \omega_{\alpha, \alpha_0, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_q} \Bigg) t^{\alpha_0} \wedge t^{\alpha_1} \wedge \dots \wedge t^{\alpha_q}. \end{aligned} \tag{3.2.10}$$

Remark 3.2.4 If d^F is the exterior differentiation operator for the generalized Lie algebroid

$$\left(\left(F, \nu, N\right), [,]_{F,h}, (\rho, \eta)\right),$$

$\omega \in \Lambda^q (F, \nu, N)$ and U is an open subset of N such that $\omega_{|U} = 0$, then $\left(d^F \omega\right)_{|U} = 0$.

Theorem 3.2.8 *The exterior differentiation operator d^F given by the previous theorem has the following properties:*

1. *For any $\omega \in \Lambda^q (F, \nu, N)$ and $\theta \in \Lambda^r (F, \nu, N)$ we obtain*

$$d^F (\omega \wedge \theta) = d^F \omega \wedge \theta + (-1)^q \omega \wedge d^F \theta. \tag{3.2.11}$$

2. *For any $z \in \Gamma (F, \nu, N)$* we obtain

$$L_z \circ d^F = d^F \circ L_z. \tag{3.2.12}$$

3. $d^F \circ d^F = 0$.

Proof.
1. Let $S = \left\{q \in \mathbb{N} : d^F (\omega \wedge \theta) = d^F \omega \wedge \theta + (-1)^q \omega \wedge d^F \theta,\ \forall \omega \in \Lambda^q (F, \nu, N)\right\}$ be. Since

$$\begin{aligned} & d^F (f \wedge \theta)(z, v) = d^F (f \cdot \theta)(z, v) \\ & = \Gamma (Th \circ \rho, h \circ \eta) z (f \omega (v)) - \Gamma (Th \circ \rho, h \circ \eta) v (f \omega (z)) - f \omega \left([z, v]_{F,h}\right) \\ & = \Gamma (Th \circ \rho, h \circ \eta) z (f) \cdot \omega (v) + f \cdot \Gamma (Th \circ \rho, h \circ \eta) z (\omega (v)) \\ & - \Gamma (Th \circ \rho, h \circ \eta) v (f) \cdot \omega (z) - f \cdot \Gamma (Th \circ \rho, h \circ \eta) v (\omega (z)) - f \omega \left([z, v]_{F,h}\right) \\ & = d^F f (z) \cdot \omega (v) - d^F f (v) \cdot \omega (z) + f \cdot d^F \omega (z, v) \\ & = \left(d^F f \wedge \omega\right)(z, v) + (-1)^0 f \cdot d^F \omega (z, v) \\ & = \left(d^F f \wedge \omega\right)(z, v) + (-1)^0 \left(f \wedge d^F \omega\right)(z, v),\ \forall z, v \in \Gamma (F, \nu, N), \end{aligned}$$

it results that

$$0 \in S. \tag{1.1}$$

In the following we prove that

$$q \in S \Longrightarrow q + 1 \in S. \tag{1.2}$$

Without restricting the generality, we consider that $\theta \in \Lambda^r(F, \nu, N)$. Since

$$
\begin{aligned}
& d^F(\omega \wedge \theta)(z_0, z_1, ..., z_{q+r}) = i_{z_0} \circ d^F(\omega \wedge \theta)(z_1, ..., z_{q+r}) \\
& = L_{z_0}(\omega \wedge \theta)(z_1, ..., z_{q+r}) - d^F \circ i_{z_0}(\omega \wedge \theta)(z_1, ..., z_{q+r}) \\
& = (L_{z_0}\omega \wedge \theta + \omega \wedge L_{z_0}\theta)(z_1, ..., z_{q+r}) \\
& - \left[d^F \circ (i_{z_0}\omega \wedge \theta + (-1)^q \omega \wedge i_{z_0}\theta)\right](z_1, ..., z_{q+r}) \\
& = \left(L_{z_0}\omega \wedge \theta + \omega \wedge L_{z_0}\theta - \left(d^F \circ i_{z_0}\omega\right) \wedge \theta\right)(z_1, ..., z_{q+r}) \\
& - \Big((-1)^{q-1} i_{z_0}\omega \wedge d^F\theta + (-1)^q d^F\omega \wedge i_{z_0}\theta\Big)(z_1, ..., z_{q+r}) \\
& - (-1)^{2q} \omega \wedge d^F \circ i_{z_0}\theta(z_1, ..., z_{q+r}) \\
& = \left(\left(L_{z_0}\omega - d^F \circ i_{z_0}\omega\right) \wedge \theta\right)(z_1, ..., z_{q+r}) \\
& + \omega \wedge \left(L_{z_0}\theta - d^F \circ i_{z_0}\theta\right)(z_1, ..., z_{q+r}) \\
& + \left((-1)^q i_{z_0}\omega \wedge d^F\theta - (-1)^q d^F\omega \wedge i_{z_0}\theta\right)(z_1, ..., z_{q+r}) \\
& = \left[\left(\left(i_{z_0} \circ d^F\right)\omega\right) \wedge \theta + (-1)^{q+1} d^F\omega \wedge i_{z_0}\theta\right](z_1, ..., z_{q+r}) \\
& + \left[\omega \wedge \left(\left(i_{z_0} \circ d^F\right)\theta\right) + (-1)^q i_{z_0}\omega \wedge d^F\theta\right](z_1, ..., z_{q+r}) \\
& = \left[i_{z_0}\left(d^F\omega \wedge \theta\right) + (-1)^q i_{z_0}\left(\omega \wedge d^F\theta\right)\right](z_1, ..., z_{q+r}) \\
& = \left[d^F\omega \wedge \theta + (-1)^q \omega \wedge d^F\theta\right](z_1, ..., z_{q+r}),
\end{aligned}
$$

for any $z_0, z_1, ..., z_{q+r} \in \Gamma(F, \nu, N)$, it results (1.2).

Using the *Peano's Axiom* and the affirmations (1.1) and (1.2) it results that $S = \mathbb{N}$.

Therefore, it results the conclusion of affirmation 1.

2. Let $z \in \Gamma(F, \nu, N)$ be arbitrary.

Let $S = \left\{q \in \mathbb{N} : \left(L_z \circ d^F\right)\omega = \left(d^F \circ L_z\right)\omega,\ \forall \omega \in \Lambda^q(F, \nu, N)\right\}$ be.

Let $f \in \mathcal{F}(N)$ be arbitrary. Since

$$
\begin{aligned}
& \left(d^F \circ L_z\right) f(v) = i_v \circ \left(d^F \circ L_z\right) f = \left(i_v \circ d^F\right) \circ L_z f \\
& = (L_v \circ L_z) f - \left(\left(d^F \circ i_v\right) \circ L_z\right) f \\
& = (L_v \circ L_z) f - L_{[z,v]_{F,h}} f + d^F \circ i_{[z,v]_{F,h}} f - d^F \circ L_z(i_v f) \\
& = (L_v \circ L_z) f - L_{[z,v]_{F,h}} f + d^F \circ i_{[z,v]_{F,h}} f - 0 \\
& = (L_v \circ L_z) f - L_{[z,v]_{F,h}} f + d^F \circ i_{[z,v]_{F,h}} f - L_z \circ d^F(i_v f) \\
& = (L_z \circ i_v)\left(d^F f\right) - L_{[z,v]_{F,h}} f + d^F \circ i_{[z,v]_{F,h}} f \\
& = (i_v \circ L_z)\left(d^F f\right) + L_{[z,v]_{F,h}} f - L_{[z,v]_{F,h}} f \\
& = i_v \circ \left(L_z \circ d^F\right) f = \left(L_z \circ d^F\right) f(v),\ \forall v \in \Gamma(F, \nu, N),
\end{aligned}
$$

it results that

$$0 \in S. \tag{2.1}$$

In the following we prove that

$$q \in S \Longrightarrow q + 1 \in S. \tag{2.2}$$

Let $\omega \in \Lambda^q(F,\nu,N)$ be arbitrary. Since

$$\begin{aligned}
&\left(d^F \circ L_z\right)\omega\left(z_0, z_1, ..., z_q\right) = i_{z_0} \circ \left(d^F \circ L_z\right)\omega\left(z_1, ..., z_q\right)\\
&= \left(i_{z_0} \circ d^F\right) \circ L_z\omega\left(z_1, ..., z_q\right)\\
&= \left[\left(L_{z_0} \circ L_z\right)\omega - \left(\left(d^F \circ i_{z_0}\right) \circ L_z\right)\omega\right]\left(z_1, ..., z_q\right)\\
&= \left[\left(L_{z_0} \circ L_z\right)\omega - L_{[z,z_0]_{F,h}}\omega\right]\left(z_1, ..., z_q\right)\\
&+ \left[d^F \circ i_{[z,z_0]_{F,h}}\omega - d^F \circ L_z\left(i_{z_0}\omega\right)\right]\left(z_1, ..., z_q\right)\\
&\overset{ip.}{=} \left[\left(L_{z_0} \circ L_z\right)\omega - L_{[z,z_0]_{F,h}}\omega\right]\left(z_1, ..., z_q\right)\\
&+ \left[d^F \circ i_{[z,z_0]_{F,h}}\omega - L_z \circ d^F\left(i_{z_0}\omega\right)\right]\left(z_1, ..., z_q\right)\\
&= \left[\left(L_z \circ i_{z_0}\right)\left(d^F\omega\right) - L_{[z,z_0]_{F,h}}\omega + d^F \circ i_{[z,z_0]_{F,h}}\omega\right]\left(z_1, ..., z_q\right)\\
&= \left[\left(i_{z_0} \circ L_z\right)\left(d^F\omega\right) + L_{[z,z_0]_{F,h}}\omega - L_{[z,z_0]_{F,h}}\omega\right]\left(z_1, ..., z_q\right)\\
&= i_{z_0} \circ \left(L_z \circ d^F\right)\omega\left(z_1, ..., z_q\right)\\
&= \left(L_z \circ d^F\right)\omega\left(z_0, z_1, ..., z_q\right),\ \forall z_0, z_1, ..., z_q \in \Gamma\left(F,\nu,N\right),
\end{aligned}$$

it results (2.2).

Using the *Peano's Axiom* and the affirmations (2.1) and (2.2) it results that $S = \mathbb{N}$. Therefore, it results the conclusion of affirmation 2.

3. It is remarked that

$$\begin{aligned}
&i_z \circ \left(d^F \circ d^F\right) = \left(i_z \circ d^F\right) \circ d^F = L_z \circ d^F - \left(d^F \circ i_z\right) \circ d^F\\
&= L_z \circ d^F - d^F \circ L_z + d^F \circ \left(d^F \circ i_z\right) = \left(d^F \circ d^F\right) \circ i_z,
\end{aligned}$$

for any $z \in \Gamma(F,\nu,N)$.

Let $\omega \in \Lambda^q(F,\nu,N)$ be arbitrary. Since

$$\begin{aligned}
&\left(d^F \circ d^F\right)\omega\left(z_1, ..., z_{q+2}\right) = i_{z_{q+2}} \circ ... \circ i_{z_1} \circ \left(d^F \circ d^F\right)\omega = ...\\
&= i_{z_{q+2}} \circ \left(d^F \circ d^F\right) \circ i_{z_{q+1}}\left(\omega\left(z_1, ..., z_q\right)\right)\\
&= i_{z_{q+2}} \circ \left(d^F \circ d^F\right)(0) = 0,\ \forall z_1, ..., z_{q+2} \in \Gamma\left(F,\nu,N\right),
\end{aligned}$$

it results the conclusion of affirmation 3. *q.e.d.*

Theorem 3.2.9 *If $((F,\nu,N),[,]_{F,h},(\rho,\eta))$ is a generalized Lie algebroid and d^F is the exterior differentiation operator for the exterior differential $\mathcal{F}(N)$-algebra $(\Lambda(F,\nu,N),+,\cdot,\wedge)$, then we obtain the structure equations of Maurer-Cartan type*

$$(\mathcal{C}_1)\qquad d^F t^\alpha = -\frac{1}{2}L^\alpha_{\beta\gamma}t^\beta \wedge t^\gamma,\ \alpha \in \overline{1,p}$$

and

$$(\mathcal{C}_2)\qquad d^F \varkappa^{\tilde{\imath}} = \theta^{\tilde{\imath}}_\alpha t^\alpha,\ \tilde{\imath} \in \overline{1,n},$$

where $\left\{t^\alpha, \alpha \in \overline{1,p}\right\}$ is the coframe of the vector bundle (F,ν,N).

This equations will be called *the structure equations of Maurer-Cartan type associated to the generalized Lie algebroid* $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.

Proof. Let $\alpha \in \overline{1,p}$ be arbitrary. Since

$$d^F t^\alpha (t_\beta, t_\gamma) = -L^\alpha_{\beta\gamma},\ \forall \beta,\gamma \in \overline{1,p}$$

it results that

$$d^F t^\alpha = -\sum_{\beta<\gamma} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma. \tag{1}$$

Since $L^\alpha_{\beta\gamma} = -L^\alpha_{\gamma\beta}$ and $t^\beta \wedge t^\gamma = -t^\gamma \wedge t^\beta$, for nay $\beta,\gamma \in \overline{1,p}$, it results that

$$\sum_{\beta<\gamma} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma = \frac{1}{2} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma \tag{2}$$

Using the equalities (1) and (2) it results the structure equation $(\mathcal{C}_1)$.

Let $\tilde{\imath} \in \overline{1,n}$ be arbitrarily. Since

$$d^F \varkappa^{\tilde{\imath}} (t_\alpha) = \theta^{\tilde{\imath}}_\alpha,\ \forall \alpha \in \overline{1,p}$$

it results the structure equation $(\mathcal{C}_2)$. *q.e.d.*

Corollary 3.2.1 *If* $\left((h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M\right)\right)$ *is the pull-back Lie algebroid associated to the generalized Lie algebroid* $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta)\right)$ *and* d^{h^*F} *is the exterior differentiation operator for the exterior differential* $\mathcal{F}(M)$*-algebra* $(\Lambda(h^*F, h^*\nu, M), +, \cdot, \wedge)$*, then we obtain the following structure equations of Maurer-Cartan type*

$$d^{h^*F} T^\alpha = -\frac{1}{2}\left(L^\alpha_{\beta\gamma} \circ h\right) T^\beta \wedge T^\gamma,\ \alpha \in \overline{1,p} \tag{$\mathcal{C}'_1$}$$

and

$$d^{h^*F} x^i = \left(\rho^i_\alpha \circ h\right) T^\alpha,\ i \in \overline{1,m}. \tag{$\mathcal{C}'_2$}$$

This equations will be called *the structure equations of Maurer-Cartan type associated to the pull-back Lie algebroid*

$$\left((h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M\right)\right).$$

Theorem 3.2.10 (of Cartan type) *Let* (E,π,M) *be an IDS of the generalized Lie algebroid* $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta)\right)$. *If* $\left\{\Theta^{r+1}, ..., \Theta^p\right\}$ *is a base for the* $\mathcal{F}(M)$*-submodule* $\left(\Gamma\left(E^\perp, \pi^\perp, M\right), +, \cdot\right)$*, then the IDS* (E,π,M) *is involutive if and only if it exists*

$$\Omega^\alpha_\beta \in \Lambda^1(h^*F, h^*\nu, M),\ \alpha,\beta \in \overline{r+1,p}$$

such that

$$d^{h^*F}\Theta^\alpha = \Sigma_{\beta\in\overline{r+1,p}} \Omega^\alpha_\beta \wedge \Theta^\beta \in \mathcal{I}\left(\Gamma\left(E^\perp, \pi^\perp, M\right)\right).$$

Proof: Let $\{S_1, ..., S_r\}$ be a base for the $\mathcal{F}(M)$-submodule $(\Gamma(E,\pi,M), +, \cdot)$
Let $\{S_{r+1}, ..., S_p\} \in \Gamma(h^*F, h^*\nu, M)$ such that

$$\{S_1, ..., S_r, S_{r+1}, ..., S_p\}$$

is a base for the $\mathcal{F}(M)$-module

$$\left(\Gamma\left(h^{*}F,h^{*}\nu,M\right),+,\cdot\right).$$

Let $\Theta^{1},...,\Theta^{r}\in\Gamma\left(\overset{*}{h^{*}F},\overset{*}{h^{*}\nu},M\right)$ such that

$$\left\{\Theta^{1},...,\Theta^{r},\Theta^{r+1},...,\Theta^{p}\right\}$$

is a base for the $\mathcal{F}(M)$-module

$$\left(\Gamma\left(\overset{*}{h^{*}F},\overset{*}{h^{*}\nu},M\right),+,\cdot\right).$$

For any $a,b\in\overline{1,r}$ and $\alpha,\beta\in\overline{r+1,p}$, we have the equalities:

$$\begin{array}{rcl}\Theta^{a}\left(S_{b}\right)&=&\delta_{b}^{a}\\ \Theta^{a}\left(S_{\beta}\right)&=&0\\ \Theta^{\alpha}\left(S_{b}\right)&=&0\\ \Theta^{\alpha}\left(S_{\beta}\right)&=&\delta_{\beta}^{\alpha}\end{array}$$

We remark that the set of the 2-forms

$$\left\{\Theta^{a}\wedge\Theta^{b},\Theta^{a}\wedge\Theta^{\beta},\Theta^{\alpha}\wedge\Theta^{\beta},\ a,b\in\overline{1,r}\wedge\alpha,\beta\in\overline{r+1,p}\right\}$$

is a base for the $\mathcal{F}(M)$-module

$$\left(\Lambda^{2}\left(h^{*}F,h^{*}\nu,M\right),+,\cdot\right).$$

Therefore, we have

(1) $$d^{h^{*}F}\Theta^{\alpha}=\Sigma_{b<c}A_{bc}^{\alpha}\Theta^{b}\wedge\Theta^{c}+\Sigma_{b,\gamma}B_{b\gamma}^{\alpha}\Theta^{b}\wedge\Theta^{\gamma}+\Sigma_{\beta<\gamma}C_{\beta\gamma}^{\alpha}\Theta^{\beta}\wedge\Theta^{\gamma},$$

where, $A_{bc}^{\alpha},B_{b\gamma}^{\alpha}$ and $C_{\beta\gamma}^{\alpha},\ a,b,c\in\overline{1,r}\wedge\alpha,\beta,\gamma\in\overline{r+1,p}$ are real local functions such that $A_{bc}^{\alpha}=-A_{cb}^{\alpha}$ and $C_{\beta\gamma}^{\alpha}=-C_{\gamma\beta}^{\alpha}$.

Using the formula

(2)
$$d^{h^{*}F}\Theta^{\alpha}\left(S_{b},S_{c}\right)=\Gamma\left(\overset{h^{*}F}{\rho},Id_{M}\right)S_{b}\left(\Theta^{\alpha}\left(S_{c}\right)\right)-\Gamma\left(\overset{h^{*}F}{\rho},Id_{M}\right)S_{c}\left(\Theta^{\alpha}\left(S_{b}\right)\right)-\Theta^{\alpha}\left(\left[S_{b},S_{c}\right]_{h^{*}F}\right),$$

we obtain that

(3) $$A_{bc}^{\alpha}=-\Theta^{\alpha}\left(\left[S_{b},S_{c}\right]_{h^{*}F}\right),\ \forall\left(b,c\in\overline{1,r}\wedge\alpha\in\overline{r+1,p}\right).$$

We admit that (E,π,M) is an involutive IDS of the generalized Lie algebroid $\left(\left(F,\nu,N\right),[,]_{F,h},\left(\rho,\eta\right)\right)$.

As

$$\left[S_{b},S_{c}\right]_{h^{*}F}\in\Gamma\left(E,\pi,M\right),\ \forall b,c\in\overline{1,r}$$

it results that

$$\Theta^{\alpha}\left(\left[S_{b},S_{c}\right]_{h^{*}F}\right)=0,\ \forall\left(b,c\in\overline{1,r}\wedge\alpha\in\overline{r+1,p}\right).$$

Therefore,

$$A_{bc}^{\alpha}=0,\ \forall\left(b,c\in\overline{1,r}\wedge\alpha\in\overline{r+1,p}\right)$$

and we obtain
$$\begin{array}{rcl} d^{h^{*}F}\Theta^{\alpha} & = & \Sigma_{b,\gamma}B_{b\gamma}^{\alpha}\Theta^{b}\wedge\Theta^{\gamma}+\frac{1}{2}C_{\beta\gamma}^{\alpha}\Theta^{\beta}\wedge\Theta^{\gamma} \\ & = & \left(B_{b\gamma}^{\alpha}\Theta^{b}+\frac{1}{2}C_{\beta\gamma}^{\alpha}\Theta^{\beta}\right)\wedge\Theta^{\gamma}. \end{array}$$

As
$$\Omega_{\gamma}^{\alpha}\overset{put}{=}B_{b\gamma}^{\alpha}\Theta^{b}+\frac{1}{2}C_{\beta\gamma}^{\alpha}\Theta^{\beta}\in\Lambda^{1}\left(h^{*}F,h^{*}\nu,M\right),\ \forall\alpha,\beta\in\overline{r+1,p}$$
it results the first implication.

Conversely, we admit that it exists
$$\Omega_{\beta}^{\alpha}\in\Lambda^{1}\left(h^{*}F,h^{*}\nu,M\right),\ \alpha,\beta\in\overline{r+1,p}$$
such that

$$d^{h^{*}F}\Theta^{\alpha}=\Sigma_{\beta\in\overline{r+1,p}}\Omega_{\beta}^{\alpha}\wedge\Theta^{\beta},\ \forall\alpha\in\overline{r+1,p}. \tag{4}$$

Using the affirmations $(1),(2)$ and (4) we obtain that
$$A_{bc}^{\alpha}=0,\ \forall\left(b,c\in\overline{1,r}\wedge\alpha\in\overline{r+1,p}\right).$$

Using the affirmation (3), we obtain
$$\Theta^{\alpha}\left(\left[S_{b},S_{c}\right]_{h^{*}F}\right)=0,\ \forall\left(b,c\in\overline{1,r}\wedge\alpha\in\overline{r+1,p}\right).$$

Therefore,
$$\left[S_{b},S_{c}\right]_{h^{*}F}\in\Gamma\left(E,\pi,M\right),\ \forall b,c\in\overline{1,r}.$$

Using the *Proposition 3.1.3.2*, we obtain the second implication. *q.e.d.*

Let $\left(\left(F^{\prime},\nu^{\prime},N^{\prime}\right),\left[,\right]_{F^{\prime},h^{\prime}},\left(\rho^{\prime},\eta^{\prime}\right)\right)$ be an another generalized Lie algebroid.

Definition 3.2.8 For any morphism $\left(\varphi,\varphi_{0}\right)$ of $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$ source and $\left(\left(F^{\prime},\nu^{\prime},N^{\prime}\right),\left[,\right]_{F^{\prime},h^{\prime}},\left(\rho^{\prime},\eta^{\prime}\right)\right)$ target we define the application
$$\begin{array}{ccc} \Lambda^{q}\left(F^{\prime},\nu^{\prime},N^{\prime}\right) & \xrightarrow{\left(\varphi,\varphi_{0}\right)^{*}} & \Lambda^{q}\left(F,\nu,N\right) \\ \omega^{\prime} & \longmapsto & \left(\varphi,\varphi_{0}\right)^{*}\omega^{\prime} \end{array},$$
where
$$\left(\left(\varphi,\varphi_{0}\right)^{*}\omega^{\prime}\right)\left(z_{1},...,z_{q}\right)=\omega^{\prime}\left(\Gamma\left(\varphi,\varphi_{0}\right)\left(z_{1}\right),...,\Gamma\left(\varphi,\varphi_{0}\right)\left(z_{q}\right)\right),$$
for any $z_{1},...,z_{q}\in\Gamma\left(F,\nu,N\right)$.

Remark 3.2.5 It is remarked that the $\mathbf{B^v}$-morphism $(Th\circ\rho, h\circ\eta)$ is a **GLA**-morphism of

$$\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$$

source and

$$\left((TN,\tau_N,N),[,]_{TN,Id_N},(Id_{TN},Id_N)\right)$$

target.

Moreover, for any $\tilde{\imath}\in\overline{1,n}$, we obtain

$$(Th\circ\rho,h\circ\eta)^*\left(d\varkappa^{\tilde{\imath}}\right)=d^F\varkappa^{\tilde{\imath}},$$

where d is the exterior differentiation operator associated to the exterior differential Lie $\mathcal{F}(N)$-algebra

$$(\Lambda(TN,\tau_N,N),+,\cdot,\wedge).$$

Theorem 3.2.11 *If* (φ,φ_0) *is a morphism of*

$$\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$$

source and

$$\left((F',\nu',N'),[,]_{F',h'},(\rho',\eta')\right)$$

target, then the following affirmations are satisfied:

1. *For any* $\omega'\in\Lambda^q(F',\nu',N')$ *and* $\theta'\in\Lambda^r(F',\nu',N')$ *we obtain*

$$(\varphi,\varphi_0)^*\left(\omega'\wedge\theta'\right)=(\varphi,\varphi_0)^*\omega'\wedge(\varphi,\varphi_0)^*\theta'. \tag{3.2.13}$$

2. *For any* $z\in\Gamma(F,\nu,N)$ *and* $\omega'\in\Lambda^q(F',\nu',N')$ *we obtain*

$$i_z\left((\varphi,\varphi_0)^*\omega'\right)=(\varphi,\varphi_0)^*\left(i_{\varphi(z)}\omega'\right). \tag{3.2.14}$$

3. *If* $N=N'$ *and*

$$(Th\circ\rho,h\circ\eta)=\left(Th'\circ\rho',h'\circ\eta'\right)\circ(\varphi,\varphi_0),$$

then we obtain

$$(\varphi,\varphi_0)^*\circ d^{F'}=d^F\circ(\varphi,\varphi_0)^*. \tag{3.2.15}$$

Proof.

1. Let $\omega'\in\Lambda^q(F',\nu',N')$ and $\theta'\in\Lambda^r(F',\nu',N')$ be arbitrary. Since

$$\begin{aligned}(\varphi,\varphi_0)^*\left(\omega'\wedge\theta'\right)(z_1,...,z_{q+r})&=\left(\omega'\wedge\theta'\right)(\Gamma(\varphi,\varphi_0)z_1,...,\Gamma(\varphi,\varphi_0)z_{q+r})\\&=\frac{1}{(q+r)!}\sum_{\sigma\in\Sigma_{q+r}}sgn(\sigma)\cdot\omega'(\Gamma(\varphi,\varphi_0)z_1,...,\Gamma(\varphi,\varphi_0)z_q)\\&\qquad\cdot\theta'(\Gamma(\varphi,\varphi_0)z_{q+1},...,\Gamma(\varphi,\varphi_0)z_{q+r})\\&=\frac{1}{(q+r)!}\sum_{\sigma\in\Sigma_{q+r}}sgn(\sigma)\cdot(\varphi,\varphi_0)^*\omega'(z_1,...,z_q)(\varphi,\varphi_0)^*\theta'(z_{q+1},...,z_{q+r})\\&=\left((\varphi,\varphi_0)^*\omega'\wedge(\varphi,\varphi_0)^*\theta'\right)(z_1,...,z_{q+r}),\end{aligned}$$

for any $z_1, ..., z_{q+r} \in \Gamma(F, \nu, N)$, it results the conclusion of affirmation 1.

2. Let $z \in \Gamma(F, \nu, N)$ and $\omega' \in \Lambda^q(F', \nu', N')$ be arbitrary. Since

$$\begin{aligned} i_z\left((\varphi, \varphi_0)^* \omega'\right)(z_2, ..., z_q) &= \omega'\left(\Gamma(\varphi, \varphi_0) z, \Gamma(\varphi, \varphi_0) z_2, ..., \Gamma(\varphi, \varphi_0) z_q\right) \\ &= i_{\Gamma(\varphi, \varphi_0) z} \omega'\left(\Gamma(\varphi, \varphi_0) z_2, ..., \Gamma(\varphi, \varphi_0) z_q\right) \\ &= (\varphi, \varphi_0)^* \left(i_{\Gamma(\varphi, \varphi_0) z} \omega'\right)(z_2, ..., z_q), \end{aligned}$$

for any $z_2, ..., z_q \in \Gamma(F, \nu, N)$, it results the conclusion of affirmation 2.

3. Let $\omega' \in \Lambda^q(F', \nu', N')$ and $z_0, ..., z_q \in \Gamma(F, \nu, N)$ be arbitrary. Since

$$\begin{aligned} &\left((\varphi, \varphi_0)^* d^{F'} \omega'\right)(z_0, ..., z_q) = \left(d^{F'} \omega'\right)\left(\Gamma(\varphi, \varphi_0) z_0, ..., \Gamma(\varphi, \varphi_0) z_q\right) \\ &= \sum_{i=0}^{q} (-1)^i \Gamma(Th' \circ \rho', h' \circ \eta')\left(\Gamma(\varphi, \varphi_0) z_i\right) \\ &\quad \cdot \omega'\left(\left(\Gamma(\varphi, \varphi_0) z_0, \Gamma(\varphi, \varphi_0) z_1, ..., \Gamma \widehat{(\varphi, \varphi_0)} z_i, ..., \Gamma(\varphi, \varphi_0) z_q\right)\right) \\ &+ \sum_{0 \leq i < j} (-1)^{i+j} \cdot \omega'\left(\Gamma(\varphi, \varphi_0)[z_i, z_j]_F, \Gamma(\varphi, \varphi_0) z_0, \Gamma(\varphi, \varphi_0) z_1, ..., \right. \\ &\qquad \left. \cdot \Gamma \widehat{(\varphi, \varphi_0)} z_i, ..., \Gamma \widehat{(\varphi, \varphi_0)} z_j, ..., \Gamma(\varphi, \varphi_0) z_q\right) \end{aligned}$$

and

$$\begin{aligned} &d^F\left((\varphi, \varphi_0)^* \omega'\right)(z_0, ..., z_q) \\ &= \sum_{i=0}^{q} (-1)^i \Gamma(Th \circ \rho, h \circ \eta)(z_i) \cdot \left((\varphi, \varphi_0)^* \omega'\right)(z_0, ..., \widehat{z_i}, ..., z_q) \\ &+ \sum_{0 \leq i < j} (-1)^{i+j} \cdot \left((\varphi, \varphi_0)^* \omega'\right)\left([z_i, z_j]_{F,h}, z_0, ..., \widehat{z_i}, ..., \widehat{z_j}, ..., z_q\right) \\ &= \sum_{i=0}^{q} (-1)^i \Gamma(Th \circ \rho, h \circ \eta)(z_i) \cdot \omega'\left(\Gamma(\varphi, \varphi_0) z_0, ..., \Gamma \widehat{(\varphi, \varphi_0)} z_i, ..., \Gamma(\varphi, \varphi_0) z_q\right) \\ &+ \sum_{0 \leq i < j} (-1)^{i+j} \cdot \omega'\left(\Gamma(\varphi, \varphi_0)[z_i, z_j]_{F,h}, \Gamma(\varphi, \varphi_0) z_0, \Gamma(\varphi, \varphi_0) z_1, ..., \right. \\ &\left. \Gamma \widehat{(\varphi, \varphi_0)} z_i, ..., \Gamma \widehat{(\varphi, \varphi_0)} z_j, ..., \Gamma(\varphi, \varphi_0) z_q\right) \end{aligned}$$

it results the conclusion of affirmation 3. *q.e.d.*

Definition 3.2.9 For any $q \in \overline{1, n}$ we define

$$\mathcal{Z}^q(F, \nu, N) = \{\omega \in \Lambda^q(F, \nu, N) : d\omega = 0\},$$

the set of *closed differential exterior q-forms* and

$$\mathcal{B}^q(F, \nu, N) = \left\{\omega \in \Lambda^q(F, \nu, N) : \exists \eta \in \Lambda^{q-1}(F, \nu, N) \mid d\eta = \omega\right\},$$

the set of *exact differential exterior q-forms.*

3.2.1 Exterior Differential Systems

We consider a generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ given by the diagrams:

$$\begin{array}{ccccccc} & & F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\ & & \downarrow\nu & & \downarrow\tau_M & & \downarrow\tau_N \\ M & \xrightarrow{h} & N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \end{array}$$

Let $\left((h^*F,h^*\nu,M),[,]_{h^*F},\left(\overset{h^*F}{\rho},Id_M\right)\right)$ be the pull-back Lie algebroid.

Definition 3.2.1.1 Any ideal $(\mathcal{I},+,\cdot)$ of the exterior differential algebra of the pull-back Lie algebroid $\left((h^*F,h^*\nu,M),[,]_{h^*F},\left(\overset{h^*F}{\rho},Id_M\right)\right)$ closed under differentiation operator d^{h^*F}, namely $d^{h^*F}\mathcal{I}\subseteq\mathcal{I}$, will be called *differential ideal of the generalized Lie algebroid* $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.

In particular, if $h=Id_N=\eta$, then any ideal $(\mathcal{I},+,\cdot)$ of the exterior differential algebra of the Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_M))$ closed under differentiation operator d^F , namely $d^F\mathcal{I}\subseteq\mathcal{I}$, will be called *differential ideal of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_M))$.

Definition 3.2.1.2 Let $(\mathcal{I},+,\cdot)$ be a differential ideal of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ or of the Lie algebroid $((F,\nu,N),[,]_F,(\rho,Id_M))$ respectively.

If it exists an IDS (E,π,M) such that for all $k\in\mathbb{N}^*$ and $\omega\in\mathcal{I}\cap\Lambda^k(h^*F,h^*\nu,M)$ we have $\omega(u_1,...,u_k)=0$, for any $u_1,...,u_k\in\Gamma(E,\pi,M)$, then we will say that $(\mathcal{I},+,\cdot)$ *is an exterior differential system (EDS) of the generalized Lie algebroid* $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.

In particular, if $h=Id_N=\eta$ and it exists an IDS (E,π,M) such that for all $k\in\mathbb{N}^*$ and $\omega\in\mathcal{I}\cap\Lambda^k(F,\nu,M)$ we have $\omega(u_1,...,u_k)=0$, for any $u_1,...,u_k\in\Gamma(E,\pi,M)$, then we will say that $(\mathcal{I},+,\cdot)$ *is an exterior differential system (EDS) of the Lie algebroid* $((F,\nu,N),[,]_F,(\rho,Id_N))$.

Theorem 3.2.1.1 *The IDS (E,π,M) of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is involutive, if and only if the ideal generated by the $\mathcal{F}(M)$-submodule $\left(\Gamma\left(E^\perp,\pi^\perp,M\right),+,\cdot\right)$ is an EDS of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.*

Proof: Let (E,π,M) be an involutive IDS of the generalized Lie algebroid $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$.
Let $\left\{\Theta^{r+1},...,\Theta^p\right\}$ be a base for the $\mathcal{F}(M)$-submodule $\left(\Gamma\left(E^\perp,\pi^\perp,M\right),+,\cdot\right)$.
We know that

$$\mathcal{I}\left(\Gamma\left(E^\perp,\pi^\perp,M\right)\right)=\cup_{q\in\mathbb{N}}\left\{\Omega_\alpha\wedge\Theta^\alpha,\ \ \{\Omega_{r+1},...,\Omega_p\}\subset\Lambda^q(h^*F,h^*\nu,M)\right\}.$$

Let

$$S=\left\{q\in\mathbb{N}:d^{h^*F}(\Omega_\alpha\wedge\Theta^\alpha)\in\mathcal{I}\left(\Gamma\left(E^\perp,\pi^\perp,M\right)\right),\ \forall\{\Omega_{r+1},...,\Omega_p\}\subset\Lambda^q(h^*F,h^*\nu,M)\right\}.$$

Let $\{\Omega_{r+1},...,\Omega_p\}\subset\Lambda^0(h^*F,h^*\nu,M)$ be arbitrary.
Using the *Theorem 3.2.10*, we obtain

$$\begin{array}{rcl} d^{h^*F}(\Omega_\alpha\wedge\Theta^\alpha) & = & d^{h^*F}\Omega_\alpha\wedge\Theta^\alpha+(-1)^0\,\Omega_\alpha\wedge d^{h^*F}\Theta^\alpha \\ & = & \left(d^{h^*F}\Omega_\alpha+\Omega_\beta\cdot\Omega_\alpha^\beta\right)\wedge\Theta^\alpha. \end{array}$$

As

$$d^{h^*F}\Omega_\beta + \Omega_\alpha \cdot \Omega_\beta^\alpha \in \Lambda^1\left(h^*F, h^*\nu, M\right)$$

it results that

$$d^{h^*F}\left(\Omega_\beta \wedge \Theta^\beta\right) \in \mathcal{I}\left(\Gamma\left(E^\perp, \pi^\perp, M\right)\right)$$

Therefore,

$$0 \in S. \tag{1}$$

In the following, we prove that

$$q \in S \Longrightarrow q+1 \in S. \tag{2}$$

Let $\{\Omega_{r+1}, ..., \Omega_p\} \subset \Lambda^{q+1}\left(h^*F, h^*\nu, M\right)$ be arbitrary.
Using the *Theorem 3.2.10*, we obtain

$$\begin{array}{rcl} d^{h^*F}\left(\Omega_\alpha \wedge \Theta^\alpha\right) & = & d^{h^*F}\Omega_\alpha \wedge \Theta^\alpha + (-1)^{q+1}\,\Omega_\beta \wedge d^{h^*F}\Theta^\beta \\ & = & \left(d^{h^*F}\Omega_\alpha + (-1)^{q+1}\,\Omega_\beta \wedge \Omega_\alpha^\beta\right) \wedge \Theta^\alpha. \end{array}$$

As

$$d^{h^*F}\Omega_\alpha + (-1)^{q+1}\,\Omega_\beta \wedge \Omega_\alpha^\beta \in \Lambda^{q+2}\left(h^*F, h^*\nu, M\right)$$

it results that

$$d^{h^*F}\left(\Omega_\beta \wedge \Theta^\beta\right) \in \mathcal{I}\left(\Gamma\left(E^\perp, \pi^\perp, M\right)\right)$$

Therefore,

$$q+1 \in S.$$

Using the **Peano's Axiom** and the affirmations (1) and (2), it results that $S = \mathbb{N}$.
Therefore,

$$d^{h^*F}\mathcal{I}\left(\Gamma\left(E^\perp, \pi^\perp, M\right)\right) \subseteq \mathcal{I}\left(\Gamma\left(E^\perp, \pi^\perp, M\right)\right).$$

Conversely, let (E, π, M) be an IDS of the generalized Lie algebroid $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$ such that the $\mathcal{F}(M)$-submodule $\left(\mathcal{I}\left(\Gamma\left(E^\perp, \pi^\perp, M\right)\right), +, \cdot\right)$ is an EDS of the generalized Lie algebroid $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$. We have:

$$d^{h^*F}\mathcal{I}\left(\Gamma\left(E^\perp, \pi^\perp, M\right)\right) \subseteq \mathcal{I}\left(\Gamma\left(E^\perp, \pi^\perp, M\right)\right). \tag{3}$$

Let $\left\{\Theta^{r+1}, ..., \Theta^p\right\}$ be a base for the $\mathcal{F}(M)$-submodule $\left(\Gamma\left(E^\perp, \pi^\perp, M\right), +, \cdot\right)$.
Using the affirmation (3), it results that it exists

$$\Omega_\beta^\alpha \in \Lambda^1\left(h^*F, h^*\nu, M\right),\ \alpha, \beta \in \overline{r+1, p}$$

such that

$$d^{h^*F}\Theta^\alpha = \Sigma_{\beta \in \overline{r+1,p}}\Omega_\beta^\alpha \wedge \Theta^\beta \in \mathcal{I}\left(\Gamma\left(E^\perp, \pi^\perp, M\right)\right).$$

Using the *Theorem 3.2.10*, it results that (E, π, M) is an involutive IDS of the generalized Lie algebroid $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$. *q.e.d.*

3.3 The generalized tangent bundle

We consider the following diagram:

$$
(3.3.1)\qquad
\begin{tikzcd}
E \arrow[d, "\pi"'] & \left(F, [,]_{F,h}, (\rho, \eta)\right) \arrow[d, "\nu"] \\
M \arrow[r, "h"] & N
\end{tikzcd}
$$

where (E, π, M) is a fiber bundle and $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$ is a generalized Lie algebroid.

We assume that the r-dimensional manifold $\mathbf{V}$ is the type fibre and the Lie group $(\mathbf{G}, \cdot)$ is the structure group for the fiber bundle (E, π, M).

We take $\left(x^i, y^a\right)$ as canonical local coordinates on (E, π, M), where $i \in \overline{1, m}$ and $a \in \overline{1, r}$.

Let

$$\left(x^i, y^a\right) \longrightarrow \left(x^{\acute{\imath}}\left(x^i\right), y^{\acute{a}}\left(x^i, y^a\right)\right)$$

be a change of coordinates on (E, π, M). Then the coordinates y^a change to $y^{\acute{a}}$ by the rule:

$$(3.3.2)\qquad y^{\acute{a}} = \frac{\partial y^{\acute{a}}}{\partial y^a} y^a.$$

In particular, if (E, π, M) is vector bundle, then the coordinates y^a change to $y^{\acute{a}}$ by the rule:

$$(3.3.2')\qquad y^{\acute{a}} = M_a^{\acute{a}} y^a.$$

Let

$$(h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M\right)$$

be the pull-back Lie algebroid of the generalized Lie algebroid

$$\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right).$$

Let

$$\left(\pi^*\left(h^*F\right), \pi^*\left(h^*\nu\right), E\right), [,]_{\pi^*\left(h^*F\right)}, \left(\overset{\pi^*\left(h^*F\right)}{\rho}, Id_E\right)$$

be the pull-back Lie algebroid of the Lie algebroid

$$(h^*F, h^*\nu, M), [,]_{h^*F}, \left(\overset{h^*F}{\rho}, Id_M\right).$$

If

$$z = z^\alpha t_\alpha \in \Gamma(F, \nu, N),$$

then, using the vector fibred $(m + r) + p$-structure $\left[\widetilde{\mathcal{AF}}_{\pi^*\left(h^*F\right)}\right]$, we obtain the section

$$\tilde{Z} = \left(z^\alpha \circ h \circ \pi\right) \tilde{T}_\alpha \in \Gamma\left(\pi^*\left(h^*F\right), \pi^*\left(h^*\nu\right), E\right)$$

such that

$$\tilde{Z}(u_x) = z(h(x)),$$

for any $u_x \in \pi^{-1}\left(U \cap h^{-1}V\right)$.

The set $\left\{\tilde{T}_\alpha,\ \alpha \in \overline{1,p}\right\}$ is a base for the module of sections

$$\left(\Gamma\left(\pi^*\left(h^*F\right), \pi^*\left(h^*\nu\right), E\right), +, \cdot\right).$$

Since $t_{\alpha'} = \Lambda_{\alpha'}^{\alpha} t_\alpha$, it results that

$$\tilde{T}_{\alpha'} = \Lambda_{\alpha'}^{\alpha} \circ (h \circ \pi)\, \tilde{T}_\alpha. \tag{3.3.3}$$

Therefore,

$$\left\| \Lambda_{\alpha}^{\alpha'} \circ (h \circ \pi) \right\| \tag{3.3.4}$$

is the matrix of coordinate transformation on $\left(\pi^*\left(h^*F\right), \pi^*\left(h^*\nu\right), E\right)$.

Let $\left(\overset{\pi^*(h^*F)}{\rho}, Id_E\right)$ be the $\mathbf{B^v}$-morphism of $\left(\pi^*\left(h^*F\right), \pi^*\left(h^*\nu\right), E\right)$ source and $\left(TE, \tau_E, E\right)$ target, where

$$\begin{array}{ccc} \pi^*\left(h^*F\right) & \xrightarrow{\overset{\pi^*(h^*F)}{\rho}} & TE \\ \tilde{Z}^\alpha \tilde{T}_\alpha(u_x) & \longmapsto & \tilde{Z}^\alpha \cdot \left(\rho_\alpha^i \circ h \circ \pi\right) \cdot \dfrac{\partial}{\partial x^i}(u_x). \end{array} \tag{3.3.5}$$

Using the operation

$$\Gamma\left(\pi^*\left(h^*F\right), \pi^*\left(h^*\nu\right), E\right)^2 \xrightarrow{[,]_{\pi^*(h^*F)}} \Gamma\left(\pi^*\left(h^*F\right), \pi^*\left(h^*\nu\right), E\right)$$

defined by

$$\begin{array}{ll} \left[\tilde{T}_\alpha, \tilde{T}_\beta\right]_{\pi^*(h^*F)} & = \left(L_{\alpha\beta}^\gamma \circ h \circ \pi\right) \tilde{T}_\gamma, \\ \left[\tilde{T}_\alpha, f\tilde{T}_\beta\right]_{\pi^*(h^*F)} & = f\left(L_{\alpha\beta}^\gamma \circ h \circ \pi\right) \tilde{T}_\gamma + \left(\rho_\alpha^i \circ h \circ \pi\right) \dfrac{\partial f}{\partial x^i} \tilde{T}_\beta, \\ \left[f\tilde{T}_\alpha, \tilde{T}_\beta\right]_{\pi^*(h^*F)} & = -\left[\tilde{T}_\beta, f\tilde{T}_\alpha\right]_{\pi^*(h^*F)}, \end{array} \tag{3.3.6}$$

for any $f \in \mathcal{F}(E)$, we obtain the following

Theorem 3.3.1 *The couple*

$$\left([,]_{\pi^*(h^*F)}, \left(\overset{\pi^*(h^*F)}{\rho}, Id_E\right)\right)$$

is a Lie algebroid structure for the vector bundle $\left(\pi^*\left(h^*F\right), \pi^*\left(h^*\nu\right), E\right)$.

It is known that the tangent bundle $\left(TE, \tau_E, E\right)$ is a vector bundle with type fibre the real space $\left(\mathbb{R}^{m+r}, +, \cdot\right)$ and structure group the Lie group $\mathbf{GL}(m+r, \mathbb{R})$.

Theorem 3.3.2 *The set*

$$\pi^*\left(h^*F\right) \oplus TE = \bigcup_{u \in E} \pi^*\left(h^*F\right)_u \oplus (TE)_u \tag{3.3.7}$$

is the total space of a vector bundle with the base E, canonical projection denoted $\overset{\oplus}{\pi}$, type fibre the real space $\left(\mathbb{R}^{p+(m+r)},+,\cdot\right)$ and structure group, a Lie subgroup of $\left(\mathbf{GL}\left(p+(m+r),\mathbb{R}\right),\cdot\right)$.

Let

$$\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^a}\right)$$

be the natural base for the sections Lie algebra $\left(\Gamma\left(TE,\tau_E,E\right),+,\cdot,[,]_{TE}\right)$.

Remark 3.3.1 The sections

$$\left(\tilde{T}_\alpha,\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^a}\right)\right) \tag{3.3.8}$$

determined the bases for the $\mathcal{F}(M)$ module $\left(\Gamma\left(\pi^*\left(h^*F\right)\oplus TE,\overset{\oplus}{\pi},E\right),+,\cdot\right)$.

The matrix of coordinate transformation on

$$\left(\pi^*\left(h^*F\right)\oplus TE,\overset{\oplus}{\pi},E\right)$$

at a change of fibred charts is

$$\left\Vert\begin{array}{ccc}\Lambda_\alpha^{\acute{\alpha}}\circ h\circ\pi & 0 & 0\\ 0 & \dfrac{\partial x^{\acute{i}}}{\partial x_i}\circ\pi & 0\\ 0 & \dfrac{\partial y^{\acute{a}}}{\partial x^i} & \dfrac{\partial y^{\acute{a}}}{\partial y^a}\end{array}\right\Vert. \tag{3.3.9}$$

In particular, if (E,π,M) is a vector bundle, then we consider that the local coordinates on (E,π,M) is changed by:

$$\left(x^i,y^a\right)\longrightarrow\left(x^{\acute{i}}\left(x^i\right),y^{\acute{a}}=M_a^{\acute{a}}\left(x^i\right)y^a\right).$$

Then the matrix of coordinate transformation on

$$\left(\pi^*\left(h^*F\right)^*F\oplus TE,\overset{\oplus}{\pi},E\right)$$

at a change of fibred charts is

$$\left\Vert\begin{array}{ccc}\Lambda_\alpha^{\acute{\alpha}}\circ h\circ\pi & 0 & 0\\ 0 & \dfrac{\partial x^{\acute{i}}}{\partial x_i}\circ\pi & 0\\ 0 & \dfrac{\partial M_a^{\acute{a}}\circ\pi}{\partial x^i}y^a & M_a^{\acute{a}}\circ\pi\end{array}\right\Vert. \tag{3.3.10}$$

For any sections

$$\tilde{Z}^\alpha\tilde{T}_\alpha\in\Gamma\left(\pi^*\left(h^*F\right),\pi^*\left(h^*F\right),E\right)$$

and

$$Y^a\frac{\partial}{\partial y^a}\in\Gamma\left(VTE,\tau_E,E\right)$$

we construct the section

$$\begin{aligned}\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y^{a}\frac{\partial}{\partial\tilde{y}^{a}} &=: \tilde{Z}^{\alpha}\left(\tilde{T}_{\alpha}\oplus\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\frac{\partial}{\partial x^{i}}\right)+Y^{a}\left(0_{\pi^{*}(h^{*}F)}\oplus\frac{\partial}{\partial y^{a}}\right)\\ &= \tilde{Z}^{\alpha}\tilde{T}_{\alpha}\oplus\left(\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\frac{\partial}{\partial x^{i}}+Y^{a}\frac{\partial}{\partial y^{a}}\right)\in\Gamma\left(\pi^{*}\left(h^{*}F\right)\oplus TE,\overset{\oplus}{\pi},E\right).\end{aligned}$$

Since we have

$$\begin{gathered}\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y^{a}\frac{\partial}{\partial\tilde{y}^{a}}=0\\ \Updownarrow\\ \tilde{Z}^{\alpha}\tilde{T}_{\alpha}=0\ \wedge\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\frac{\partial}{\partial x^{i}}+Y^{a}\frac{\partial}{\partial y^{a}}=0,\end{gathered}$$

it implies $\tilde{Z}^{\alpha}=0,\ \alpha\in\overline{1,p}$ and $Y^{a}=0,\ a\in\overline{1,r}$.

Therefore the sections $\frac{\partial}{\partial\tilde{z}^{1}},...,\frac{\partial}{\partial\tilde{z}^{p}},\frac{\partial}{\partial\tilde{y}^{1}},...,\frac{\partial}{\partial\tilde{y}^{r}}$ are linearly independent.

We consider the vector subbundle $((\rho,\eta)TE,(\rho,\eta)\tau_{E},E)$ of the vector bundle $\left(\pi^{*}\left(h^{*}F\right)\oplus TE,\overset{\oplus}{\pi},E\right)$, for which the $\mathcal{F}(E)$-module of sections is the $\mathcal{F}(E)$-submodule of $\left(\Gamma\left(\pi^{*}\left(h^{*}F\right)\oplus TE,\overset{\oplus}{\pi},E\right),+,\cdot\right)$, generated by the family of sections $\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^{a}}\right)$.

The base sections

$$\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^{a}}\right)\overset{put}{=}\left(\tilde{\partial}_{\alpha},\dot{\tilde{\partial}}_{a}\right)\tag{3.3.11}$$

will be called the *natural* (ρ,η)*-base.*

The matrix of coordinate transformation on $((\rho,\eta)TE,(\rho,\eta)\tau_{E},E)$ at a change of fibred charts is

$$\left\Vert\begin{array}{cc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\pi & 0\\ \left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\frac{\partial y^{\acute{a}}}{\partial x^{i}} & \frac{\partial y^{\acute{a}}}{\partial y^{a}}\end{array}\right\Vert.\tag{3.3.12}$$

In particular, if (E,π,M) is a vector bundle, then the matrix of coordinate transformation on $((\rho,\eta)TE,(\rho,\eta)\tau_{E},E)$ at a change of fibred charts is

$$\left\Vert\begin{array}{cc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\pi & 0\\ \left(\rho_{a}^{i}\circ h\circ\pi\right)\frac{\partial M_{b}^{\acute{a}}\circ\pi}{\partial x_{i}}y^{b} & M_{a}^{\acute{a}}\circ\pi\end{array}\right\Vert.\tag{3.3.13}$$

Next, we consider the operation

$$\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)^{2}\ \xrightarrow{[,]_{(\rho,\eta)TE}}\ \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$$

defined by

$$\begin{aligned}&\left[\left(\tilde{Z}_{1}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{1}^{a}\frac{\partial}{\partial\tilde{y}^{a}}\right),\left(\tilde{Z}_{2}^{\beta}\frac{\partial}{\partial\tilde{z}^{\beta}}+Y_{2}^{b}\frac{\partial}{\partial\tilde{y}^{b}}\right)\right]_{(\rho,\eta)TE}\\ &=\left[\tilde{Z}_{1}^{\alpha}\tilde{T}_{a},\tilde{Z}_{2}^{\beta}\tilde{T}_{\beta}\right]_{\pi^{*}(h^{*}F)}\oplus\left[\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\tilde{Z}_{1}^{\alpha}\frac{\partial}{\partial x^{i}}+Y_{1}^{a}\frac{\partial}{\partial y^{a}},\right.\\ &\qquad\left.\left(\rho_{\beta}^{i}\circ h\circ\pi\right)\tilde{Z}_{2}^{\beta}\frac{\partial}{\partial x^{i}}+Y_{2}^{b}\frac{\partial}{\partial y^{b}}\right]_{TE},\end{aligned}\tag{3.3.14}$$

for any $\left(\tilde{Z}_1^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y_1^a \frac{\partial}{\partial \tilde{y}^a}\right)$ and $\left(\tilde{Z}_2^\beta \frac{\partial}{\partial \tilde{z}^\beta} + Y_2^b \frac{\partial}{\partial \tilde{y}^b}\right)$.

Let $(\tilde{\rho}, Id_E)$ be the $\mathbf{B^v}$-morphism of $((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E)$ source and (TE, τ_E, E) target, where

$$
\begin{array}{c}
(\rho,\eta)\,TE \xrightarrow{\tilde{\rho}} TE \\
\left(\tilde{Z}^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y^a \frac{\partial}{\partial \tilde{y}^a}\right)(u_x) \longmapsto \left(\tilde{Z}^\alpha \left(\rho_\alpha^i \circ h \circ \pi\right) \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial y^a}\right)(u_x)
\end{array}
\tag{3.3.15}
$$

Lemma 3.3.1 *The operation $[,]_{(\rho,\eta)TE}$ is a Lie bracket, namely it satisfies*

$$
\left[\tilde{U}, f\tilde{Z}\right]_{(\rho,\eta)TE} = f\left[\tilde{U}, \tilde{Z}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\tilde{U}\right)(f)\,\tilde{Z}
\tag{3.3.16}
$$

for any $\tilde{U}, \tilde{Z} \in \Gamma\left((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E\right)$ and $f \in F(E)$.

Proof. For any $f \in \mathcal{F}(E)$, we obtain:

$$
\begin{array}{l}
\left[\frac{\partial}{\partial \tilde{z}^\alpha}, f\frac{\partial}{\partial \tilde{z}^\beta}\right]_{(\rho,\eta)TE} = \left[T_\alpha, fT_\beta\right]_{\pi^*(h^*F)} \oplus \left[\rho_\alpha^i \circ h \circ \pi \frac{\partial}{\partial x^i}, f\cdot\left(\rho_\beta^j \circ h \circ \pi\right)\frac{\partial}{\partial x^j}\right]_{TE} \\
= \left(f\left[\tilde{T}_\alpha, \tilde{T}_\beta\right]_{\pi^*(h^*F)} + \Gamma\left(\overset{\pi^*(h^*F)}{\rho}, Id_E\right)\left(\tilde{T}_\alpha\right) f\cdot \tilde{T}_\beta\right) \\
\qquad \oplus \left(f\left[\rho_\alpha^i \circ h \circ \pi \frac{\partial}{\partial x^i}, \rho_\beta^j \circ h \circ \pi \frac{\partial}{\partial x^j}\right]_{TE}\right. \\
\left. + \Gamma\left(Id_{TE}, Id_E\right)\left(\rho_\alpha^i \circ h \circ \pi \frac{\partial}{\partial x^i}\right) f\cdot \rho_\beta^j \circ h \circ \pi \frac{\partial}{\partial x^j}\right) \\
= f\left(\left[\tilde{T}_\alpha, \tilde{T}_\beta\right]_{\pi^*(h^*F)} \oplus \left[\rho_\alpha^i \circ h \circ \pi \frac{\partial}{\partial x^i}, \rho_\beta^j \circ h \circ \pi \frac{\partial}{\partial x^j}\right]_{TE}\right) \\
+ \rho_\alpha^i \circ h \circ \pi \frac{\partial f}{\partial x^i}\left(\tilde{T}_\beta \oplus \rho_\beta^j \circ h \circ \pi \frac{\partial}{\partial x^j}\right) \\
= f\left[\frac{\partial}{\partial \tilde{z}^\alpha}, \frac{\partial}{\partial \tilde{z}^\beta}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\frac{\partial}{\partial \tilde{z}^\alpha}\right) f\cdot \frac{\partial}{\partial \tilde{z}^\beta};
\end{array}
\tag{1}
$$

$$
\begin{array}{l}
\left[\frac{\partial}{\partial \tilde{z}^\alpha}, f\frac{\partial}{\partial \tilde{y}^b}\right]_{(\rho,\eta)TE} = \left[\tilde{T}_\alpha, 0\right]_{\pi^*(h^*F)} \oplus \left[\rho_\alpha^i \circ h \circ \pi \frac{\partial}{\partial x^i}, f\frac{\partial}{\partial y^b}\right]_{TE} \\
= 0_{\pi^*(h^*F)} \oplus \left(f\left[\rho_\alpha^i \circ h \circ \pi \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^b}\right]_{TE} + \rho_\alpha^i \circ h \circ \pi \frac{\partial f}{\partial x^i}\frac{\partial}{\partial y^b}\right) \\
= 0_{\pi^*(h^*F)} \oplus \left(-f\frac{\partial\left(\rho_\alpha^i \circ h \circ \pi\right)}{\partial y^b}\frac{\partial}{\partial x^i} + \rho_\alpha^i \circ h \circ \pi \frac{\partial f}{\partial x^i}\frac{\partial}{\partial y^b}\right) \\
\overset{(2.2.3)}{=} 0_{\pi^*(h^*F)} \oplus \left(0_{TE} + \rho_\alpha^i \circ h \circ \pi \frac{\partial f}{\partial x^i}\frac{\partial}{\partial y^b}\right) \\
= 0_{\pi^*(h^*F)} \oplus \left(\rho_\alpha^i \circ h \circ \pi\right)\frac{\partial f}{\partial x^i}\frac{\partial}{\partial y^b} \\
= \left(\rho_\alpha^i \circ h \circ \pi\right)\frac{\partial f}{\partial x^i}\left(0_{\pi^*(h^*F)} \oplus \frac{\partial}{\partial y^b}\right) \\
= f\left[\frac{\partial}{\partial \tilde{z}^\alpha}, \frac{\partial}{\partial \tilde{y}^b}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\frac{\partial}{\partial \tilde{z}^\alpha}\right) f\frac{\partial}{\partial \tilde{y}^b};
\end{array}
\tag{2}
$$

$$\begin{aligned}
&\left[\frac{\partial}{\partial\tilde{y}^a}, f\frac{\partial}{\partial\tilde{z}^\beta}\right]_{(\rho,\eta)TE} = \left[0, \tilde{T}_\beta\right]_{\pi^*(h^*F)} \oplus \left[\tfrac{\partial}{\partial y^a}, f\left(\rho^j_\beta\circ h\circ\pi\right)\tfrac{\partial}{\partial x^j}\right]_{TE}\\
&= 0_{\pi^*(h^*F)} \oplus \left(f\left[\frac{\partial}{\partial y^a}, \left(\rho^j_\beta\circ h\circ\pi\right)\frac{\partial}{\partial x^j}\right]_{TE} + \frac{\partial f}{\partial y^a}\left(\rho^j_\beta\circ h\circ\pi\right)\frac{\partial}{\partial x^j}\right)\\
&= 0_{\pi^*(h^*F)} \oplus \left(f\left(\rho^j_\beta\circ h\circ\pi\right)\left[\frac{\partial}{\partial y^a}, \frac{\partial}{\partial x^j}\right]_{TE}\right.\\
(3)\quad &\left.+f\frac{\partial\left(\rho^j_\beta\circ h\circ\pi\right)}{\partial y^a}\frac{\partial}{\partial x^j} + \frac{\partial f}{\partial y^a}\left(\rho^j_\beta\circ h\circ\pi\right)\frac{\partial}{\partial x^j}\right)\\
&\overset{(2.2.3)}{=} 0_{\pi^*(h^*F)} \oplus \left(0_{TE} + 0_{TE} + \frac{\partial f}{\partial y^a}\left(\rho^j_\beta\circ h\circ\pi\right)\frac{\partial}{\partial x^j}\right)\\
&= f\left[\frac{\partial}{\partial\tilde{y}^a}, \frac{\partial}{\partial\tilde{z}^\beta}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\frac{\partial}{\partial\tilde{y}^a}\right) f\frac{\partial}{\partial\tilde{z}^\beta};
\end{aligned}$$

$$\begin{aligned}
&\left[\frac{\partial}{\partial\tilde{y}^a}, f\frac{\partial}{\partial\tilde{y}^b}\right]_{(\rho,\eta)TE} = [0,0]_{\pi^*(h^*F)} \oplus \left[\tfrac{\partial}{\partial y^a}, f\tfrac{\partial}{\partial y^b}\right]_{TE}\\
&= 0_{\pi^*(h^*F)} \oplus \left(f\left[\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right]_{TE} + \frac{\partial f}{\partial y^a}\frac{\partial}{\partial y^b}\right)\\
(4)\quad &= 0_{\pi^*(h^*F)} \oplus \left(0_{TE} + \frac{\partial f}{\partial y^a}\frac{\partial}{\partial y^b}\right) = \left(0_{\pi^*(h^*F)} \oplus \frac{\partial f}{\partial y^a}\frac{\partial}{\partial y^b}\right)\\
&= \frac{\partial f}{\partial y^a}\frac{\partial}{\partial\tilde{y}^b} = f\left[\frac{\partial}{\partial\tilde{y}^a}, \frac{\partial}{\partial\tilde{y}^b}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\frac{\partial}{\partial\tilde{y}^a}\right) f\cdot\frac{\partial}{\partial\tilde{y}^b}.
\end{aligned}$$

In general, after some calculations, we obtain

$$(5)\qquad \left[\tilde{U}, f\tilde{Z}\right]_{(\rho,\eta)TE} = f\left[\tilde{U}, \tilde{Z}\right]_{(\rho,\eta)TE} + \Gamma\left(\tilde{\rho}, Id_E\right)\left(\tilde{U}\right)(f)\,\tilde{Z}$$

for any $\tilde{U}, \tilde{Z} \in \Gamma\left(\left(\rho,\eta\right)TE, \left(\rho,\eta\right)\tau_E, E\right)$ and $f \in \mathcal{F}(E)$. *q.e.d.*

Lemma 3.3.2 *For any* $\tilde{U}, \tilde{V} \in \Gamma\left(\left(\rho,\eta\right)TE, \left(\rho,\eta\right)\tau_E, E\right)$, *we have:*

$$(3.3.17)\qquad \left[\tilde{U}, \tilde{V}\right]_{(\rho,\eta)TE} = -\left[\tilde{V}, \tilde{U}\right]_{(\rho,\eta)TE}.$$

In particular, we obtain:

$$(3.3.18)\qquad \left[\tilde{U}, \tilde{U}\right]_{(\rho,\eta)TE} = 0_{(\rho,\eta)TE},\ \forall\tilde{U} \in \Gamma\left(\left(\rho,\eta\right)TE, \left(\rho,\eta\right)\tau_E, E\right).$$

Proof. Using relations (3.1.1.9) and (3.1.2.2), we obtain

$$\begin{aligned}
&\left(L^\gamma_{\alpha\beta}\circ h\circ\pi\right)\left(\rho^k_\gamma\circ h\circ\pi\right)\\
(1)\quad &\qquad = \left(\rho^i_\alpha\circ h\circ\pi\right)\frac{\partial\left(\rho^k_\beta\circ h\circ\pi\right)}{\partial x^i} - \left(\rho^j_\beta\circ h\circ\pi\right)\frac{\partial\left(\rho^k_\alpha\circ h\circ\pi\right)}{\partial x^j}.
\end{aligned}$$

Using relation (1), we obtain

$$
\begin{aligned}
&\left[\frac{\partial}{\partial \tilde{z}^{\alpha}},\frac{\partial}{\partial \tilde{z}^{\beta}}\right]_{(\rho,\eta)TE}=\\
&=\left(\left[\tilde{T}_{\alpha},\tilde{T}_{\beta}\right]_{\pi^*(h^*F)}\oplus\left[\rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial}{\partial x^{i}},\rho_{\beta}^{j}\circ h\circ\pi\frac{\partial}{\partial x^{j}}\right]_{TE}\right)\\
&=L_{\alpha\beta}^{\gamma}\circ(h\circ\pi)\tilde{T}_{\gamma}\oplus\left(\rho_{\alpha}^{i}\circ h\circ\pi\frac{\partial\left(\rho_{\beta}^{k}\circ h\circ\pi\right)}{\partial x^{i}}-\rho_{\beta}^{j}\circ h\circ\pi\frac{\partial\left(\rho_{\alpha}^{k}\circ h\circ\pi\right)}{\partial x^{j}}\right)\frac{\partial}{\partial x^{k}}\\
&=L_{\alpha\beta}^{\gamma}\circ(h\circ\pi)\left(\tilde{T}_{\gamma}\oplus\rho_{\gamma}^{k}\circ h\circ\pi\frac{\partial}{\partial x^{k}}\right)=L_{\alpha\beta}^{\gamma}\circ(h\circ\pi)\frac{\partial}{\partial \tilde{z}^{\gamma}}.
\end{aligned}
\tag{2}
$$

Moreover, we obtain

$$
\begin{aligned}
\left[\frac{\partial}{\partial \tilde{z}^{\alpha}},\frac{\partial}{\partial \tilde{y}^{b}}\right]_{(\rho,\eta)TE}&=\left[\tilde{T}_{\alpha},0\right]_{\pi^*(h^*F)}\oplus\left[\rho_{\alpha}^{i}\circ h\circ\pi\tfrac{\partial}{\partial x^{i}},\tfrac{\partial}{\partial y^{b}}\right]_{TE}\\
&=0_{\pi^*(h^*F)}\oplus\frac{-\partial\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)}{\partial y^{b}}\frac{\partial}{\partial x^{i}}=0_{\pi^*(h^*F)}\oplus 0_{TE};
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
\left[\frac{\partial}{\partial \tilde{y}^{a}},\frac{\partial}{\partial \tilde{z}^{\beta}}\right]_{(\rho,\eta)TE}&=\left[0,\tilde{T}_{\beta}\right]_{\pi^*(h^*F)}\oplus\left[\frac{\partial}{\partial y^{a}},\rho_{\beta}^{j}\circ h\circ\pi\frac{\partial}{\partial x^{j}}\right]_{TE}\\
&=0_{\pi^*(h^*F)}\oplus\frac{\partial\left(\rho_{\beta}^{j}\circ h\circ\pi\right)}{\partial y^{a}}\frac{\partial}{\partial x^{j}}=0_{\pi^*(h^*F)}\oplus 0_{TE};
\end{aligned}
\tag{4}
$$

$$
\left[\frac{\partial}{\partial \tilde{y}^{a}},\frac{\partial}{\partial \tilde{y}^{b}}\right]_{(\rho,\eta)TE}=[0,0]_{\pi^*(h^*F)}\oplus\left[\frac{\partial}{\partial y^{a}},\frac{\partial}{\partial y^{b}}\right]_{TE}=0_{\pi^*(h^*F)}\oplus 0_{TE}.
\tag{5}
$$

Using relations (2), (3), (4) and (5), we obtain:

$$
\begin{aligned}
\left[\frac{\partial}{\partial \tilde{z}^{\alpha}},\frac{\partial}{\partial \tilde{z}^{\beta}}\right]_{(\rho,\eta)TE}&=-\left[\frac{\partial}{\partial \tilde{z}^{\beta}},\frac{\partial}{\partial \tilde{z}^{\alpha}}\right]_{(\rho,\eta)TE},\\
\left[\frac{\partial}{\partial \tilde{z}^{\alpha}},\frac{\partial}{\partial \tilde{y}^{b}}\right]_{(\rho,\eta)TE}&=-\left[\frac{\partial}{\partial \tilde{y}^{b}},\frac{\partial}{\partial \tilde{z}^{\alpha}}\right]_{(\rho,\eta)TE},\\
\left[\frac{\partial}{\partial \tilde{y}^{a}},\frac{\partial}{\partial \tilde{z}^{\beta}}\right]_{(\rho,\eta)TE}&=-\left[\frac{\partial}{\partial \tilde{z}^{\beta}},\frac{\partial}{\partial \tilde{y}^{a}}\right]_{(\rho,\eta)TE},\\
\left[\frac{\partial}{\partial \tilde{y}^{a}},\frac{\partial}{\partial \tilde{y}^{b}}\right]_{(\rho,\eta)TE}&=-\left[\frac{\partial}{\partial \tilde{y}^{b}},\frac{\partial}{\partial \tilde{y}^{a}}\right]_{(\rho,\eta)TE}.
\end{aligned}
\tag{6}
$$

In general, for any $\tilde{U},\tilde{V}\in\Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right)$, we have:

$$
\left[\tilde{U},\tilde{V}\right]_{(\rho,\eta)TE}=-\left[\tilde{V},\tilde{U}\right]_{(\rho,\eta)TE}.
\tag{7}
$$

Since equality (7) implies

$$
2\left[\tilde{U},\tilde{U}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE},\ \forall\tilde{U}\in\Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right),
$$

we obtain:

$$
\left[\tilde{U},\tilde{U}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE},\ \forall\tilde{U}\in\Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right).
$$

q.e.d.

Lemma 3.3.3 *We have the Jacobi identity:*

$$\begin{aligned}(3.3.19)\qquad &\left[\tilde{U},\left[\tilde{V},\tilde{Z}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}+\left[\tilde{Z},\left[\tilde{U},\tilde{V}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}\\ &+\left[\tilde{V},\left[\tilde{Z},\tilde{U}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE}.\end{aligned}$$

for any $\tilde{U},\tilde{V},\tilde{Z}\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right).$

Proof. After some calculations, using the sections of natural (ρ,η)-base, we obtain the following Jacobi identities:

$$\begin{aligned}(1)\quad &\left[\frac{\partial}{\partial\tilde{z}^{\alpha}},\left[\frac{\partial}{\partial\tilde{z}^{\beta}},\frac{\partial}{\partial\tilde{z}^{\gamma}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}+\left[\frac{\partial}{\partial\tilde{z}^{\beta}},\left[\frac{\partial}{\partial\tilde{z}^{\gamma}},\frac{\partial}{\partial\tilde{z}^{\alpha}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}\\ &+\left[\frac{\partial}{\partial\tilde{z}^{\gamma}},\left[\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{z}^{\beta}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE},\end{aligned}$$

$$\begin{aligned}(2)\quad &\left[\frac{\partial}{\partial\tilde{y}^{a}},\left[\frac{\partial}{\partial\tilde{y}^{b}},\frac{\partial}{\partial\tilde{z}^{\gamma}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}+\left[\frac{\partial}{\partial\tilde{y}^{b}},\left[\frac{\partial}{\partial\tilde{z}^{\gamma}},\frac{\partial}{\partial\tilde{y}^{a}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}\\ &+\left[\frac{\partial}{\partial\tilde{z}^{\gamma}},\left[\frac{\partial}{\partial\tilde{y}^{a}},\frac{\partial}{\partial\tilde{y}^{b}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE},\end{aligned}$$

$$\begin{aligned}(3)\quad &\left[\frac{\partial}{\partial\tilde{y}^{a}},\left[\frac{\partial}{\partial\tilde{y}^{b}},\frac{\partial}{\partial\tilde{y}^{c}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}+\left[\frac{\partial}{\partial\tilde{y}^{b}},\left[\frac{\partial}{\partial\tilde{y}^{c}},\frac{\partial}{\partial\tilde{y}^{a}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}\\ &+\left[\frac{\partial}{\partial\tilde{y}^{c}},\left[\frac{\partial}{\partial\tilde{y}^{a}},\frac{\partial}{\partial\tilde{y}^{b}}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE}.\end{aligned}$$

After some calculations, we obtain the Jacobi identity

$$\begin{aligned}&\left[\tilde{U},\left[\tilde{V},\tilde{Z}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}+\left[\tilde{Z},\left[\tilde{U},\tilde{V}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}\\ &+\left[\tilde{V},\left[\tilde{Z},\tilde{U}\right]_{(\rho,\eta)TE}\right]_{(\rho,\eta)TE}=0_{(\rho,\eta)TE},\end{aligned}$$

for any $\tilde{U},\tilde{V},\tilde{Z}\in\Gamma\left(\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)\right)$ *q.e.d.*

Lemma 3.3.4 *The* **Mod**-*morphism*

$$\Gamma\left(\tilde{\rho},Id_{E}\right)$$

is a **Liealg**-*morphism of*

$$\left(\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right),+,\cdot,\left[,\right]_{(\rho,\eta)TE}\right)$$

source and

$$\left(\Gamma\left(TE,\tau_E,E\right),+,\cdot,[,]_{TE}\right)$$

target.

Proof. Indeed, we have:

$$
\begin{array}{ll}
 & \left[\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{z}^\alpha},\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{z}^\beta}\right]_{TE} \\
 & =\left[\left(\rho_\alpha^i\circ h\circ\pi\right)\frac{\partial}{\partial x^i},\left(\rho_\beta^j\circ h\circ\pi\right)\frac{\partial}{\partial x^j}\right]_{TE} \\
(1) & =\left(\left(\rho_\alpha^i\circ h\circ\pi\right)\frac{\partial\left(\rho_\beta^k\circ h\circ\pi\right)}{\partial x^i}-\left(\rho_\beta^j\circ h\circ\pi\right)\frac{\partial\left(\rho_\alpha^k\circ h\circ\pi\right)}{\partial x^j}\right)\frac{\partial}{\partial x^k} \\
 & \underset{(3.1.1.9)}{\overset{(2.2.2)}{=}}\left(L_{\alpha\beta}^\gamma\circ h\circ\pi\right)\left(\rho_\gamma^k\circ h\circ\pi\right)\frac{\partial}{\partial x^k}=\left(L_{\alpha\beta}^\gamma\circ h\circ\pi\right)\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{z}^\gamma} \\
 & =\Gamma\left(\tilde{\rho},Id_E\right)\left[\frac{\partial}{\partial\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{z}^\beta}\right]_{(\rho,\eta)TE},
\end{array}
$$

$$
\begin{array}{ll}
 & \left[\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{z}^\alpha},\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{y}^b}\right]_{TE}=\left[\left(\rho_\alpha^i\circ h\circ\pi\right)\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^b}\right]_{TE} \\
(2) & =-\frac{\partial\left(\rho_\alpha^i\circ h\circ\pi\right)}{\partial y^b}\frac{\partial}{\partial x^i}=0_{TE}=\Gamma\left(\tilde{\rho},Id_E\right)\left(0_{(\rho,\eta)TE}\right) \\
 & =\Gamma\left(\tilde{\rho},Id_E\right)\left[\frac{\partial}{\partial\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{y}^b}\right]_{(\rho,\eta)TE},
\end{array}
$$

$$
\begin{array}{ll}
 & \left[\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{y}^a},\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{z}^\beta}\right]_{TE}=\left[\frac{\partial}{\partial y^a},\left(\rho_\beta^j\circ h\circ\pi\right)\frac{\partial}{\partial x^j}\right]_{TE} \\
(3) & =\frac{\partial\left(\rho_\beta^j\circ h\circ\pi\right)}{\partial y^a}\frac{\partial}{\partial x^j}=0_{TE}=\Gamma\left(\tilde{\rho},Id_E\right)\left(0_{(\rho,\eta)TE}\right) \\
 & =\Gamma\left(\tilde{\rho},Id_E\right)\left[\frac{\partial}{\partial\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{y}^b}\right]_{(\rho,\eta)TE}
\end{array}
$$

and

$$
\begin{array}{ll}
 & \left[\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{y}^a},\Gamma\left(\tilde{\rho},Id_E\right)\frac{\partial}{\partial\tilde{y}^b}\right]_{TE}=\left[\frac{\partial}{\partial y^a},\frac{\partial}{\partial y^b}\right]_{TE} \\
(4) & =0_{TE}=\Gamma\left(\tilde{\rho},Id_E\right)\left(0_{(\rho,\eta)TE}\right)=\Gamma\left(\tilde{\rho},Id_E\right)\left[\frac{\partial}{\partial\tilde{y}^a},\frac{\partial}{\partial\tilde{y}^b}\right]_{(\rho,\eta)TE}.
\end{array}
$$

In general, for any $\tilde{U},\tilde{Z}\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$, we obtain:

$$\left[\Gamma\left(\tilde{\rho},Id_E\right)(\tilde{U}),\Gamma\left(\tilde{\rho},Id_E\right)(\tilde{Z})\right]_{TE}=\Gamma\left(\tilde{\rho},Id_E\right)\left(\left[\tilde{U},\tilde{Z}\right]_{(\rho,\eta)TE}\right).$$

q.e.d.

Using *Lemmas 3.3.1, 3.3.2, 3.3.3* and *3.3.4,* we obtain the following

Theorem 3.3.4 *The couple*

$$\left([,]_{(\rho,\eta)TE},\left(\tilde{\rho},Id_E\right)\right)$$

is a Lie algebroid structure for the vector bundle

$$\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right).$$

Remark 3.3.2 In particular, if $h = Id_M$ and $[,]_{TM}$ is the usual Lie bracket, it results that the Lie algebroid

$$\left(\left((Id_{TM}, Id_M)\,TE, (Id_{TM}, Id_M)\,\tau_E, E\right), [,]_{(Id_{TM},Id_M)TE}, \left(\widetilde{Id_{TM}}, Id_E\right)\right)$$

is isomorphic with the usual Lie algebroid

$$\left((TE, \tau_E, E), [,]_{TE}, (Id_{TE}, Id_E)\right).$$

This is a reason for which the Lie algebroid

$$\left(\left((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E\right), [,]_{(\rho,\eta)TE}, (\tilde{\rho}, Id_E)\right),$$

will be called the *Lie algebroid generalized tangent bundle.*

The vector bundle $((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E)$ will be called the *generalized tangent bundle.*

3.3.1 The generalized tangent bundle of dual vector bundle

Let (E, π, M) be a vector bundle. We build the generalized tangent bundle of dual vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ using the diagram:

(3.3.1.1)
$$\begin{array}{ccc} \overset{*}{E} & & \left(F, [,]_{F,h}, (\rho,\eta)\right) \\ \overset{*}{\pi}\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array},$$

where $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta)\right)$ is a generalized Lie algebroid.

We take $\left(x^i, p_a\right)$ as canonical local coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, where $i \in \overline{1,m}$ and $a \in \overline{1,r}$.

Consider

$$\left(x^i, p_a\right) \longrightarrow \left(x^{\acute{i}}\left(x^i\right), p_{\acute{a}}\left(x^i, p_a\right)\right)$$

a change of coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$. Then the coordinates p_a change to $p_{\acute{a}}$ by the rule:

(3.3.1.2)
$$p_{\acute{a}} = M_{\acute{a}}^{a} p_a.$$

Let

$$\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_a}\right)$$

be the natural base for the sections Lie algebra $\left(\Gamma\left(T\overset{*}{E}, \tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot, [,]_{T\overset{*}{E}}\right)$.

The sections

(3.3.1.3)
$$\left(\tilde{T}_\alpha, \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_a}\right)\right)$$

determine a base for the module $\Gamma\left(\overset{**}{\pi}(h^*F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right)$.

The matrix of coordinate transformation on

$$\left(\overset{**}{\pi}(h^*F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right)$$

at a change of fibred charts is

$$(3.3.1.4)\qquad \left\Vert \begin{array}{ccc} \Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\overset{*}{\pi} & 0 & 0\\ 0 & \dfrac{\partial x^{\acute{i}}}{\partial x_i}\circ\overset{*}{\pi} & 0\\ 0 & \dfrac{\partial M_{\acute{a}}^{a}\circ\overset{*}{\pi}}{\partial x^i}p_a & M_{\acute{a}}^{a}\circ\overset{*}{\pi} \end{array}\right\Vert .$$

For any sections

$$\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\in\Gamma\left(\overset{**}{\pi}(h^*F),\overset{**}{\pi}(h^*\nu),\overset{*}{E}\right)$$

and

$$Y_a\frac{\partial}{\partial p_a}\in\Gamma\left(VT\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

we construct the section

$$\begin{array}{c}\tilde{Z}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_a\dfrac{\partial}{\partial\tilde{p}_a}=:\tilde{Z}^{\alpha}\left(T_{\alpha}\oplus\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial}{\partial x^i}\right)+Y_a\left(0_{\overset{**}{\pi}(h^*F)}\oplus\dfrac{\partial}{\partial p_a}\right)\\ =\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\oplus\left(\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial}{\partial x^i}+Y_a\dfrac{\partial}{\partial p_a}\right)\in\Gamma\left(\overset{**}{\pi}(h^*F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right).\end{array}$$

Since we have

$$\begin{array}{c}\tilde{Z}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_a\dfrac{\partial}{\partial\tilde{p}_a}=0_{\overset{**}{\pi}(h^*F)\oplus T\overset{*}{E}}\\ \Updownarrow\\ \tilde{Z}^{\alpha}\tilde{T}_{\alpha}=0_{\overset{**}{\pi}(h^*F)}\wedge\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial}{\partial x^i}+Y_a\dfrac{\partial}{\partial p_a}=0_{T\overset{*}{E}},\end{array}$$

it implies $\tilde{Z}^{\alpha}=0,\ \alpha\in\overline{1,p}$ and $Y_a=0,\ a\in\overline{1,r}$.

Therefore, the sections

$$\frac{\partial}{\partial\tilde{z}^1},...,\frac{\partial}{\partial\tilde{z}^p},\frac{\partial}{\partial\tilde{p}_1},...,\frac{\partial}{\partial\tilde{p}_r}$$

are linearly independent.

We consider the vector subbundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

of vector bundle

$$\left(\overset{**}{\pi}(h^*F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right),$$

for which the $\mathcal{F}\left(\overset{*}{E}\right)$-module of sections is the $\mathcal{F}\left(\overset{*}{E}\right)$-submodule of

$$\left(\Gamma\left(\overset{*}{\pi}{}^{*}\left(h^{*}F\right)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right),+,\cdot\right),$$

generated by the family of sections $\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{p}_{a}}\right)$.

The base sections

$$\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{p}_{a}}\right)\overset{put}{=}\left(\tilde{\partial}_{\alpha},\dot{\tilde{\partial}}^{a}\right) \tag{3.3.1.5}$$

will be called the *natural* (ρ,η)*-base.*

The matrix of coordinate transformation on $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ at a change of fibred charts is

$$\left\|\begin{array}{cc}\Lambda_{\alpha}^{\acute{\alpha}}\circ h\circ\overset{*}{\pi} & 0\\ \left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial M_{\acute{a}}^{b}\circ\overset{*}{\pi}}{\partial x_{i}}p_{b} & M_{\acute{a}}^{a}\circ\overset{*}{\pi}\end{array}\right\|. \tag{3.3.1.6}$$

We consider the operation $[,]_{(\rho,\eta)T\overset{*}{E}}$ defined by

$$\begin{array}{l}\left[\left(\tilde{Z}_{1}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{1a}\dfrac{\partial}{\partial\tilde{p}_{a}}\right),\left(\tilde{Z}_{2}^{\beta}\dfrac{\partial}{\partial\tilde{z}^{\beta}}+Y_{2b}\dfrac{\partial}{\partial\tilde{p}_{b}}\right)\right]_{(\rho,\eta)T\overset{*}{E}}=\\ =\left[\tilde{Z}_{1}^{\alpha}\tilde{T}_{a},\tilde{Z}_{2}^{\beta}\tilde{T}_{\beta}\right]_{\overset{*}{\pi}{}^{*}(h^{*}F)}\oplus\left[\left(\rho_{\alpha}^{i}\circ h\circ\pi^{*}\right)\tilde{Z}_{1}^{\alpha}\dfrac{\partial}{\partial x^{i}}+Y_{1a}\dfrac{\partial}{\partial p_{a}},\right.\\ \qquad\qquad\left.\left(\rho_{\beta}^{i}\circ h\circ\pi^{*}\right)\tilde{Z}_{2}^{\beta}\dfrac{\partial}{\partial x^{i}}+Y_{2b}\dfrac{\partial}{\partial p_{b}}\right]_{T\overset{*}{E}},\end{array} \tag{3.3.1.7}$$

for any sections $\left(\tilde{Z}_{1}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{1a}\frac{\partial}{\partial\tilde{p}_{a}}\right)$ and $\left(\tilde{Z}_{2}^{\beta}\frac{\partial}{\partial\tilde{z}^{\beta}}+Y_{2b}\frac{\partial}{\partial\tilde{p}_{b}}\right)$.

Let $\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)$ be the $\mathbf{B^{v}}$-morphism of $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ source and $\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right)$ target, where

$$\begin{array}{c}(\rho,\eta)\,T\overset{*}{E}\xrightarrow{\tilde{\rho}^{*}}T\overset{*}{E}\\ \left(\tilde{Z}^{\alpha}\dfrac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{a}\dfrac{\partial}{\partial\tilde{p}_{a}}\right)(\overset{*}{u}_{x})\longmapsto\left(\tilde{Z}^{\alpha}\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\dfrac{\partial}{\partial x^{i}}+Y_{a}\dfrac{\partial}{\partial p_{a}}\right)(\overset{*}{u}_{x})\end{array} \tag{3.3.1.8}$$

The Lie algebroid generalized tangent bundle of the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be denoted

$$\left(\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),[,]_{(\rho,\eta)T\overset{*}{E}},\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\right).$$

3.4 (Linear) (ρ, η)-connections

The theory of (linear) connections constitutes undoubtedly one of most beautiful and most important chapter of differential geometry, which has been widely explored in the literature (see [8, 11, 14, 26, 31, 41, 42, 45, 46, 47, 50, 51, 59, 60, 62, 63]).

In the following, we consider the diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \left(F, [,]_{F,h}, (\rho, \eta)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E, \pi, M) \in |\mathbf{B}|$ and $\left((F, \nu, N), [,]_{F,h}, (\rho, \eta)\right)$ is a generalized Lie algebroid.

Let

$$\left(((\rho, \eta) TE, (\rho, \eta) \tau_E, E), [,]_{(\rho,\eta)TE}, (\tilde{\rho}, Id_E)\right)$$

be the Lie algebroid generalized tangent bundle of fiber bundle (E, π, M).

We consider the $\mathbf{B^v}$-morphism

$$((\rho, \eta) \pi!, Id_E)$$

given by the commutative diagram

$$\begin{tikzcd} (\rho, \eta) TE \arrow[r, "(\rho,\eta)\pi!"] \arrow[d, "(\rho,\eta)\tau_E"'] & \pi^* (h^* F) \arrow[d, "pr_1"] \\ E \arrow[r, "id_E"] & E \end{tikzcd} \tag{3.4.1}$$

Using the components, this is defined as:

$$(\rho, \eta) \pi! \left(\left(\tilde{Z}^\alpha \frac{\partial}{\partial \tilde{z}^\alpha} + Y^a \frac{\partial}{\partial \tilde{y}^a}\right)(u_x)\right) = \left(\tilde{Z}^\alpha \tilde{T}_\alpha\right)(u_x), \tag{3.4.2}$$

for any $\left(\tilde{Z}^\alpha \dfrac{\partial}{\partial \tilde{z}^\alpha} + Y^a \dfrac{\partial}{\partial \tilde{y}^a}\right) \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$.

We define the *tangent* (ρ, η)*-application* as being a $\mathbf{B^v}$-morphism

$$((\rho, \eta) T\pi, h \circ \pi) = (pr_2, h \circ \pi) \circ ((\rho, \eta) \pi!, Id_E) \tag{3.4.3}$$

of $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ source and (F, ν, N) target.

Definition 3.4.1 The kernel of the tangent (ρ, η)-application

$$((\rho, \eta) T\pi, h \circ \pi)$$

is written as

$$(V (\rho, \eta) TE, (\rho, \eta) \tau_E, E)$$

and will be called *the vertical subbundle.*

The set $\left\{\frac{\partial}{\partial\tilde{y}^a},\ a\in\overline{1,r}\right\}$ is a base for the $\mathcal{F}(E)$-module

$$\left(\Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),+,\cdot\right).$$

Proposition 3.4.1 *The short sequence of vector bundles*

$$\begin{tikzcd} 0 \arrow[r, hook, "i"] \arrow[d] & V(\rho,\eta)TE \arrow[r, hook, "i"] \arrow[d] & (\rho,\eta)TE \arrow[r, "(\rho,\eta)\pi!"] \arrow[d] & \pi^*\left(h^{*F}\right) \arrow[r] \arrow[d] & 0 \arrow[d] \\ E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \arrow[r, "Id_E"] & E \end{tikzcd} \tag{3.4.4}$$

is exact.

Definition 3.4.2 *A* **Man**-morphism $(\rho,\eta)\Gamma$ of $(\rho,\eta)TE$ source and $V(\rho,\eta)TE$ target defined by

$$(\rho,\eta)\Gamma\left(\tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y^a\frac{\partial}{\partial\tilde{y}^a}\right)(u_x)=\left(Y^a+(\rho,\eta)\Gamma^a_\alpha\tilde{Z}^\alpha\right)\frac{\partial}{\partial\tilde{y}^a}(u_x), \tag{3.4.5}$$

such that the $\mathbf{B^v}$-morphism $((\rho,\eta)\Gamma,Id_E)$ is a split to the left in the previous exact sequence, will be called (ρ,η)*-connection for the fiber bundle* (E,π,M).

The differentiable real local functions $(\rho,\eta)\Gamma^a_\alpha$ will be called the *components of* (ρ,η)*-connection* $(\rho,\eta)\Gamma$.

The (ρ,Id_M)-connection for the fiber bundle (E,π,M) will be called ρ*-connection for the fiber bundle* (E,π,M) and will be denoted $\rho\Gamma$.

The (Id_{TM},Id_M)-connection for the fiber bundle (E,π,M) will be called *connection for the fiber bundle* (E,π,M) and will be denoted Γ.

Definition 3.4.3 If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the fiber bundle (E,π,M), then the kernel of the $\mathbf{B^v}$-morphism $((\rho,\eta)\Gamma,Id_E)$ is written as

$$\left(H\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$$

and will be called the *horizontal vector subbundle.*

Definition 3.4.4 If $(E,\pi,M)\in|\mathbf{B}|$, then the **B**-morphism (Π,π) defined by the commutative diagram

$$\begin{tikzcd} V\left(\rho,\eta\right)TE \arrow[r, "\Pi"] \arrow[d, "(\rho,\eta)\tau_E"'] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"] & M \end{tikzcd} \tag{3.4.6}$$

such that the components of the image of vector $Y^a\frac{\partial}{\partial\tilde{y}^a}(u_x)$ are the real numbers $Y^1(u_x),...,Y^r(u_x)$ will be called the *canonical projection* **B***-morphism.*

Example 3.4.1 If $(E,\pi,M)\in|\mathbf{B^v}|$, then the $\mathbf{B^v}$-morphism (Π,π) defined by the commutative diagram (3.4.6), where Π is defined by

$$\Pi\left(Y^a\frac{\partial}{\partial\tilde{y}^a}(u_x)\right)=Y^a(u_x)\,s_a(x), \tag{3.4.7}$$

is canonical projection $\mathbf{B^v}$-morphism.

The set $\left\{s_a, a \in \overline{1,r}\right\}$ is the base of $\mathcal{F}(M)$-module of sections $(\Gamma(E,\pi,M),+,\cdot)$.

Theorem 3.4.1 *If* $(\rho,\eta)\Gamma$ *is a* (ρ,η)*-connection for the fiber bundle* (E,π,M)*, then its components satisfy the law of transformation*

$$(3.4.8) \qquad (\rho,\eta)\Gamma_{\acute{\gamma}}^{\acute{a}} = \frac{\partial y^{\acute{a}}}{\partial y^{a}}\left[\rho_{\gamma}^{i}\circ(h\circ\pi)\frac{\partial y^{a}}{\partial x^{i}} + (\rho,\eta)\Gamma_{\gamma}^{a}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ(h\circ\pi).$$

If $(\rho,\eta)\Gamma$ *is a* (ρ,η)*-connection for the vector bundle* (E,π,M)*, then its components satisfy the law of transformation*

$$(3.4.8') \qquad (\rho,\eta)\Gamma_{\acute{\gamma}}^{\acute{a}} = M_{a}^{\acute{a}}\circ\pi\left[\rho_{\gamma}^{i}\circ(h\circ\pi)\frac{\partial M_{\acute{b}}^{a}\circ\pi}{\partial x^{i}}y^{\acute{b}} + (\rho,\eta)\Gamma_{\gamma}^{a}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ(h\circ\pi).$$

If $\rho\Gamma$ *is a* ρ*-connection for the vector bundle* (E,π,M) *and* $h = Id_M$*, then relations* $(3.4.8')$ *become*

$$(3.4.8'') \qquad \rho\Gamma_{\acute{\gamma}}^{\acute{a}} = M_{a}^{\acute{a}}\circ\pi\left[\rho_{\gamma}^{i}\circ\pi\frac{\partial M_{\acute{b}}^{a}\circ\pi}{\partial x^{i}}y^{\acute{b}} + \rho\Gamma_{\gamma}^{a}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ\pi.$$

In particular, if $(\rho,\eta) = (Id_{TM}, Id_M)$*, then the relations* $(3.4.8'')$ *become*

$$(3.4.8''') \qquad \Gamma_{\acute{k}}^{\acute{i}} = \frac{\partial x^{\acute{i}}}{\partial x^{i}}\circ\pi\left[\frac{\partial}{\partial x^{k}}\left(\frac{\partial x^{i}}{\partial x^{\acute{j}}}\circ\pi\right)y^{\acute{j}} + \Gamma_{k}^{i}\right]\frac{\partial x^{k}}{\partial x^{\acute{k}}}\circ\pi.$$

Proof. Let (Π,π) be the canonical projection $\mathbf{B}$-morphism.

Obviously, the components of

$$\Pi\circ(\rho,\eta)\Gamma\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}} + Y^{\acute{a}}\frac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x)$$

are the real numbers

$$\left(Y^{\acute{a}} + (\rho,\eta)\Gamma_{\acute{\gamma}}^{\acute{a}}\tilde{Z}^{\acute{\gamma}}\right)(u_x).$$

Since

$$\begin{aligned}\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}} + Y^{\acute{a}}\frac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x) &= \tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\frac{\partial}{\partial\tilde{z}^{\alpha}}(u_x)\\ &+\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\pi\frac{\partial y^{a}}{\partial x^{\acute{i}}} + \frac{\partial y^{a}}{\partial y^{\acute{a}}}Y^{\acute{a}}\right)\frac{\partial}{\partial\tilde{y}^{a}}(u_x),\end{aligned}$$

it results that the components of

$$\Pi\circ(\rho,\eta)\Gamma\left(\tilde{Z}^{\acute{\alpha}}\frac{\partial}{\partial\tilde{z}^{\acute{\alpha}}} + Y^{\acute{a}}\frac{\partial}{\partial\tilde{y}^{\acute{a}}}\right)(u_x)$$

are the real numbers

$$\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\pi\frac{\partial y^{a}}{\partial x^{\acute{i}}} + \frac{\partial y^{a}}{\partial y^{\acute{a}}}Y^{\acute{a}} + (\rho,\eta)\Gamma_{\alpha}^{a}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\right)(u_x)\frac{\partial y^{\acute{a}}}{\partial y^{a}},$$

where

$$\left\|\frac{\partial y^{a}}{\partial y^{\acute{a}}}\right\| = \left\|\frac{\partial y^{\acute{a}}}{\partial y^{a}}\right\|^{-1}.$$

Therefore, we have:

$$\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ\pi\frac{\partial y^{a}}{\partial x^{\acute{i}}}+\frac{\partial y^{a}}{\partial y^{\acute{a}}}Y^{\acute{a}}+\left(\rho,\eta\right)\Gamma_{\alpha}^{a}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi\right)\frac{\partial y^{\acute{a}}}{\partial y^{a}}=Y^{\acute{a}}+\left(\rho,\eta\right)\Gamma_{\acute{\alpha}}^{\acute{a}}\tilde{Z}^{\acute{\alpha}}.$$

After some calculations we obtain:

$$\left(\rho,\eta\right)\Gamma_{\acute{\alpha}}^{\acute{a}}=\frac{\partial y^{\acute{a}}}{\partial y^{a}}\left(\rho_{\alpha}^{i}\circ\left(h\circ\pi\right)\frac{\partial y^{a}}{\partial x^{i}}+\left(\rho,\eta\right)\Gamma_{\alpha}^{a}\right)\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\pi. \qquad q.e.d.$$

Remark 3.4.1 If we have a set of real local functions $\left(\rho,\eta\right)\Gamma_{\gamma}^{a}$ which satisfies the relations of passing (3.4.8) , then we have a (ρ,η)-connection $\left(\rho,\eta\right)\Gamma$ for the fiber bundle $\left(E,\pi,M\right)$.

Example 3.4.1 If Γ is a classical connection for the vector bundle (E,π,M) on components Γ_{k}^{a}, then the differentiable real local functions

$$\left(\rho,\eta\right)\Gamma_{\gamma}^{a}=\left(\rho_{\gamma}^{k}\circ h\circ\pi\right)\Gamma_{k}^{a}$$

are the components of a (ρ,η)-connection $\left(\rho,\eta\right)\Gamma$ for the vector bundle (E,π,M) which will be called the *(ρ,η)-connection associated to the connection* Γ.

Definition 3.4.5 If $\left(\rho,\eta\right)\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M), then for any

$$z=z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right)$$

the application

$$(3.4.9)\qquad \begin{array}{ccc}\Gamma\left(E,\pi,M\right) & \stackrel{\left(\rho,\eta\right)D_{v}}{\longrightarrow} & \Gamma\left(E,\pi,M\right)\\ u=u^{a}s_{a} & \longmapsto & \left(\rho,\eta\right)D_{z}u\end{array}$$

where

$$\left(\rho,\eta\right)D_{z}u=z^{\alpha}\circ h\left(\rho_{\alpha}^{i}\circ h\frac{\partial u^{a}}{\partial x^{i}}+\left(\rho,\eta\right)\Gamma_{\alpha}^{a}\circ u\right)s_{a}$$

will be called the *covariant (ρ,η)-derivative associated to (ρ,η)-connection $\left(\rho,\eta\right)\Gamma$ with respect to the section z.*

If $h=Id_{M}$ and $\eta=Id_{M}$, then we obtain the *covariant ρ-derivative associated to ρ-connection $\rho\Gamma$ with respect to the section z.*

In addition, if $\rho=Id_{TM}$, then we obtain the *covariant derivative associated to connection Γ with respect to the vector field z.*

Remark 3.4.2 If $\left(\rho,\eta\right)\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M), then the operator

$$\begin{array}{ccc}\Gamma\left(F,\nu,N\right)\times\Gamma\left(E,\pi,M\right) & \stackrel{\left(\rho,\eta\right)D}{\longrightarrow} & \Gamma\left(E,\pi,M\right)\\ \left(z,u\right) & \longmapsto & \left(\rho,\eta\right)D_{z}u\end{array}$$

satisfies the following properties:

(i) $\left(\rho,\eta\right)D$ is $\mathbb{R}$-bilinear;

(ii) $\left(\rho,\eta\right)D_{f_{1}z_{1}+f_{2}z_{2}}u=f_{1}\left(\rho,\eta\right)D_{z_{1}}u+f_{2}\left(\rho,\eta\right)D_{z_{2}}u$;

(iii) if $u \in \Gamma(E,\pi,M)$ is null on a nonempty subset of M, then $(\rho,\eta) D_z u$ is null on the same nonempty subset, for any $z \in \Gamma(F,\nu,N)$.

Definition 3.4.6 We will say that the *(ρ,η)-connection $(\rho,\eta)\Gamma$ is homogeneous* or *linear* if the local real functions $(\rho,\eta)\Gamma_{\gamma}^{a}$ are homogeneous or linear on the fibre of the fiber bundle (E,π,M).

Remark 3.4.3 If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the fiber bundle (E,π,M), then for each local vector $(m+r)$-chart (U,s_U) and for each local vector $(n+p)$-chart (V,t_V) such that $U\cap h^{-1}(V)\neq\phi$, it exists the differentiable real functions $\rho\Gamma_{b\gamma}^{a}$ defined on $U\cap h^{-1}(V)$ such that

$$(\rho,\eta)\Gamma_{\gamma}^{a}\circ u=(\rho,\eta)\Gamma_{b\gamma}^{a}\cdot u^{b},\forall u=u^{b}s_{b}\in\Gamma(E,\pi,M). \tag{3.4.10}$$

The differentiable real local functions $(\rho,\eta)\Gamma_{b\alpha}^{a}$ will be called the *Christoffel coefficients of linear (ρ,η)-connection $(\rho,\eta)\Gamma$*.

Theorem 3.4.2 *If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the fiber bundle (E,π,M), then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma_{\acute{b}\acute{\gamma}}^{\acute{a}}=\frac{\partial y^{\acute{a}}}{\partial y^{a}}\left[\rho_{\gamma}^{k}\circ h\frac{\partial}{\partial x^{k}}\left(\frac{\partial y^{a}}{\partial y^{\acute{b}}}\right)+(\rho,\eta)\Gamma_{b\gamma}^{a}\frac{\partial y^{b}}{\partial y^{\acute{b}}}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ h. \tag{3.4.11}$$

If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M), then its components satisfy the law of transformation

$$(\rho,\eta)\Gamma_{\acute{b}\acute{\gamma}}^{\acute{a}}=M_{a}^{\acute{a}}\left[\rho_{\gamma}^{k}\circ h\frac{\partial M_{\acute{b}}^{a}}{\partial x^{k}}+(\rho,\eta)\Gamma_{b\gamma}^{a}M_{\acute{b}}^{b}\right]\Lambda_{\acute{\gamma}}^{\gamma}\circ h. \tag{3.4.11$'$}$$

If $\rho\Gamma$ is a ρ-connection for the vector bundle (E,π,M) and $h=Id_M$, then the relations (3.4.11′) *become*

$$\rho\Gamma_{\acute{b}\acute{\gamma}}^{\acute{a}}=M_{a}^{\acute{a}}\left[\rho_{\gamma}^{k}\frac{\partial M_{\acute{b}}^{a}}{\partial x^{k}}+\rho\Gamma_{b\gamma}^{a}M_{\acute{b}}^{b}\right]\Lambda_{\acute{\gamma}}^{\gamma}. \tag{3.4.11$''$}$$

In particular, if $(\rho,\eta)=(Id_{TM},Id_M)$, then the relations (3.4.11″) *become*

$$\Gamma_{\acute{j}\acute{k}}^{\acute{i}}=\frac{\partial x^{\acute{i}}}{\partial x^{i}}\left[\frac{\partial}{\partial x^{k}}\left(\frac{\partial x^{i}}{\partial x^{\acute{j}}}\right)+\Gamma_{jk}^{i}\frac{\partial x^{j}}{\partial x^{\acute{j}}}\right]\frac{\partial x^{k}}{\partial x^{\acute{k}}}. \tag{3.4.11$'''$}$$

Definition 3.4.7 We say that the (linear) (ρ,η)-connection $(\rho,\eta)\Gamma$ for the fiber bundle (E,π,M) is differentiable of C^r class, if its components are differentiable of C^r class.

Definition 3.4.8 If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M), then for any

$$z=z^{\alpha}t_{\alpha}\in\Gamma(F,\nu,N)$$

the application

$$\begin{array}{ccc}\Gamma(E,\pi,M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma(E,\pi,M)\\ u=u^{a}s_{a} & \longmapsto & (\rho,\eta)D_{z}u\end{array} \tag{3.4.12}$$

defined by

$$(\rho,\eta)D_{z}u{=}z^{\alpha}{\circ}h\left(\rho_{\alpha}^{i}{\circ}h\frac{\partial u^{a}}{\partial x^{i}}{+}(\rho,\eta)\Gamma_{b\alpha}^{a}\cdot u^{b}\right)s_{a},$$

will be called the *covariant* (ρ,η)*-derivative associated to linear* (ρ,η)*-connection* $(\rho,\eta)\Gamma$ *with respect to the section* z.

If $h=Id_M$ and $\eta=Id_M$, then we obtain the *covariant* ρ*-derivative associated to linear* ρ*-connection* $\rho\Gamma$ *with respect to the section* z.

In addition, if $\rho=Id_{TM}$, then we obtain the *covariant derivative associated to linear connection* Γ *with respect to the vector field* z.

3.4.1 (Linear) (ρ,η)-connections for dual of vector bundles

Let (E,π,M) be a vector bundle.

We consider the following diagram:

(3.4.1.1)
$$\begin{array}{ccc} \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \overset{*}{\pi}\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array},$$

where $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let

$$\left(\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),[,]_{(\rho,\eta)T\overset{*}{E}},\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\right)$$

be the Lie algebroid generalized tangent bundle of the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

We consider the $\mathbf{B^v}$-morphism $\left((\rho,\eta)\,\overset{*}{\pi}!,Id_{\overset{*}{E}}\right)$ given by the commutative diagram

(3.4.1.2)
$$\begin{array}{ccc} (\rho,\eta)\,T\overset{*}{E} & \xrightarrow{(\rho,\eta)\overset{*}{\pi}!} & \overset{*}{\pi}{}^{*}\,(h^{*}F) \\ {\scriptstyle (\rho,\eta)\tau_{\overset{*}{E}}}\downarrow & & \downarrow{\scriptstyle pr_1} \\ \overset{*}{E} & \xrightarrow{id_{\overset{*}{E}}} & \overset{*}{E} \end{array}$$

Using the components, this is defined as:

(3.4.1.3)
$$(\rho,\eta)\,\overset{*}{\pi}!\left(\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{a}\frac{\partial}{\partial\tilde{p}_{a}}\right)\left(\overset{*}{u}_{x}\right)=\left(\tilde{Z}^{\alpha}\tilde{T}_{\alpha}\right)\left(\overset{*}{u}_{x}\right),$$

for any $\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y_{a}\frac{\partial}{\partial\tilde{p}_{a}}\in\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

We define the *tangent* (ρ,η)*-application* as being a $\mathbf{B^v}$-morphism

(3.4.1.4)
$$\left((\rho,\eta)\,T\overset{*}{\pi},h\circ\overset{*}{\pi}\right)=\left(pr_{2},h\circ\overset{*}{\pi}\right)\circ\left((\rho,\eta)\,\overset{*}{\pi}!,Id_{\overset{*}{E}}\right)$$

of $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ source and (F,ν,N) target.

Definition 3.4.1.1 The kernel of the tangent (ρ,η)-application

$$\left((\rho,\eta)\,T\overset{*}{\pi},h\circ\overset{*}{\pi}\right)$$

is written as

$$\left(V\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and will be called the *vertical subbundle*.

The set $\left\{\dfrac{\partial}{\partial\tilde{p}_a},\ a\in\overline{1,r}\right\}$ is a base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module

$$\left(\Gamma\left(V\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right).$$

Proposition 3.4.1.1 *The short sequence of vector bundles*

$$\begin{tikzcd}
0 \arrow[r, hook, "i"] \arrow[d] & V(\rho,\eta)T\overset{*}{E} \arrow[r, hook, "i"] \arrow[d] & (\rho,\eta)T\overset{*}{E} \arrow[r, "(\rho,\eta)\overset{*}{\pi}!"] \arrow[d] & \overset{*}{\pi}^{*}\left(h^{*}F\right) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\overset{*}{E} \arrow[r, "Id_{\overset{*}{E}}"] & \overset{*}{E} \arrow[r, "Id_{\overset{*}{E}}"] & \overset{*}{E} \arrow[r, "Id_{\overset{*}{E}}"] & \overset{*}{E} \arrow[r, "Id_{\overset{*}{E}}"] & \overset{*}{E}
\end{tikzcd}$$

is exact.

Definition 3.4.1.2 *A* **Man**-morphism $(\rho,\eta)\,\overset{*}{\Gamma}$ of $(\rho,\eta)\,T\overset{*}{E}$ source and $V\,(\rho,\eta)\,T\overset{*}{E}$ target defined by

$$(3.4.1.5)\qquad (\rho,\eta)\,\Gamma\left(\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y_b\frac{\partial}{\partial\tilde{p}_b}\right)\left(\overset{*}{u}_x\right)=\left(Y_b-(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\tilde{Z}^{\alpha}\right)\frac{\partial}{\partial\tilde{p}_b}\left(\overset{*}{u}_x\right),$$

such that the $\mathbf{B^v}$-morphism $\left((\rho,\eta)\,\overset{*}{\Gamma},Id_{\overset{*}{E}}\right)$ is a split to the left in the previous exact sequence, will be called (ρ,η)-*connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

The differentiable real local functions $(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}$ will be called the *components of* (ρ,η)-*connection* $(\rho,\eta)\,\overset{*}{\Gamma}$.

The (ρ,Id_M)-connection for the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be called ρ-*connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and will be denoted $\rho\overset{*}{\Gamma}$.

The (Id_{TM},Id_M)-connection for the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ will be called *connection for the dual vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and will be denoted $\overset{*}{\Gamma}$.

Let $\left\{s^a,\ a\in\overline{1,r}\right\}$ be the dual base of the base $\left\{s_a,a\in\overline{1,r}\right\}$.

The $\mathbf{B^v}$-morphism $\left(\overset{*}{\Pi},\overset{*}{\pi}\right)$ defined by the commutative diagram

$$
\text{(3.4.1.6)}\qquad
\begin{array}{ccc}
V(\rho,\eta)\,T\overset{*}{E} & \xrightarrow{\ \overset{*}{\Pi}\ } & \overset{*}{E} \\
{\scriptstyle (\rho,\eta)\tau_{\overset{*}{E}}}\downarrow & & \downarrow{\scriptstyle \overset{*}{\pi}} \\
\overset{*}{E} & \xrightarrow{\ \overset{*}{\pi}\ } & M
\end{array}\ ,
$$

where, $\overset{*}{\Pi}$ is defined by

$$
\text{(3.4.1.7)}\qquad \overset{*}{\Pi}\left(Y_a\frac{\partial}{\partial\tilde{p}_a}\left(\overset{*}{u}_x\right)\right) = Y_a\left(\overset{*}{u}_x\right)s^a\left(\overset{*}{\pi}\left(\overset{*}{u}_x\right)\right),
$$

is canonical projection $\mathbf{B^v}$-morphism.

Theorem 3.4.1.1 *If* $(\rho,\eta)\overset{*}{\Gamma}$ *is a* (ρ,η)*-connection for the vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, *then its components satisfy the law of transformation*

$$
\text{(3.4.1.8)}\qquad (\rho,\eta)\overset{*}{\Gamma}_{b'\gamma'} = M_{b'}^{b}\circ\overset{*}{\pi}\left[-\rho^i_{\gamma}\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_b^{a'}\circ\overset{*}{\pi}}{\partial x^i}p_{a'} + (\rho,\eta)\overset{*}{\Gamma}_{b\gamma}\right]\Lambda^{\gamma}_{\gamma'}\circ\left(h\circ\overset{*}{\pi}\right).
$$

In particular, if $h = Id_M$*, then the relations* (3.4.1.8) *become*

$$
\text{(3.4.1.8}'\text{)}\qquad (\rho,\eta)\overset{*}{\Gamma}_{b'\gamma'} = M_{b'}^{b}\circ\overset{*}{\pi}\left[-\rho^i_{\gamma}\circ\overset{*}{\pi}\cdot\frac{\partial M_b^{a'}\circ\overset{*}{\pi}}{\partial x^i}p_{a'} + (\rho,\eta)\overset{*}{\Gamma}_{b\gamma}\right]\Lambda^{\gamma}_{\gamma'}\circ\overset{*}{\pi}.
$$

In particular, if $(\rho,\eta) = (Id_{TM}, Id_M)$*, then the relations* (3.4.1.8′) *become*

$$
\text{(3.4.1.8}''\text{)}\qquad \overset{*}{\Gamma}_{j'k'} = \frac{\partial x^j}{\partial x^{j'}}\circ\overset{*}{\pi}\left[-\frac{\partial}{\partial x^i}\left(\frac{\partial x^{i'}}{\partial x^j}\circ\overset{*}{\pi}\right)p_{i'} + \overset{*}{\Gamma}_{jk}\right]\frac{\partial x^k}{\partial x^{k'}}\circ\overset{*}{\pi}.
$$

Proof. Let $\left(\overset{*}{\Pi},\overset{*}{\pi}\right)$ be the canonical projection $\mathbf{B}$-morphism.

Obviously, the components of

$$
\Pi^*\circ(\rho,\eta)\overset{*}{\Gamma}\left(\tilde{Z}^{\alpha'}\frac{\partial}{\partial\tilde{z}^{\alpha'}} + Y_{b'}\frac{\partial}{\partial\tilde{p}_{b'}}\right)\left(\overset{*}{u}_x\right)
$$

are the real numbers

$$
\left(Y_{b'} - (\rho,\eta)\overset{*}{\Gamma}_{b'\gamma'}\tilde{Z}^{\gamma'}\right)\left(\overset{*}{u}_x\right).
$$

Since

$$
\begin{aligned}
\left(\tilde{Z}^{\alpha'}\frac{\partial}{\partial\tilde{z}^{\alpha'}} + Y_{b'}\frac{\partial}{\partial\tilde{p}_{b'}}\right)\left(\overset{*}{u}_x\right) &= \tilde{Z}^{\alpha'}\Lambda^{\alpha}_{\alpha'}\circ h\circ\overset{*}{\pi}\frac{\partial}{\partial\tilde{z}^{\alpha}}\left(\overset{*}{u}_x\right)\\
&+\left(\tilde{Z}^{\alpha'}\rho^{i'}_{\alpha'}\circ h\circ\overset{*}{\pi}\frac{\partial M_b^{a'}\circ\pi}{\partial x^{i'}}p_{a'} + M_b^{b'}Y_{b'}\right)\frac{\partial}{\partial\tilde{p}_b}\left(\overset{*}{u}_x\right),
\end{aligned}
$$

it results that the components of

$$
\Pi^*\circ(\rho,\eta)\overset{*}{\Gamma}\left(\tilde{Z}^{\alpha'}\frac{\partial}{\partial\tilde{z}^{\alpha'}} + Y_{b'}\frac{\partial}{\partial\tilde{p}_{b'}}\right)\left(\overset{*}{u}_x\right)
$$

are the real numbers

$$\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ \overset{*}{\pi}\frac{\partial M_{b}^{\acute{a}}\circ\overset{*}{\pi}}{\partial x^{\acute{i}}}p_{\acute{a}}+M_{b}^{\acute{b}}\circ\overset{*}{\pi}Y_{\acute{b}}-\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\overset{*}{\pi}\right)M_{\acute{b}}^{b}\circ\overset{*}{\pi}\left(\overset{*}{u}_{x}\right),$$

where $\left\|M_{\acute{b}}^{b}\right\|=\left\|M_{b}^{\acute{b}}\right\|^{-1}$.

Therefore, we have:

$$\begin{aligned}&\left(\tilde{Z}^{\acute{\alpha}}\rho_{\acute{\alpha}}^{\acute{i}}\circ h\circ \overset{*}{\pi}\frac{\partial M_{b}^{\acute{a}}\circ\overset{*}{\pi}}{\partial x^{\acute{i}}}p_{\acute{a}}+M_{b}^{\acute{b}}\circ\overset{*}{\pi}Y_{\acute{b}}-\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\tilde{Z}^{\acute{\alpha}}\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\overset{*}{\pi}\right)M_{\acute{b}}^{b}\circ\overset{*}{\pi}\\&\qquad=Y_{\acute{b}}-\left(\rho,\eta\right)\overset{*}{\Gamma}_{\acute{b}\acute{\alpha}}\tilde{Z}^{\acute{\alpha}}.\end{aligned}$$

After some calculations we obtain:

$$\left(\rho,\eta\right)\overset{*}{\Gamma}_{\acute{b}\acute{\alpha}}=M_{\acute{b}}^{b}\circ\overset{*}{\pi}\left(-\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_{b}^{\acute{a}}\circ\overset{*}{\pi}}{\partial x^{i}}p_{\acute{a}}+\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\right)\Lambda_{\acute{\alpha}}^{\alpha}\circ h\circ\overset{*}{\pi}.\qquad q.e.d.$$

Remark 3.4.1.1 If we have a set of real local functions $\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\gamma}$ which satisfies the relations of passing $(3.4.1.8)$, then we have a (ρ,η)-connection $\left(\rho,\eta\right)\overset{*}{\Gamma}$ for the fiber bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

Example 3.4.1.1 If $\overset{*}{\Gamma}$ is a classical connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ on components $\overset{*}{\Gamma}_{bk}$, then the differentiable real local functions

$$\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\gamma}=\left(\rho_{\gamma}^{k}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\Gamma}_{bk}$$

are the components of a (ρ,η)-connection $\left(\rho,\eta\right)\overset{*}{\Gamma}$ for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ which will be called the *(ρ,η)-connection associated to the connection* $\overset{*}{\Gamma}$.

Definition 3.4.1.3 If $\left(\rho,\eta\right)\overset{*}{\Gamma}$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then for any

$$z=z^{\alpha}t_{\alpha}\in\Gamma\left(F,\nu,N\right)$$

the application

$$(3.4.1.9)\qquad \begin{array}{ccc}\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)&\xrightarrow{(\rho,\eta)D_{z}}&\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)\\ \overset{*}{u}=u_{a}s^{a}&\longmapsto&\left(\rho,\eta\right)D_{z}\overset{*}{u}\end{array}$$

defined by

$$\left(\rho,\eta\right)D_{z}\overset{*}{u}=z^{\alpha}\circ h\left(\rho_{\alpha}^{i}\circ h\frac{\partial u_{b}}{\partial x^{i}}-\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\circ\overset{*}{u}\right)s^{b},$$

will be called the *covariant* (ρ,η)*-derivative associated to* (ρ,η)*-connection* $(\rho,\eta)\overset{*}{\Gamma}$ *with respect to the section* z*.*

If $h = Id_M$ and $\eta = Id_M$, then we obtain the *covariant* ρ*-derivative associated to* ρ*-connection* $\rho\overset{*}{\Gamma}$ *with respect to the section* z*.*

In addition, if $\rho = Id_{TM}$, then we obtain the *covariant derivative associated to connection* $\overset{*}{\Gamma}$ *with respect to the vector field* z*.*

Definition 3.4.1.4 We will say that the (ρ,η)*-connection* $(\rho,\eta)\overset{*}{\Gamma}$ *is homogeneous* or *linear* if the local real functions $(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}$ are homogeneous or linear on the fibre of vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ respectively.

Remark 3.4.1.2 If $(\rho,\eta)\overset{*}{\Gamma}$ is a linear (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then for each local vector $(m+r)$-chart $\left(U,\overset{*}{s}_U\right)$ and for each local vector $(n+p)$-chart (V,t_V) such that $U\cap h^{-1}(V)\neq\phi$, there exists the differentiable real functions $\rho\Gamma^a_{b\gamma}$ defined on $U\cap h^{-1}(V)$ such that

$$(3.4.1.10)\qquad (\rho,\eta)\overset{*}{\Gamma}_{b\gamma}\circ\overset{*}{u} = (\rho,\eta)\Gamma^a_{b\gamma}\cdot u_a,$$

for any $\overset{*}{u} = u_a s^a \in \Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

The differentiable real local functions $(\rho,\eta)\Gamma^a_{b\alpha}$ will be called the *Christoffel coefficients of linear* (ρ,η)*-connection* $(\rho,\eta)\Gamma$.

Theorem 3.4.1.2 *If* $(\rho,\eta)\Gamma$ *is a linear* (ρ,η)*-connection for the vector bundle* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, *then its components satisfy the law of transformation*

$$(3.4.1.11)\qquad (\rho,\eta)\Gamma^{a'}_{b'\gamma'} = M^b_{b'}\left[-\rho^i_\gamma\circ h\frac{\partial M^{a'}_b}{\partial x^i} + (\rho,\eta)\Gamma^a_{b\gamma}M^{a'}_a\right]\Lambda^\gamma_{\gamma'}\circ h.$$

In particular, if $(\rho,\eta) = (Id_{TM}, Id_M)$ *and* $h = Id_M$*, then the relations* (3.4.1.11) *become*

$$(3.4.1.11')\qquad \Gamma^{i'}_{jk'} = \frac{\partial x^j}{\partial x^{j'}}\left[-\frac{\partial}{\partial x^i}\left(\frac{\partial x^{i'}}{\partial x^j}\right) + \Gamma^i_{jk}\frac{\partial x^{i'}}{\partial x^i}\right]\frac{\partial x^k}{\partial x^{k'}}.$$

Remark 3.4.1.3 Since

$$\frac{\partial M^{a'}_b}{\partial x^i}M^b_{b'} + \frac{\partial M^b_{b'}}{\partial x^i}M^{a'}_b = 0,$$

it results that the relations (3.4.11) are equivalent with the relations (3.4.1.11′).

Definition 3.4.1.5 If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then for any

$$z = z^\alpha t_\alpha \in \Gamma(F,\nu,N)$$

the application

$$\begin{array}{rcl}\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right) & \xrightarrow{(\rho,\eta)D_z} & \Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)\\ \overset{*}{u}=u_a s^a & \longmapsto & (\rho,\eta)D_z\overset{*}{u}\end{array}\tag{3.4.1.12}$$

defined by

$$(\rho,\eta)D_z\overset{*}{u}=z^\alpha\circ h\left(\rho_\alpha^i\circ h\frac{\partial u_b}{\partial x^i}-(\rho,\eta)\Gamma_{b\alpha}^a\cdot u_a\right)s^b$$

will be called the *covariant* (ρ,η)*-derivative associated to linear* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$ *with respect to the section* z.

If $h=Id_M$ and $\eta=Id_M$, then we obtain the *covariant* ρ*-derivative associated to linear* ρ*-connection* $\rho\Gamma$ *with respect to the section* z.

In addition, if $\rho=Id_{TM}$, then we obtain the *covariant derivative associated to linear connection* Γ *with respect to vector field* z.

Note. In the next we use the same notation $(\rho,\eta)\,\Gamma$ for the linear (ρ,η)-connection for the vector bundle (E,π,M) or for its dual $\left(\overset{*}{E},\overset{*}{\pi},M\right)$

Remark 3.4.1.4 If $(\rho,\eta)\,\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M) or for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ then, the tensor fields algebra

$$\left(\mathcal{T}\left(E,\pi,M\right),+,\cdot,\otimes\right)$$

is endowed with the (ρ,η)-derivative

$$\begin{array}{rcl}\Gamma\left(F,\nu,N\right)\times\mathcal{T}\left(E,\pi,M\right) & \xrightarrow{(\rho,\eta)D} & \mathcal{T}\left(E,\pi,M\right)\\ (z,T) & \longmapsto & (\rho,\eta)\,D_zT\end{array}\tag{3.4.1.13}$$

defined for a tensor field $T\in\mathcal{T}_q^p\left(E,\pi,M\right)$ by the relation:

$$\begin{array}{c}(\rho,\eta)\,D_zT\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)=\Gamma\left(\rho,\eta\right)(z)\left(T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)\right)\\ -T\left((\rho,\eta)\,D_z\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,u_q\right)-...-T\left(\overset{*}{u}_1,...,(\rho,\eta)\,D_z\overset{*}{u}_p,u_1,...,u_q\right)\\ -T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,(\rho,\eta)\,D_zu_1,...,u_q\right)-...-T\left(\overset{*}{u}_1,...,\overset{*}{u}_p,u_1,...,(\rho,\eta)\,D_zu_q\right).\end{array}\tag{3.4.1.14}$$

Moreover, it satisfies the condition

$$(\rho,\eta)\,D_{f_1z_1+f_2z_2}T=f_1\,(\rho,\eta)\,D_{z_1}T+f_2\,(\rho,\eta)\,D_{z_2}T.\tag{3.4.1.15}$$

Consequently, if the tensor algebra $\left(\mathcal{T}\left(E,\pi,M\right),+,\cdot,\otimes\right)$ is endowed with a (ρ,η)-derivative (3.4.1.13) defined for a tensor field $T\in\mathcal{T}\,_q^p\left(E,\pi,M\right)$ by (3.4.1.14) which satisfies the condition (3.4.1.15), then we can endowed (E,π,M) with a linear (ρ,η)-connection $(\rho,\eta)\,\Gamma$ such that its components are defined by the equality:

$$(\rho,\eta)\,D_{t_\alpha}s_b=(\rho,\eta)\,\Gamma_{b\alpha}^a s_a$$

or

$$(\rho,\eta)\,D_{t_\alpha}s^a=-\,(\rho,\eta)\,\Gamma_{b\alpha}^a s^b.$$

The (ρ,η)-derivative (3.4.1.13) will be called the *covariant* (ρ,η)*-derivative.*

After some calculations, we obtain:

$$\begin{array}{ll} & (\rho,\eta)\,D_z\left(T_{b_1,\dots,b_q}^{a_1,\dots,a_p}s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\right)\\ & =z^\alpha\circ h\left(\rho_\alpha^i\circ h\dfrac{\partial T_{b_1,\dots,b_q}^{a_1,\dots,a_p}}{\partial x^i}+(\rho,\eta)\,\Gamma_{a\alpha}^{a_1}T_{b_1,\dots,b_q}^{a,a_2\dots,a_p}\right.\\ (3.4.1.16) & +(\rho,\eta)\,\Gamma_{a\alpha}^{a_2}T_{b_1,\dots,b_q}^{a_1,a\dots,a_p}+\dots+(\rho,\eta)\,\Gamma_{a\alpha}^{a_p}T_{b_1,\dots,b_q}^{a_1,a_2\dots,a}-\dots\\ & -(\rho,\eta)\,\Gamma_{b_1\alpha}^{b}T_{b,b_2,\dots,b_q}^{a_1,a_2\dots,a_p}-(\rho,\eta)\,\Gamma_{b_2\alpha}^{b}T_{b_1,b,\dots,b_q}^{a_1,a_2\dots,a_p}-\dots\\ & \left.-(\rho,\eta)\,\Gamma_{b_q\alpha}^{b}T_{b_1,b_2,\dots,b}^{a_1,a_2\dots,a_p}\right)s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\\ & \overset{put}{=}z^\alpha\circ hT_{b_1,\dots,b_q|\alpha}^{a_1,\dots,a_p}s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}.\end{array}$$

If $(\rho,\eta)\,\Gamma$ is the linear (ρ,η)-connection associated to linear connection Γ, namely $(\rho,\eta)\,\Gamma_{b\alpha}^{a}=\left(\rho_\alpha^k\circ h\right)\Gamma_{bk}^{a}$, then

$$T_{b_1,\dots,b_q|\alpha}^{a_1,\dots,a_p}=\left(\rho_\alpha^k\circ h\right)T_{b_1,\dots,b_q|k}^{a_1,\dots,a_p}. \tag{3.4.1.17}$$

In particular, if $h=Id_M$, then we obtain the formula:

$$\begin{array}{ll} & (\rho,\eta)\,D_z\left(T_{b_1,\dots,b_q}^{a_1,\dots,a_p}s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\right)\\ & =z^\alpha\left(\rho_\alpha^i\dfrac{\partial T_{b_1,\dots,b_q}^{a_1,\dots,a_p}}{\partial x^i}+(\rho,\eta)\,\Gamma_{a\alpha}^{a_1}T_{b_1,\dots,b_q}^{a,a_2\dots,a_p}\right.\\ (3.4.1.18) & +(\rho,\eta)\,\Gamma_{a\alpha}^{a_2}T_{b_1,\dots,b_q}^{a_1,a\dots,a_p}+\dots+(\rho,\eta)\,\Gamma_{a\alpha}^{a_p}T_{b_1,\dots,b_q}^{a_1,a_2\dots,a}-\dots\\ & -(\rho,\eta)\,\Gamma_{b_1\alpha}^{b}T_{b,b_2,\dots,b_q}^{a_1,a_2\dots,a_p}-(\rho,\eta)\,\Gamma_{b_2\alpha}^{b}T_{b_1,b,\dots,b_q}^{a_1,a_2\dots,a_p}-\dots\\ & \left.-(\rho,\eta)\,\Gamma_{b_q\alpha}^{b}T_{b_1,b_2,\dots,b}^{a_1,a_2\dots,a_p}\right)s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\\ & \overset{put}{=}z^\alpha T_{b_1,\dots,b_q|\alpha}^{a_1,\dots,a_p}s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}.\end{array}$$

4 The geometry of base of the Lie algebroid generalized tangent bundle for a vector bundle

In this section, we present new applications of generalized Lie algebroids in the study of the geometry of vector bundles using the theory of generalized linear connections.

4.1 Torsion and curvature. Formulas of Ricci type

We apply the theory for the diagram:

$$\begin{array}{cc} E & \left(F,[,]_{F,h},(\rho,Id_M)\right)\\ \pi\downarrow & \downarrow\nu\\ M & \xrightarrow{h} M \end{array}\quad, \tag{4.1.1}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,M),[,]_{F,h},(\rho,Id_M)\right)\in|\mathbf{GLA}|$.

Let $\rho\Gamma$ be a linear ρ-connection for the vector bundle (E,π,M) by components $\rho\Gamma_{b\alpha}^{a}$.

Using the components of the linear ρ-connection $\rho\Gamma$, then we obtain a linear ρ-connection $\rho\dot{\Gamma}$ for the vector bundle (E,π,M) given by the diagram:

$$\begin{array}{ccc} E & & \left(h^{*}F,[,]_{h^{*}F},\left(\overset{h^{*}F}{\rho},Id_{M}\right)\right) \\ \pi\downarrow & & \downarrow h^{*}\nu \\ M & \xrightarrow{Id_{M}} & M \end{array}. \tag{4.1.2}$$

If $(E,\pi,M)=(F,\nu,N)$, then, using the components of the linear ρ-connection $\rho\Gamma$, we can consider a linear ρ-connection $\rho\ddot{\Gamma}$ for the vector bundle $(h^{*}E,h^{*}\pi,M)$ given by the diagram:

$$\begin{array}{ccc} h^{*}E & & \left(h^{*}E,[,]_{h^{*}E},\left(\overset{h^{*}E}{\rho},Id_{M}\right)\right) \\ h^{*}\pi\downarrow & & \downarrow h^{*}\pi \\ M & \xrightarrow{Id_{M}} & M \end{array}, \tag{4.1.3}$$

In the following, we will use the exterior differentiation operators $d,\ d^{E}$ and $d^{h^{*}E}$ respectively for the exterior differential $\mathcal{F}(M)$-algebras $(\Lambda(TM,\tau_{M},M),+,\cdot,\wedge)$, $(\Lambda(E,\pi,M),+,\cdot,\wedge)$ and $((h^{*}E,h^{*}\pi,M),+,\cdot,\wedge)$ respectively.

Definition 4.1.1 If $(E,\pi,M)=(F,\nu,N)$, then the application

$$\begin{array}{ccc} \Gamma(h^{*}E,h^{*}\pi,M)^{2} & \xrightarrow{(\rho,h)\mathbb{T}} & \Gamma(h^{*}E,h^{*}\pi,M) \\ (U,V) & \longrightarrow & \rho\mathbb{T}(U,V) \end{array} \tag{4.1.4}$$

defined by:

$$(\rho,h)\mathbb{T}(U,V)=\rho\ddot{D}_{U}V-\rho\ddot{D}_{V}U-[U,V]_{h^{*}E}, \tag{4.1.5}$$

for any $U,V\in\Gamma(h^{*}E,h^{*}\pi,M)$, will be called *$(\rho,h)$-torsion associated to linear ρ-connection $\rho\Gamma$.*

Remark 4.1.1 In particular, if $h=Id_{M}$, then we obtain the application

$$\begin{array}{ccc} \Gamma(E,\pi,M)^{2} & \xrightarrow{\rho\mathbb{T}} & \Gamma(E,\pi,M) \\ (u,v) & \longrightarrow & \rho\mathbb{T}(u,v) \end{array} \tag{4.1.4'}$$

defined by:

$$\rho\mathbb{T}(u,v)=\rho D_{u}v-\rho D_{v}u-[u,v]_{E}, \tag{4.1.5'}$$

for any $u,v\in\Gamma(E,\pi,M)$, which will be called *ρ-torsion associated to linear ρ-connection $\rho\Gamma$.*

Moreover, if $\rho=Id_{TM}$, then we obtain the torsion $\mathbb{T}$ associated to linear connection Γ.

Proposition 4.1.1 *The (ρ,h)-torsion $(\rho,h)\mathbb{T}$ associated to linear ρ-connection $\rho\Gamma$ is $\mathbb{R}$-bilinear and antisymmetric.*

If

$$(\rho,h)\mathbb{T}(S_{a},S_{b})\overset{put}{=}(\rho,h)\mathbb{T}^{c}{}_{ab}S_{c}$$

then

$$(4.1.6)\qquad (\rho,h)\,\mathbb{T}^{c}{}_{ab}=\rho\Gamma^{c}_{ab}-\rho\Gamma^{c}_{ba}-L^{c}_{ab}\circ h.$$

In particular, if $h=Id_M$ *and* $\rho\mathbb{T}\left(s_a,s_b\right)\overset{put}{=}\rho\mathbb{T}^{c}_{ab}s_c$*, then*

$$(4.1.6')\qquad \rho\mathbb{T}^{c}{}_{ab}=\rho\Gamma^{c}_{ab}-\rho\Gamma^{c}_{ba}-L^{c}_{ab}.$$

Moreover, if $\rho=Id_{TM}$*, then the equality* $(4.1.6')$ *becomes:*

$$(4.1.6'')\qquad \mathbb{T}^{i}{}_{jk}=\Gamma^{i}_{jk}-\Gamma^{i}_{kj}.$$

Definition 4.1.2 The application

$$(4.1.7)\qquad \begin{array}{ccc}\left(\Gamma\left(h^{*}F,h^{*}\nu,M\right)\right)^{2}\times\Gamma(E,\pi,M) & \xrightarrow{(\rho,h)\mathbb{R}} & \Gamma(E,\pi,M)\\ ((Z,V),u) & \longrightarrow & \rho\mathbb{R}(Z,V)u\end{array}$$

defined by

$$(4.1.8)\qquad (\rho,h)\,\mathbb{R}\left(Z,V\right)u=\rho\dot{D}_{Z}\left(\rho\dot{D}_{V}u\right)-\rho\dot{D}_{V}\left(\rho\dot{D}_{Z}u\right)-\rho\dot{D}_{[Z,V]_{h^{*}F}}u,$$

for any $Z,V\in\Gamma\left(h^{*}F,h^{*}\nu,M\right),\ u\in\Gamma\left(E,\pi,M\right)$, will be called (ρ,h)-*curvature associated to linear* ρ-*connection* $\rho\Gamma$.

Remark 4.1.1 In particular, if $h=Id_M$, then we obtain the application

$$(4.1.7')\qquad \begin{array}{ccc}\Gamma\left(F,\nu,M\right)^{2}\times\Gamma(E,\pi,M) & \xrightarrow{\rho\mathbb{R}} & \Gamma(E,\pi,M)\\ ((z,v),u) & \longrightarrow & \rho\mathbb{R}(z,v)u\end{array}$$

defined by

$$(4.1.8')\qquad \rho\mathbb{R}\left(z,v\right)u=\rho D_{z}\left(\rho D_{v}u\right)-\rho D_{v}\left(\rho D_{z}u\right)-\rho D_{[z,v]_{F}}u,$$

for any $z,v\in\Gamma\left(F,\nu,M\right),\ u\in\Gamma\left(E,\pi,M\right)$, which will be called ρ-*curvature associated to linear* ρ-*connection* $\rho\Gamma$.

Moreover, if $\rho=Id_{TM}$, then we obtain the curvature $\mathbb{R}$ associated to linear connection Γ.

Proposition 4.1.2 *The* (ρ,h)*-curvature* $(\rho,h)\,\mathbb{R}$ *associated to linear* ρ*-connection* $\rho\Gamma$*, is* $\mathbb{R}$*-linear in each argument and antisymmetric in the first two arguments.*

If

$$(\rho,h)\,\mathbb{R}\left(T_{\beta},T_{\alpha}\right)s_{b}\overset{put}{=}(\rho,h)\,\mathbb{R}^{a}_{b\ \alpha\beta}s_{a},$$

then

$$(4.1.9)\quad (\rho,h)\,\mathbb{R}^{a}_{b\ \alpha\beta}=\rho^{j}_{\beta}\circ h\frac{\partial\rho\Gamma^{a}_{b\alpha}}{\partial x^{j}}+\rho\Gamma^{a}_{e\beta}\rho\Gamma^{e}_{b\alpha}-\rho^{i}_{\alpha}\circ h\frac{\partial\rho\Gamma^{a}_{b\beta}}{\partial x^{i}}-\rho\Gamma^{a}_{e\alpha}\rho\Gamma^{e}_{b\beta}+\rho\Gamma^{a}_{b\gamma}L^{\gamma}_{\alpha\beta}\circ h.$$

In particular, if $h=Id_M$ *and* $\rho\mathbb{R}\left(t_{\beta},t_{\alpha}\right)s_{b}\overset{put}{=}\rho\mathbb{R}^{a}_{b\ \alpha\beta}s_{a}$*, then*

$$(4.1.9')\qquad \rho\mathbb{R}^{a}_{b\ \alpha\beta}=\rho^{j}_{\beta}\frac{\partial\rho\Gamma^{a}_{b\alpha}}{\partial x^{j}}+\rho\Gamma^{a}_{e\beta}\rho\Gamma^{e}_{b\alpha}-\rho^{i}_{\alpha}\frac{\partial\rho\Gamma^{a}_{b\beta}}{\partial x^{i}}-\rho\Gamma^{a}_{e\alpha}\rho\Gamma^{e}_{b\beta}+\rho\Gamma^{a}_{b\gamma}L^{\gamma}_{\alpha\beta}.$$

Moreover, if $\rho = Id_{TM}$*, then equality* (4.1.9′) *becomes:*

$$\mathbb{R}^{a}_{b\ hk} = \frac{\partial \Gamma^{a}_{bh}}{\partial x^{k}} + \Gamma^{a}_{ek}\Gamma^{e}_{bh} - \frac{\partial \Gamma^{a}_{bk}}{\partial x^{h}} - \Gamma^{a}_{eh}\Gamma^{e}_{bk}. \tag{4.1.9''}$$

Theorem 4.1.1 *For any* $u^{a}s_{a} \in \Gamma(E, \pi, M)$ *we shall use the notation*

$$u^{a}{}_{|\alpha\beta} = \rho^{j}_{\beta} \circ h \frac{\partial}{\partial x^{j}}\left(u^{a_1}{}_{|\alpha}\right) + \rho\Gamma^{a_1}_{b_\beta} u^{b}{}_{|\alpha}, \tag{4.1.10}$$

and we verify the formulas:

$$u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} = u^{a}(\rho, h)\,\mathbb{R}^{a_1}_{a\ \alpha\beta} - u^{a_1}{}_{|\gamma} L^{\gamma}_{\alpha\beta} \circ h. \tag{4.1.11}$$

After some calculations, we obtain

$$(\rho, h)\,\mathbb{R}^{a_1}_{a\ \alpha\beta} = u_{a}\left(u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} + u^{a_1}{}_{|\gamma} L^{\gamma}_{\alpha\beta} \circ h\right), \tag{4.1.12}$$

where $u_{a}s^{a} \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ *such that* $u_{a}u^{b} = \delta^{b}_{a}$.

In particular, if $h = Id_{M}$*, then the relations* (4.1.12) *become*

$$\rho\mathbb{R}^{a_1}_{a\ \alpha\beta} = u_{a}\left(u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} + u^{a_1}{}_{|\gamma} L^{\gamma}_{\alpha\beta}\right). \tag{4.1.12'}$$

Moreover, if $\rho = id_{TM}$*, then the relations* (4.1.12′) *become*

$$\mathbb{R}^{a_1}_{a\ ij} = u_{a}\left(u^{a_1}{}_{|ij} - u^{a_1}{}_{|ji}\right). \tag{4.1.12''}$$

Proof. Since

$$\begin{aligned}
u^{a_1}{}_{|\alpha\beta} &= \rho^{j}_{\beta} \circ h\left(\frac{\partial}{\partial x^{j}}\left(\rho^{i}_{\alpha} \circ h \frac{\partial u^{a_1}}{\partial x^{i}} + \rho\Gamma^{a_1}_{a\alpha} u^{a}\right)\right) \\
&+ \rho\Gamma^{a_1}_{b\beta}\left(\rho^{i}_{\alpha} \circ h \frac{\partial u^{b}}{\partial x^{i}} + \rho\Gamma^{b}_{a\alpha} u^{a}\right) \\
&= \rho^{j}_{\beta} \circ h \frac{\partial \rho^{i}_{\alpha} \circ h}{\partial x^{j}} \frac{\partial u^{a_1}}{\partial x^{i}} + \rho^{j}_{\beta} \circ h \rho^{i}_{\alpha} \circ h \frac{\partial}{\partial x^{j}}\left(\frac{\partial u^{a_1}}{\partial x^{i}}\right) \\
&+ \rho^{j}_{\beta} \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^{j}} u^{a} + \rho^{j}_{\beta} \circ h \rho\Gamma^{a_1}_{a\alpha} \frac{\partial u^{a}}{\partial x^{j}} \\
&+ \rho^{i}_{\alpha} \circ h \rho\Gamma^{a_1}_{b\beta} \frac{\partial u^{b}}{\partial x^{i}} + \rho\Gamma^{a_1}_{b\beta}\rho\Gamma^{b}_{a\alpha} u^{a}
\end{aligned}$$

and

$$\begin{aligned}
u^{a_1}{}_{|\beta\alpha} &= \rho^{i}_{\alpha} \circ h\left(\frac{\partial}{\partial x^{i}}\left(\rho^{j}_{\beta} \circ h \frac{\partial u^{a_1}}{\partial x^{j}} + \rho\Gamma^{a_1}_{a\beta} u^{a}\right)\right) \\
&+ \rho\Gamma^{a_1}_{b\alpha}\left(\rho^{j}_{\beta} \circ h \frac{\partial u^{b}}{\partial x^{j}} + \rho\Gamma^{b}_{a\beta} u^{a}\right) \\
&= \rho^{i}_{\alpha} \circ h \frac{\partial \rho^{j}_{\beta} \circ h}{\partial x^{i}} \frac{\partial u^{a_1}}{\partial x^{j}} + \rho^{j}_{\beta} \circ h \rho^{i}_{\alpha} \circ h \frac{\partial}{\partial x^{i}}\left(\frac{\partial u^{a_1}}{\partial x^{j}}\right) \\
&+ \rho^{i}_{\alpha} \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^{i}} u^{a} + \rho^{i}_{\alpha} \circ h \rho\Gamma^{a_1}_{a\beta} \frac{\partial u^{a}}{\partial x^{i}} \\
&+ \rho^{j}_{\beta} \circ h \rho\Gamma^{a_1}_{b\alpha} \frac{\partial u^{b}}{\partial x^{j}} + \rho\Gamma^{a_1}_{b\alpha}\rho\Gamma^{b}_{a\beta} u^{a},
\end{aligned}$$

it results that

$$\begin{aligned}
u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} &= \rho^j_\beta \circ h \frac{\partial \rho^i_\alpha \circ h}{\partial x^j} \frac{\partial u^{a_1}}{\partial x^i} - \rho^i_\alpha \circ h \frac{\partial \rho^j_\beta \circ h}{\partial x^i} \frac{\partial u^{a_1}}{\partial x^j} \\
&+ \left(\rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial^2 u^{a_1}}{\partial x^i \partial x^j} - \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial^2 u^{a_1}}{\partial x^j \partial x^i} \right) \\
&+ \left(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} u^a \right) \\
&+ \left(\rho^j_\beta \circ h \rho\Gamma^{a_1}_{a\alpha} \frac{\partial u^a}{\partial x^j} - \rho^j_\beta \circ h \rho\Gamma^{a_1}_{b\alpha} \frac{\partial u^b}{\partial x^j} \right) \\
&+ \left(\rho^i_\alpha \circ h \rho\Gamma^{a_1}_{b\beta} \frac{\partial u^b}{\partial x^i} - \rho^i_\alpha \circ h \rho\Gamma^{a_1}_{a\beta} \frac{\partial u^a}{\partial x^i} \right) \\
&+ \rho\Gamma^{a_1}_{b\beta} \rho\Gamma^b_{a\alpha} u^a - \rho\Gamma^{a_1}_{b\alpha} \rho\Gamma^b_{a\beta} u^a.
\end{aligned}$$

After some calculations, we obtain:

$$\begin{aligned}
u^{a_1}{}_{|\alpha\beta} - u^{a_1}{}_{|\beta\alpha} &= L^\gamma_{\beta\alpha} \circ h \rho^k_\gamma \circ h \frac{\partial u^{a_1}}{\partial x^k} \\
&+ \left(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} u^a \right) \\
&+ \rho\Gamma^{a_1}_{b\beta} \rho\Gamma^b_{a\alpha} u^a - \rho\Gamma^{a_1}_{b\alpha} \rho\Gamma^b_{a\beta} u^a.
\end{aligned}$$

Since

$$\begin{aligned}
u^a (\rho, h) \mathbb{R}^{a_1}_{a\ \alpha\beta} &= u^a \left(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} + \rho\Gamma^{a_1}_{e\beta} \rho\Gamma^e_{a\alpha} - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} \right. \\
&\qquad \left. - \rho\Gamma^{a_1}_{e\alpha} \rho\Gamma^e_{a\beta} - \rho\Gamma^{a_1}_{a\gamma} L^\gamma_{\beta\alpha} \circ h \right).
\end{aligned}$$

and

$$u^{a_1}{}_{|\gamma} L^\gamma_{\alpha\beta} \circ h = \left(\rho^k_\gamma \circ h \frac{\partial u^{a_1}}{\partial x^k} + \rho\Gamma^{a_1}_{a\gamma} u^a \right) L^\gamma_{\alpha\beta} \circ h$$

it results that

$$\begin{aligned}
u^a (\rho, h) \mathbb{R}^{a_1}_{a\ \alpha\beta} - u^{a_1}{}_{|\gamma} L^\gamma_{\alpha\beta} \circ h &= -L^\gamma_{\alpha\beta} \circ h \rho^k_\gamma \circ h \frac{\partial u^{a_1}}{\partial x^k} \\
&+ \left(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} u^a \right) \\
&+ \rho\Gamma^{a_1}_{b\beta} \rho\Gamma^b_{a\alpha} u^a - \rho\Gamma^{a_1}_{b\alpha} \rho\Gamma^b_{a\beta} u^a.
\end{aligned}$$

q.e.d.

Lemma 4.1.1 *If* $(E, \pi, M) = (F, \nu, N)$*, then, for any*

$$u^a s_a \in \Gamma(E, \pi, M),$$

we have that $u^a{}_{|c}$, $a, c \in \overline{1, n}$ are the *components of a tensor field of* $(1,1)$ *type.*

Proof. Let U and U' be two vector local $(m+n)$-charts such that $U \cap U' \neq \phi$.

Since $u^{a'}(x) = M_a^{a'}(x)\, u^a(x)$, for any $x \in U \cap U'$, it results that

$$\rho_{c'}^{k'} \circ h(x) \frac{\partial u^{a'}(x)}{\partial x^{k'}} = \rho_{c'}^{k'} \circ h(x) \frac{\partial}{\partial x^{k'}} \left(M_a^{a'}(x) \right) u^a(x) + M_a^{a'}(x)\, \rho_{c'}^{k'} \circ h(x) \frac{\partial u^a(x)}{\partial x^{k'}}. \quad (1)$$

Since , for any $x \in U \cap U'$, we have

$$\rho\Gamma_{b'c'}^{a'}(x) = M_a^{a'}(x) \left(\rho_c^k \circ h(x) \frac{\partial}{\partial x^k} (M_{b'}^a(x)) + \rho\Gamma_{bc}^a(x) M_{b'}^b(x) \right) M_{c'}^c(x), \quad (2)$$

and

$$0 = \frac{\partial}{\partial x^{k'}} \left(M_a^{a'}(x)\, M_{b'}^a(x) \right) = \frac{\partial}{\partial x^{k'}} \left(M_a^{a'}(x) \right) M_{b'}^a(x) + M_a^{a'}(x) \frac{\partial}{\partial x^{k'}} \left(M_{b'}^a(x) \right) \quad (3)$$

it results that

$$\begin{aligned} \rho\Gamma_{b'c'}^{a'}(x)\, u^{b'}(x) = &-\rho_{c'}^{k'} \circ h(x) \frac{\partial}{\partial x^{k'}} \left(M_a^{a'}(x) \right) u^a(x) \\ &+ M_a^{a'}(x)\, \rho\Gamma_{bc}^a(x)\, u^b(x)\, M_{c'}^c(x). \end{aligned} \quad (4)$$

Summing the equalities (1) and (4), it results the conclusion of lemma. *q.e.d.*

Theorem 4.1.2 *If* $(E,\pi,M) = (F,\nu,N)$*, then, for any*

$$u^a s_a \in \Gamma(E,\pi,M),$$

we shall use the notation

$$u^{a_1}{}_{|a|b} = u^{a_1}{}_{|ab} - \rho\Gamma_{ab}^d u^{a_1}{}_{|d} \tag{4.1.13}$$

and we verify the formulas of Ricci type

$$u^{a_1}{}_{|a|b} - u^{a_1}{}_{|b|a} + (\rho,h)\, \mathbb{T}^d{}_{ab} u^{a_1}{}_{|d} = u^d (\rho,h)\, \mathbb{R}_{d\ ab}^{a_1} - u^{a_1}{}_{|c} L_{ab}^c \circ h \tag{4.1.14}$$

In particular, if $h = Id_M$*, then the relations* (4.1.14) *become*

$$u^{a_1}{}_{|a|b} - u^{a_1}{}_{|b|a} + \rho\mathbb{T}^d{}_{ab} u^{a_1}{}_{|d} = u^d \rho\mathbb{R}_{d\ ab}^{a_1} - u^{a_1}{}_{|c} L_{ab}^c \tag{4.1.14'}$$

Moreover, if $\rho = id_{TM}$*, then the relations* (4.1.14′) *become*

$$u^{i_1}{}_{|i|j} - u^{i_1}{}_{|i|j} + \mathbb{T}^k{}_{ij} u^{i_1}{}_{|k} = u^k \mathbb{R}_{k\ ij}^{i_1} \tag{4.1.14''}$$

Theorem 4.1.3 *For any* $u_a s^a \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$ *we shall use the notation*

$$u_{b_1|\alpha\beta} = \rho_\beta^j \circ h \tfrac{\partial}{\partial x^j} \left(u_{b_1|\alpha} \right) - \rho\Gamma_{b_1\beta}^b u_{b|\alpha} \tag{4.1.15}$$

and we verify the formulas:

$$u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} = -u_b (\rho,h)\, \mathbb{R}_{b_1\ \alpha\beta}^b - u_{b_1|\gamma} L_{\alpha\beta}^\gamma \circ h \tag{4.1.16}$$

After some calculations, we obtain

$$(\rho,h)\,\mathbb{R}^{b}_{b_1\ \ \alpha\beta}=u^b\left(-u_{b_1|\alpha\beta}+u_{b_1|\beta\alpha}-u_{b_1|\gamma}L^{\gamma}_{\alpha\beta}\circ h\right), \tag{4.1.17}$$

where $u^a s_a\in\Gamma(E,\pi,M)$ *such that* $u_a u^b=\delta^b_a$.

In particular, if $h=Id_M$*, then the relations* (4.1.17) *become*

$$\rho\mathbb{R}^{b}_{b_1\ \ \alpha\beta}=u^b\left(-u_{b_1|\alpha\beta}+u_{b_1|\beta\alpha}-u_{b_1|\gamma}L^{\gamma}_{\alpha\beta}\right). \tag{4.1.17$'$}$$

Moreover, if $\rho=id_{TM}$ *then the relations* (4.1.17′) *become*

$$\mathbb{R}^{b}_{b_1\ \ ij}=u^b\left(-u_{b_1|ij}+u_{b_1|ji}\right). \tag{4.1.17$''$}$$

Proof. Since

$$\begin{aligned}u_{b_1|\alpha\beta}&=\rho^j_\beta\circ h\left(\frac{\partial}{\partial x^j}\left(\rho^i_\alpha\circ h\frac{\partial u_{b_1}}{\partial x^i}-\rho\Gamma^b_{b_1\alpha}u_b\right)\right)\\&-\rho\Gamma^b_{b_1\beta}\left(\rho^i_\alpha\circ h\frac{\partial u_b}{\partial x^i}-\rho\Gamma^a_{b\alpha}u_a\right)\\&=\rho^j_\beta\circ h\frac{\partial\rho^i_\alpha\circ h}{\partial x^j}\frac{\partial u_{b_1}}{\partial x^i}+\rho^j_\beta\circ h\rho^i_\alpha\circ h\frac{\partial}{\partial x^j}\left(\frac{\partial u_{b_1}}{\partial x^i}\right)\\&-\rho^j_\beta\circ h\frac{\partial\rho\Gamma^b_{b_1\alpha}}{\partial x^j}u_b-\rho^j_\beta\circ h\rho\Gamma^b_{b_1\alpha}\frac{\partial u_b}{\partial x^j}\\&-\rho^i_\alpha\circ h\rho\Gamma^b_{b_1\beta}\frac{\partial u_b}{\partial x^i}+\rho\Gamma^b_{b_1\beta}\rho\Gamma^a_{b\alpha}u_a\end{aligned}$$

and

$$\begin{aligned}u_{b_1|\beta\alpha}&=\rho^i_\alpha\circ h\left(\frac{\partial}{\partial x^i}\left(\rho^j_\beta\circ h\frac{\partial u_{b_1}}{\partial x^j}-\rho\Gamma^b_{b_1\beta}u_b\right)\right)\\&-\rho\Gamma^b_{b_1\alpha}\left(\rho^j_\beta\circ h\frac{\partial u_b}{\partial x^j}-\rho\Gamma^a_{b\beta}u_a\right)\\&=\rho^i_\alpha\circ h\frac{\partial\rho^j_\beta\circ h}{\partial x^i}\frac{\partial u_{b_1}}{\partial x^i}+\rho^j_\beta\circ h\rho^i_\alpha\circ h\frac{\partial}{\partial x^i}\left(\frac{\partial u_{b_1}}{\partial x^j}\right)\\&-\rho^i_\alpha\circ h\frac{\partial\rho\Gamma^b_{b_1\beta}}{\partial x^i}u_b-\rho^i_\alpha\circ h\rho\Gamma^b_{b_1\beta}\frac{\partial u_b}{\partial x^i}\\&-\rho^j_\beta\circ h\rho\Gamma^b_{b_1\alpha}\frac{\partial u_b}{\partial x^i}+\rho\Gamma^b_{b_1\alpha}\rho\Gamma^a_{b\beta}u_a\end{aligned}$$

it results that

$$\begin{aligned}u_{b_1|\alpha\beta}-u_{b_1|\beta\alpha}&=\rho^j_\beta\circ h\frac{\partial\rho^i_\alpha\circ h}{\partial x^j}\frac{\partial u_{b_1}}{\partial x^i}-\rho^i_\alpha\circ h\frac{\partial\rho^j_\beta\circ h}{\partial x^i}\frac{\partial u_{b_1}}{\partial x^j}\\&+\rho^j_\beta\circ h\rho^i_\alpha\circ h\frac{\partial}{\partial x^j}\left(\frac{\partial u_{b_1}}{\partial x^i}\right)-\rho^j_\beta\circ h\rho^i_\alpha\circ h\frac{\partial}{\partial x^i}\left(\frac{\partial u_{b_1}}{\partial x^j}\right)\\&+\rho^i_\alpha\circ h\frac{\partial\rho\Gamma^b_{b_1\beta}}{\partial x^i}u_b-\rho^j_\beta\circ h\frac{\partial\rho\Gamma^b_{b_1\alpha}}{\partial x^j}u_b\\&+\rho^j_\beta\circ h\rho\Gamma^b_{b_1\alpha}\frac{\partial u_b}{\partial x^j}-\rho^j_\beta\circ h\rho\Gamma^b_{b_1\alpha}\frac{\partial u_b}{\partial x^j}\\&+\rho^i_\alpha\circ h\rho\Gamma^b_{b_1\alpha}\frac{\partial u_b}{\partial x^i}-\rho^i_\alpha\circ h\rho\Gamma^b_{b_1\alpha}\frac{\partial u_b}{\partial x^i}\\&+\rho\Gamma^b_{b_1\beta}\rho\Gamma^a_{b\alpha}u_a-\rho\Gamma^b_{b_1\alpha}\rho\Gamma^a_{b\beta}u_a.\end{aligned}$$

After some calculations, we obtain:

$$\begin{aligned} u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} &= L^{\gamma}_{\beta\alpha} \circ h\rho^k_{\gamma} \circ h\tfrac{\partial u_{b_1}}{\partial x^k} \\ &+ \left(\rho^i_{\alpha} \circ h\tfrac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} u_b - \rho^j_{\beta} \circ h\tfrac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} u_b\right) \\ &+ \rho\Gamma^b_{b_1\beta}\rho\Gamma^a_{b\alpha}u_a - \rho\Gamma^b_{b_1\alpha}\rho\Gamma^a_{b\beta}u_a. \end{aligned}$$

Since

$$\begin{aligned} u_b\left(\rho, h\right)\mathbb{R}^b_{b_1\alpha\beta} = u_b \Bigg(& \rho^j_{\beta} \circ h\frac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} + \rho\Gamma^b_{e\beta}\rho\Gamma^e_{b_1\alpha} \\ & - \rho^i_{\alpha} \circ h\frac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} - \rho\Gamma^b_{e\alpha}\rho\Gamma^e_{b_1\beta} - \rho\Gamma^b_{b_1\gamma}L^{\gamma}_{\beta\alpha} \circ h\Bigg) \end{aligned}$$

and

$$u_{b_1|\gamma}L^{\gamma}_{\alpha\beta} \circ h = \left(\rho^k_{\gamma} \circ h\frac{\partial u_{b_1}}{\partial x^k} - \rho\Gamma^b_{b_1\gamma}u_b\right) L^{\gamma}_{\alpha\beta} \circ h$$

it results that

$$\begin{aligned} -u_b\left(\rho, h\right)\mathbb{R}^b_{b_1,\alpha\beta} - u_{b_1|\gamma}L^{\gamma}_{\alpha\beta} \circ h = & -L^{\gamma}_{\alpha\beta} \circ h\rho^k_{\gamma} \circ h\frac{\partial u_{b_1}}{\partial x^k} \\ & + \left(\rho^i_{\alpha} \circ h\frac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} u_b - \rho^j_{\beta} \circ h\frac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} u_b\right) \\ & + \rho\Gamma^b_{b_1\beta}\rho\Gamma^a_{b\alpha}u_a - \rho\Gamma^b_{b_1\alpha}\rho\Gamma^a_{b\beta}u_a. \end{aligned}$$

q.e.d.

Lemma 4.1.2 *If* $(E, \pi, M) = (F, \nu, N)$, *then, for any*

$$u_b s^b \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right),$$

we have that $u_{b\ |c}$, $b, c \in \overline{1, n}$ are the *components of a tensor field of* $(0, 2)$ *type.*

Proof. Let U and U' be two vector local $(m+n)$-charts such that $U \cap U' \neq \phi$.
Since $u_{b'}(x) = M^b_{b'}(x)\, u_b(x)$, for any $x \in U \cap U'$, it results that

$$\begin{aligned} \rho^{k'}_{c'} \circ h\left(x\right)\frac{\partial u_{b'}\left(x\right)}{\partial x^{k'}} &= \rho^{k'}_{c'} \circ h\left(x\right)\frac{\partial}{\partial x^{k'}}\left(M^b_{b'}\left(x\right)\right)u_b\left(x\right) \\ &+ M^b_{b'}\left(x\right)\rho^{k'}_{c'} \circ h\left(x\right)\frac{\partial u_b\left(x\right)}{\partial x^{k'}}. \end{aligned} \tag{1}$$

Since, for any $x \in U \cap U'$, we have

$$\begin{aligned} \rho\Gamma^{a'}_{b'c'}\left(x\right) &= M^{a'}_{a}\left(x\right)\left(\rho^k_c \circ h\left(x\right)\frac{\partial}{\partial x^k}\left(M^a_{b'}\left(x\right)\right)\right. \\ &\left. + \rho\Gamma^a_{bc}\left(x\right)M^b_{b'}\left(x\right)\right)M^c_{c'}\left(x\right), \end{aligned} \tag{2}$$

and

$$
(3) \qquad \begin{aligned} 0 &= \frac{\partial}{\partial x^{k'}}\left(M_a^{a'}(x)\, M_{b'}^a(x)\right) \\ &= \frac{\partial}{\partial x^{k'}}\left(M_a^{a'}(x)\right) M_{b'}^a(x) + M_a^{a'}(x)\, \frac{\partial}{\partial x^{k'}}\left(M_{b'}^a(x)\right) \end{aligned}
$$

it results that

$$
(4) \qquad \begin{aligned} \rho\Gamma_{b'c'}^{a'}(x)\, u_{a'}(x) = -\rho_{c'}^{k'} \circ h(x)\, \frac{\partial}{\partial x^{k'}}\left(M_{b'}^b(x)\right) u_b(x) \\ +M_{b'}^b(x)\, \rho\Gamma_{bc}^a(x)\, u_a(x)\, M_{c'}^c(x)\,. \end{aligned}
$$

Summing the equalities (1) and (4), it results the conclusion of lemma. *q.e.d.*

Theorem 4.1.4 *If* $(E, \pi, M) = (F, \nu, N)$*, then, for any*

$$
u_b s^b \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right),
$$

we shall use the notation

$$
(4.1.18) \qquad u_{b_1\ |a|b} = u_{b_1\ |ab} - \rho\Gamma_{ab}^d u_{b_1\ |d}
$$

and we verify the formulas of Ricci type

$$
(4.1.19) \qquad u_{b_1\ |a|b} - u_{b_1\ |b|a} + (\rho, h)\, \mathbb{T}_{\ ab}^{d} u_{b_1\ |d} = -u_d\, (\rho, h)\, \mathbb{R}_{b_1\ ab}^{d} - u_{b_1\ |d} L_{ab}^d \circ h
$$

In particular, if $h = Id_M$*, then the relations* (4.1.19) *become*

$$
(4.1.19') \qquad u_{b_1\ |a|b} - u_{b_1\ |b|a} + \rho\mathbb{T}_{\ ab}^{d} u_{b_1\ |d} = -u_d \rho\mathbb{R}_{b_1\ ab}^{d} - u_{b_1\ |d} L_{ab}^d.
$$

Moreover, if $\rho = id_{TM}$ *then the relations* $(4.1.19')$ *become*

$$
(4.1.19'') \qquad u_{j_1\ |i|j} - u_{j_1\ |j|i} + \mathbb{T}_{\ ij}^{h} u_{j_1\ |h} = u_h \mathbb{R}_{j_1\ ij}^{h}.
$$

Theorem 4.1.5 *For any tensor field*

$$
T_{b_1,\dots,b_q}^{a_1,\dots,a_p} s_{a_1} \otimes \dots \otimes s_{a_p} \otimes s^{b_1} \otimes \dots \otimes s^{b_q}\ ,
$$

we verify the equality:

$$
(4.1.20) \qquad \begin{aligned} & T_{b_1,\dots,b_q|\alpha\beta}^{a_1,\dots,a_p} - T_{b_1,\dots,b_q|\beta\alpha}^{a_1,\dots,a_p} = T_{b_1,\dots,b_q}^{aa_2,\dots,a_p}(\rho, h)\, \mathbb{R}_{a\ \alpha\beta}^{a_1} + \dots \\ & +T_{b_1,\dots,b_q}^{a_1,\dots,a_{p-1}a}(\rho, h)\, \mathbb{R}_{a\ \alpha\beta}^{a_p} - T_{b,b_2,\dots,b_q}^{a_1,\dots,a_p}(\rho, h)\, \mathbb{R}_{b_1\ \alpha\beta}^{b} - \dots \\ & -T_{b_1,\dots,b_{q-1}b}^{a_1,\dots,a_p}(\rho, h)\, \mathbb{R}_{b_q\ \alpha\beta}^{b} - T_{b_1,\dots,b_q|\gamma}^{a_1,\dots,a_p} L_{\alpha\beta}^{\gamma} \circ h. \end{aligned}
$$

In particular, if $h = Id_M$*, then the relations* (4.1.20) *become*

$$
(4.1.20') \qquad \begin{aligned} & T_{b_1,\dots,b_q|\alpha\beta}^{a_1,\dots,a_p} - T_{b_1,\dots,b_q|\beta\alpha}^{a_1,\dots,a_p} = T_{b_1,\dots,b_q}^{aa_2,\dots,a_p} \rho\mathbb{R}_{a\ \alpha\beta}^{a_1} + \dots \\ & +T_{b_1,\dots,b_q}^{a_1,\dots,a_{p-1}a} \rho\mathbb{R}_{a\ \alpha\beta}^{a_p} - T_{b,b_2,\dots,b_q}^{a_1,\dots,a_p} \rho\mathbb{R}_{b_1\ \alpha\beta}^{b} - \dots \\ & -T_{b_1,\dots,b_{q-1}b}^{a_1,\dots,a_p} \rho\mathbb{R}_{b_q\ \alpha\beta}^{b} - T_{b_1,\dots,b_q|\gamma}^{a_1,\dots,a_p} L_{\alpha\beta}^{\gamma}. \end{aligned}
$$

Theorem 4.1.6 *If* $(E,\pi,M)=(F,\nu,N)$*, then we obtain the following formulas of Ricci type:*

$$\begin{array}{ll}(4.1.21) & \begin{array}{l}T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q\ |b|c}-T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|c|b}+(\rho,h)\,\mathbb{T}^d{}_{bc}T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|d}\\ =T^{aa_2,\ldots,a_p}_{b_1,\ldots,b_q}\,(\rho,h)\,\mathbb{R}^{a_1}_{a\ bc}+\ldots+T^{a_1,\ldots,a_{p-1}a}_{b_1,\ldots,b_q}\,(\rho,h)\,\mathbb{R}^{a_p}_{a\ bc}\\ -T^{a_1,\ldots,a_p}_{b,b_2,\ldots,b_q}\,(\rho,h)\,\mathbb{R}^{b}_{b_1\ bc}-\ldots-T^{a_1,\ldots,a_p}_{b_1,\ldots,b_{q-1}b}\,(\rho,h)\,\mathbb{R}^{b}_{b_q\ bc}-T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|d}L^d_{bc}\circ h.\end{array}\end{array}$$

In particular, if $h=Id_M$*, then the relations* (4.1.21) *become*

$$\begin{array}{ll}(4.1.21') & \begin{array}{l}T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q\ |b|c}-T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|c|b}+\rho\mathbb{T}^d{}_{bc}T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|d}\\ =T^{aa_2,\ldots,a_p}_{b_1,\ldots,b_q}\rho\mathbb{R}^{a_1}_{a\ bc}+\ldots+T^{a_1,\ldots,a_{p-1}a}_{b_1,\ldots,b_q}\rho\mathbb{R}^{a_p}_{a\ bc}\\ -T^{a_1,\ldots,a_p}_{b,b_2,\ldots,b_q}\rho\mathbb{R}^{b}_{b_1\ bc}-\ldots-T^{a_1,\ldots,a_p}_{b_1,\ldots,b_{q-1}b}\rho\mathbb{R}^{b}_{b_q\ bc}-T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|d}L^d_{bc}.\end{array}\end{array}$$

We observe that if the structure functions of generalized Lie algebroid

$$\left((F,\nu,M)\,,[,]_{F,h}\,,(\rho,Id_M)\right),$$

the (ρ,h)-torsion associated to linear ρ-connection $\rho\Gamma$ and the (ρ,h)-curvature associated to linear ρ-connection $\rho\Gamma$ are null, then we have the equality:

$$(4.1.22)\qquad T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|b|c}=T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|c|b},$$

which generalizes the Schwartz equality.

4.2 Torsion and curvature forms. Identities of Cartan and Bianchi type

We apply the theory of the generalized linear connections for the diagram:

$$(4.2.1)\qquad \begin{array}{ccc} E & & \left(F,[,]_{F,h}\,,(\rho,Id_M)\right)\\ \pi\downarrow & & \downarrow\nu\\ M & \xrightarrow{h} & M\end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,M)\,,[,]_{F,h}\,,(\rho,Id_M)\right)\in|\mathbf{GLA}|$.

Let $\rho\Gamma$ be a linear ρ-connection for the vector bundle (E,π,M).

Definition 4.2.1 For each $a,b\in\overline{1,n}$, we obtain the scalar 1-forms

$$(4.2.2)\qquad \Omega^a_b=\rho\Gamma^a_{b\alpha}T^\alpha$$

and

$$(4.2.2')\qquad \omega^a_b=\rho\Gamma^a_{b\alpha}t^\alpha$$

which will be called the *form of linear* ρ*-connection* $\rho\dot\Gamma$ *and* $\rho\Gamma$ *respectively.*

Definition 4.2.2 If $(E,\pi,M)=(F,\nu,M)$, then the vector valued 2-form

$$(4.2.3)\qquad (\rho,h)\,\mathbb{T}=((\rho,h)\,\mathbb{T}^c{}_{ab}S_c)\,S^a\wedge S^b$$

will be called the *vector valued form of* (ρ, h)*-torsion* $(\rho, h)\,\mathbb{T}$.

In particular, if $h = Id_M$, then the vector valued 2-form

$$\rho\mathbb{T} = \left(\rho\mathbb{T}^c{}_{ab}s_c\right) s^a \wedge s^b \tag{4.2.3'}$$

will be called the *vector form of* ρ*-torsion* $\rho\mathbb{T}$.

Moreover, if $\rho = Id_{TM}$, then the vector valued form (4.2.3′) becomes:

$$\mathbb{T} = \left(\mathbb{T}^i{}_{jk}\frac{\partial}{\partial x^i}\right) dx^j \wedge dx^k. \tag{4.2.3''}$$

Definition 4.2.3 For each $c \in \overline{1,n}$ we obtain the *scalar* 2*-form of* (ρ, h)*-torsion* $(\rho, h)\,\mathbb{T}$

$$(\rho, h)\,\mathbb{T}^c = (\rho, h)\,\mathbb{T}^c{}_{ab}S^a \wedge S^b. \tag{4.2.4}$$

In particular, if $h = Id_M$, then, for each $c \in \overline{1,n}$, we obtain the *scalar* 2*-form of* ρ*-torsion* $\rho\mathbb{T}$

$$\rho\mathbb{T}^c = \rho\mathbb{T}^c{}_{ab}s^a \wedge s^b. \tag{4.2.4'}$$

Moreover, if $\rho = Id_{TM}$, then the scalar 2-form (4.2.4′) becomes:

$$\mathbb{T}^i = \mathbb{T}^i{}_{jk}dx^j \wedge dx^k. \tag{4.2.4''}$$

Definition 4.2.3 The vector mixed form

$$(\rho, h)\,\mathbb{R} = \left(\left((\rho, h)\,\mathbb{R}^a_{b\ \alpha\beta}s_a\right) T^\alpha \wedge T^\beta\right) s^b \tag{4.2.5}$$

will be called the *vector valued form of* (ρ, h)*-curvature* $(\rho, h)\,\mathbb{R}$.

In particular, if $h = Id_M$, then the vector mixed form

$$\rho\mathbb{R} = \left(\left(\rho\mathbb{R}^a_{b\ \alpha\beta}s_a\right) t^\alpha \wedge t^\beta\right) s^b \tag{4.2.5'}$$

will be called the *vector valued form of* ρ*-curvature* $\rho\mathbb{R}$.

Moreover, if $\rho = Id_{TM}$, then the vector form (4.2.5′) becomes:

$$\mathbb{R} = \left(\left(\mathbb{R}^a_{b\ hk}s_a\right) dx^h \wedge dx^k\right) s^b. \tag{4.2.5''}$$

Definition 4.2.4 For each $a, b \in \overline{1,n}$ we obtain the *scalar* 2*-form of* (ρ, h)*-curvature* $(\rho, h)\,\mathbb{R}$

$$(\rho, h)\,\mathbb{R}^a_b = (\rho, h)\,\mathbb{R}^a_{b\ \alpha\beta}T^\alpha \wedge T^\beta. \tag{4.2.6}$$

In particular, if $h = Id_M$, then, for each $a, b \in \overline{1,n}$, we obtain the *scalar* 2*-form of* ρ*-curvature* $\rho\mathbb{R}$

$$\rho\mathbb{R}^a_b = \rho\mathbb{R}^a_{b\ \alpha\beta}t^\alpha \wedge t^\beta. \tag{4.2.6'}$$

Moreover, if $\rho = Id_{TM}$, then the scalar form (4.2.6′) becomes:

$$\mathbb{R}^a_b = \mathbb{R}^a_{b\ hk}dx^h \wedge dx^k. \tag{4.2.6''}$$

Theorem 4.2.1 *The identities*

$$(C_1) \qquad (\rho, h)\,\mathbb{T}^a = d^{h^*F} S^a + \Omega^a_b \wedge S^b,$$

and

$$(C_2) \qquad (\rho, h)\,\mathbb{R}^a_b = d^{h^*F} \Omega^a_b + \Omega^a_c \wedge \Omega^c_b$$

hold good. These will be called the first respectively the second identity of Cartan type.

Proof. To prove the first identity we consider that $(E, \pi, M) = (F, \nu, M)$. Therefore, $\Omega^a_b = \rho\Gamma^a_{bc} S^c$. Since

$$\begin{aligned}
& d^{h^*F} S^a(U,V)S_a = ((\Gamma(\overset{h^*F}{\rho}, Id_M)U)S^a(V) \\
& -(\Gamma(\overset{h^*F}{\rho}, Id_M)V)(S^a(U)) - S^a([U,V]_{h^*F}))S_a \\
& = (\Gamma(\overset{h^*F}{\rho}, Id_M)U)(V^a) - (\Gamma(\overset{h^*F}{\rho}, Id_M)V)(U^a) - S^a([U,V]_{h^*F})S_a \\
& = \rho\ddot{D}_U V - V^b \rho\ddot{D}_U S_b - \rho\ddot{D}_V U - U^b \rho\ddot{D}_V S_b - [U,V]_{h^*F} \\
& = (\rho, h)\,\mathbb{T}(U,V) - (\rho\Gamma^a_{bc} V^b U^c - \rho\Gamma^a_{bc} U^b V^c)S_a \\
& = ((\rho, h)\,\mathbb{T}^a(U,V) - \Omega^a_b \wedge S^b(U,V))S_a,
\end{aligned}$$

it results the first identity.

To prove the second identity, we consider that $(E, \pi, M) \neq (F, \nu, M)$. Since

$$\begin{aligned}
& (\rho, h)\,\mathbb{R}^a_b\,(Z,W)\,s_a = (\rho, h)\,\mathbb{R}\,((W,Z)\,, s_b) \\
& = \rho\dot{D}_Z\left(\rho\dot{D}_W s_b\right) - \rho\dot{D}_W\left(\rho\dot{D}_Z s_b\right) - \rho\dot{D}_{[Z,W]_{h^*F}} s_b \\
& = \rho\dot{D}_Z\left(\Omega^a_b\,(W)\,s_a\right) - \rho\dot{D}_W\left(\Omega^a_b\,(Z)\,s_a\right) - \Omega^a_b\left([Z,W]_{h^*F}\right)s_a \\
& + \left(\Omega^a_c\,(Z)\,\Omega^c_b\,(W) - \Omega^a_c\,(W)\,\Omega^c_b\,(Z)\right)s_a \\
& = \left(d^{h^*F}\Omega^a_b\,(Z,W) + \Omega^a_c \wedge \Omega^c_b\,(Z,W)\right)s_a
\end{aligned}$$

it results the second identity.

Corollary 4.2.1 *In particular, if $h = Id_M$, then the identities (C_1) and (C_2) become*

$$(C_1') \qquad \rho\mathbb{T}^a = d^F s^a + \omega^a_b \wedge s^b,$$

and

$$(C_2') \qquad \rho\mathbb{R}^a_b = d^F \omega^a_b + \omega^a_c \wedge \omega^c_b$$

respectively.

Moreover, if $\rho = Id_{TM}$, then the identities (C_1') and (C_2') become:

$$(C_1'') \qquad \mathbb{T}^i = ddx^i + \omega^i_j \wedge dx^j = \omega^i_j \wedge dx^j$$

and

$$(C_2'') \qquad \mathbb{R}^i_j = d\omega^i_j + \omega^i_h \wedge \omega^h_j,$$

respectively. *q.e.d.*

Theorem 4.2.2 *The identities*

$$(B_1) \qquad d^{h^*F}(\rho,h)\,\mathbb{T}^a = (\rho,h)\,\mathbb{R}^a_b \wedge S^b - \Omega^a_c \wedge (\rho,h)\,\mathbb{T}^c$$

and

$$(B_2) \qquad d^{h^*F}(\rho,h)\,\mathbb{R}^a_b = (\rho,h)\,\mathbb{R}^a_c \wedge \Omega^c_b - \Omega^a_c \wedge (\rho,h)\,\mathbb{R}^c_b,$$

hold good. We will called these the first respectively the second identity of Bianchi type.

If the (ρ,h)-torsion is null, then the first identity of Bianchi type becomes:

$$(\tilde{B}_1) \qquad (\rho,h)\,\mathbb{R}^a_b \wedge s^b = 0.$$

Proof. We consider $(E,\pi,M) = (F,\nu,M)$. Using the first identity of Cartan type and the equality $d^{h^*F} \circ d^{h^*F} = 0$, we obtain:

$$d^{h^*F}(\rho,h)\,\mathbb{T}^a = d^{h^*F}\Omega^a_b \wedge S^b - \Omega^a_c \wedge d^{h^*F} S^c.$$

Using the second identity of Cartan type and the previous identity, we obtain:

$$d^{h^*F}(\rho,h)\,\mathbb{T}^a = ((\rho,h)\,\mathbb{R}^a_b - \Omega^a_c \wedge \Omega^c_b) \wedge S^b - \Omega^a_c \wedge \Big((\rho,h)\,\mathbb{T}^c - \Omega^c_b \wedge S^b\Big).$$

After some calculations, we obtain the first identity of Bianchi type.

Using the second identity of Cartan type and the equality $d^{h^*F} \circ d^{h^*F} = 0$, we obtain:

$$d^{h^*F}\Omega^a_c \wedge \Omega^c_b - \Omega^a_c \wedge d^{h^*F}\Omega^c_b = d^{h^*F}(\rho,h)\,\mathbb{R}^a_b.$$

Using the second of Cartan type and the previous identity, we obtain:

$$d^{h^*F}(\rho,h)\,\mathbb{R}^a_b = ((\rho,h)\,\mathbb{R}^a_c - \Omega^a_e \wedge \Omega^e_c) \wedge \Omega^c_b - \Omega^a_c \wedge ((\rho,h)\,\mathbb{R}^c_b - \Omega^c_e \wedge \Omega^e_b).$$

After some calculations, we obtain the second identity of Bianchi type. *q.e.d.*

Corollary 4.2.2 *In particular, if $h = Id_M$, then the identities (B_1) and (B_2) become*

$$(B_1') \qquad d^F \rho\mathbb{T}^a = \rho\mathbb{R}^a_b \wedge s^b - \omega^a_c \wedge \rho\mathbb{T}^c$$

and

$$(B_2') \qquad d^F \rho\mathbb{R}^a_b = \rho\mathbb{R}^a_c \wedge \omega^c_b - \omega^a_c \wedge \rho\mathbb{R}^c_b,$$

respectively.

Moreover, if $\rho = Id_{TM}$, then the identities (B_1') and (B_2') become:

$$(B_1'') \qquad d\mathbb{T}^i = \mathbb{R}^i_j \wedge dx^j - \omega^i_k \wedge \mathbb{T}^k$$

and

$$(B_2'') \qquad d\mathbb{R}^i_j = \mathbb{R}^i_h \wedge \omega^h_j - \omega^i_h \wedge \mathbb{R}^h_j,$$

respectively.

Theorem 4.2.3 *If* (E,π,M)=(F,ν,M)*, then the following relations hold good*

$$(\tilde{B}_1)\qquad \sum_{cyclic(u_1,u_2,u_3)}\Big\{\rho\ddot{D}_{U_1}\left(\left(\rho,h\right)\mathbb{T}\left(U_2,U_3\right)\right)-\left(\rho,h\right)\mathbb{R}\left(U_1,U_2\right)U_3 + \left(\rho,h\right)\mathbb{T}\left(\left(\rho,h\right)\ \mathbb{T}\left(U_1,U_2\right),U_3\right)\Big\}=0,$$

and

$$(\tilde{B}_2)\qquad \sum_{cyclic(u_1,u_2,u_3,u)}\Big\{\rho\ddot{D}_{U_1}\left(\left(\rho,h\right)\mathbb{R}\left(U_2,U_3\right)U\right)-\left(\rho,h\right)\mathbb{R}\left(\left(\rho,h\right)\mathbb{T}\left(U_1,U_2\right),U_3\right)U\Big\}=0.$$

respectively. This identities will be called the first respectively the second identity of Bianchi type.

In particular, if $h=Id_M$*, then the identities* $(\tilde{B}_1)$ *and* $(\tilde{B}_2)$ *become*

$$(\tilde{B}_1')\qquad \sum_{cyclic(u_1,u_2,u_3)}\left\{\rho D_{u_1}\left(\rho\mathbb{T}\left(u_2,u_3\right)\right)-\rho\mathbb{R}\left(u_1,u_2\right)u_3+\rho\mathbb{T}\left(\rho\ \mathbb{T}\left(u_1,u_2\right),u_3\right)\right\}=0,$$

$$(\tilde{B}_2')\qquad \sum_{cyclic(u_1,u_2,u_3,u)}\left\{\rho D_{u_1}\left(\rho\mathbb{R}\left(u_2,u_3\right)u\right)-\rho\mathbb{R}\left(\rho\mathbb{T}\left(u_1,u_2\right),u_3\right)u\right\}=0.$$

which will be called the first respectively the second identity of Bianchi type.

Remark 4.2.1 On components, the identities of Bianchi type $(\tilde{B}_1)$ and $(\tilde{B}_2)$ become:

$$(\tilde{B}_1'')\qquad \sum_{cyclic(a_1,a_2,a_3)}\Big\{\left(\rho,h\right)\mathbb{T}^a{}_{a_2a_3|a_1}+\left(\rho,h\right)\mathbb{T}^a{}_{ga_3}\cdot\left(\rho,h\right)\mathbb{T}^g{}_{a_1a_2}\Big\} = \sum_{cyclic(a_1,a_2,a_3)}\left(\rho,h\right)\mathbb{R}^a_{a_3\ a_1a_2}$$

and

$$(\tilde{B}_2'')\qquad \sum_{cyclic(a_1,a_2,a_3)}\Big\{\left(\rho,h\right)\mathbb{R}^a_{b\ a_2a_3|a_1}+\left(\rho,h\right)\mathbb{R}^a_{b\ ga_3}\cdot\left(\rho,h\right)\mathbb{T}^g{}_{a_1a_2}\Big\}=0.$$

If the (ρ,h)-torsion is null, then the identities of Bianchi type become:

$$(\tilde{B}_1''')\qquad \sum_{cyclic(a_1\ a_2,a_3)}\left(\rho,h\right)\mathbb{R}^a_{a_3,a_1a_2}=0$$

and

$$(\tilde{B}_2''')\qquad \sum_{cyclic(a_1,a_2,a_3)}\left(\rho,h\right)\mathbb{R}^a_{b\ a_2a_3|_{a_1}}=0.$$

4.3 (Pseudo)metrizable vector bundles

We will apply our theory for the diagram:

$$(4.3.1)\qquad \begin{array}{ccc} E & & \left(F,[,]_{F,h},(\rho,Id_M)\right) \\ \pi\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & M \end{array},$$

where $(E, \pi, M) \in |\mathbf{B^v}|$ and $\left((F, \nu, M), [,]_{F,h}, (\rho, Id_M)\right) \in |\mathbf{GLA}|$.

Definition 4.3.1 We will say that the *vector bundle* (E, π, M) *is endowed with a pseudometrical structure* if it exists

$$g = g_{ab} s^a \otimes s^b \in \mathcal{T}\,{}^0_2\,(E, \pi, M)$$

such that for each $x \in M$, the matrix $\|g_{ab}(x)\|$ is nondegenerate and symmetric.

Moreover, if for each $x \in M$ the matrix $\|g_{ab}(x)\|$ has constant signature, then we will say that *the vector bundle* (E, π, M) *is endowed with a metrical structure.*

If

$$g = g_{ab} s^a \otimes s^b \in \mathcal{T}\,{}^0_2\,(E, \pi, M)$$

is a (pseudo) metrical structure, then, for any $a, b \in \overline{1, r}$ and for any vector local $(m+r)$-chart (U, s_U) of (E, π, M), we consider the real functions

$$U \xrightarrow{\tilde{g}^{ba}} \mathbb{R}$$

such that

$$\left\|\tilde{g}^{ba}(x)\right\| = \|g_{ab}(x)\|^{-1},$$

for any $\forall x \in U$.

Definition 4.3.2 Let (E, π, M) be a vector bundle endowed with a (pseudo)metrical structure g and with a linear ρ-connection $\rho\Gamma$.

We will say that the *linear ρ-connection $\rho\Gamma$ is compatible with the (pseudo)metrical structure g* if

$$\rho D_z g = 0,\ \forall z \in \Gamma(F, \nu, M). \tag{4.3.2}$$

Definition 4.3.3 We will say that the vector bundle (E, π, M) is ρ-*(pseudo)metrizable,* if it exists a (pseudo)metrical structure

$$g \in \mathcal{T}_2^0\,(E, \pi, M)$$

and a linear ρ-connection $\rho\Gamma$ for (E, π, M) compatible with g. The id_{TM}-(pseudo)metrizable vector bundles will be called *(pseudo)metrizable vector bundles.*

In particular, if (TM, τ_M, M) is a (pseudo)metrizable vector bundle, then we will say that (TM, τ_M, M) *is a* (pseudo)Riemannian space*,* and the manifold M will be called the *(pseudo)Riemannian manifold.*

The linear connection of a (pseudo)Riemannian space will be called *(pseudo)Riemannian linear connection.*

Theorem 4.3.1 *If $(E, \pi, M) = (F, \nu, M)$ and $g \in \mathcal{T}_2^0\,(h^*E, h^*\pi, M)$ is a (pseudo)metrical structure, then the local real functions*

$$\begin{aligned}\rho\Gamma^a_{bc} &= \frac{1}{2}\tilde{g}^{ad}\left(\rho^k_c \circ h\frac{\partial g_{bd}}{\partial x^k} + \rho^j_b \circ h\frac{\partial g_{dc}}{\partial x^j} - \rho^h_d \circ h\frac{\partial g_{bc}}{\partial x^h}\right. \\ &\quad \left. + g_{ec}L^e_{bd} \circ h + g_{be}L^e_{dc} \circ h - g_{de}L^e_{bc} \circ h\right).\end{aligned} \tag{4.3.3}$$

are the components of a linear ρ-connection $\rho\Gamma$ for the vector bundle $(h^*E, h^*\pi, M)$ compatible with g such that $(\rho, h)\,\mathbb{T} = 0$.

Therefore, the vector bundle $(h^*E, h^*\pi, M)$ becomes ρ-(pseudo)metrizable.

The linear ρ-connection $\rho\Gamma$ will be called the *linear ρ-connection of Levi-Civita type.*

Proof. Since

$$\begin{aligned}\left(\rho\ddot{D}_U g\right) V\otimes Z &= \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)\left((g(V\otimes Z)) - g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right)\right.\\ &\left. - g\left(V\otimes\left(\rho\ddot{D}_U Z\right)\right)\right),\ \forall U,V,Z\in\Gamma\left(h^*E, h^*\pi, M\right).\end{aligned}$$

It results that, for any $U,V,Z\in\Gamma\left(h^*E,h^*\pi,M\right)$, we obtain the equalities:

$$(1)\qquad \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)\left(g\left(V\otimes Z\right)\right) = g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right) + g\left(V\otimes\left(\rho\ddot{D}_U Z\right)\right),$$

$$(2)\qquad \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)\left(g\left(U\otimes V\right)\right) = g\left(\left(\rho\ddot{D}_Z U\right)\otimes V\right) + g\left(U\otimes\left(\rho\ddot{D}_Z V\right)\right),$$

$$(3)\qquad \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)\left(g\left(Z\otimes U\right)\right) = g\left(\left(\rho\ddot{D}_V Z\right)\otimes U\right) + g\left(Z\otimes\left(\rho\ddot{D}_V U\right)\right).$$

We observe that $(1)+(3)-(2)$ is equivalent with the equality:

$$\begin{aligned}&g\left(\left(\rho\ddot{D}_U V + \rho\ddot{D}_V U\right)\otimes Z\right) + g\left(\left(\rho\ddot{D}_V Z - \rho\ddot{D}_Z V\right)\otimes U\right)\\ &\quad + g\left(\left(\rho\ddot{D}_U Z - \rho\ddot{D}_Z U\right)\otimes V\right) = \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)\left(g\left(V\otimes Z\right)\right)\\ &\quad + \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)\left(g\left(Z\otimes U\right)\right) - \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)\left(g\left(U\otimes V\right)\right).\end{aligned}$$

Using the condition $(\rho,h)\,\mathbb{T}=0$, which is equivalent with the equality

$$\rho\ddot{D}_U V - \rho\ddot{D}_V U - [U,V]_{h^*E} = 0,$$

we obtain:

$$\begin{aligned}2g\left(\left(\rho\ddot{D}_U V\right)\otimes Z\right) &= \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(U)\cdot\left(g\left(V\otimes Z\right)\right)\\ &+ \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(V)\left(g\left(Z\otimes U\right)\right) - \Gamma\left(\overset{h^*E}{\rho}, Id_M\right)(Z)\left(g\left(U\otimes V\right)\right)\\ &+ g\left([U,V]_{h^*E}\otimes Z\right) - g\left([U,Z]_{h^*E}\otimes V\right)\\ &- g\left([V,Z]_{h^*E}\otimes U\right),\ \forall U,V,Z\in\Gamma\left(h^*E,h^*\pi,M\right).\end{aligned}$$

Therefore, we obtain the equality:

$$\begin{aligned}2g\left(\left(\rho\Gamma_{ba}^{d}S_d\right)\otimes S_c\right) &= \rho_a^i\circ h\frac{\partial g\left(S_b\otimes S_c\right)}{\partial x^i} + \rho_b^j\circ h\frac{\partial g\left(S_c\otimes S_a\right)}{\partial x^j} - \rho_c^k\circ h\frac{\partial g\left(S_a\otimes S_b\right)}{\partial x^k}\\ &+ g\left(\left(L_{ab}^d\circ h\right)S_d\otimes S_c\right) - g\left(\left(L_{ac}^d\circ h\right)S_d\otimes S_b\right) - g\left(\left(L_{bc}^d\circ h\right)S_d\otimes S_a\right),\end{aligned}$$

which is equivalent with:

$$\begin{aligned}2g_{dc}\rho\Gamma_{ba}^d &= \rho_a^i\circ h\frac{\partial g_{bc}}{\partial x^i} + \rho_b^j\circ h\frac{\partial g_{ca}}{\partial x^j} - \rho_c^k\circ h\frac{\partial g_{ab}}{\partial x^k} + \left(L_{ab}^d\circ h\right)g_{dc}\\ &- \left(L_{ac}^d\circ h\right)g_{db} - \left(L_{bc}^d\circ h\right)g_{da}.\end{aligned}$$

Finally, we obtain:

$$\begin{aligned}\rho\Gamma_{ba}^d &= \frac{1}{2}\tilde{g}^{dc}\left(\rho_a^i\circ h\frac{\partial g_{bc}}{\partial x^i} + \rho_b^j\circ h\frac{\partial g_{ca}}{\partial x^j} - \rho_c^k\circ h\frac{\partial g_{ab}}{\partial x^k}\right.\\ &\left. + \left(L_{ab}^d\circ h\right)g_{dc} - \left(L_{ac}^d\circ h\right)g_{db} - \left(L_{bc}^d\circ h\right)g_{da}\right),\end{aligned}$$

where $\left\|\tilde{g}^{dc}(x)\right\| = \|g_{cd}(x)\|^{-1}$, for any $x \in M$. *q.e.d.*

Corollary 4.3.1 *In particular, if $h = Id_M$, $(E,\pi,M) = (F,\nu,M)$ and $g \in \mathcal{T}\,{}^{0}_{2}(E,\pi,M)$ is a (pseudo)metrical structure, then the local real functions*

$$\rho\Gamma^a_{bc} = \frac{1}{2}\tilde{g}^{ad}\left(\rho^k_c\frac{\partial g_{bd}}{\partial x^k} + \rho^j_b\frac{\partial g_{dc}}{\partial x^j} - \rho^h_d\frac{\partial g_{bc}}{\partial x^h} + g_{ec}L^e_{bd} + g_{be}L^e_{dc} - g_{de}L^e_{bc}\right). \tag{4.3.3'}$$

are the components of a linear ρ-connection $\rho\Gamma$ for the vector bundle (E,π,M) compatible with g such that $\rho\mathbb{T} = 0$.

Therefore, the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

The linear ρ-connection $\rho\Gamma$ will be called the *linear ρ-connection of Levi-Civita type.*

In particular, if $\rho = Id_{TM}$, we obtain the classical Levi-Civita linear connection.

Theorem 4.3.2. *If $(E,\pi,M) = (F,\nu,M)$, $g \in \mathcal{T}\,{}^{0}_{2}(h^*E,h^*\pi,M)$ is a pseudo(metrical) structure and $\mathbb{T} \in \mathcal{T}\,{}^{1}_{2}(h^*E,h^*\pi,M)$ such that its components are skew symmetric in the lover indices, then the local real functions*

$$\rho\overset{\circ}{\Gamma}{}^a_{bc} = \rho\Gamma^a_{bc} + \frac{1}{2}\tilde{g}^{ad}\left(g_{de}\mathbb{T}^e_{bc} - g_{be}\mathbb{T}^e_{dc} + g_{ec}\mathbb{T}^e_{bd}\right), \tag{4.3.4}$$

are the components of a linear ρ-connection compatible with the (pseudo) metrical structure g, where $\rho\Gamma^a_{bc}$ are the components of linear ρ-connection of Levi-Civita type. Therefore, the vector bundle $(h^*E,h^*\pi,M)$ becomes ρ-(pseudo)metrizable.

In addition, the tensor field $\mathbb{T}$ is the (ρ,h)-torsion tensor field.

Corollary 4.3.2 *In particular, if $h = Id_M$, $(E,\pi,M) = (F,\nu,M)$, $g \in \mathcal{T}\,{}^{0}_{2}(E,\pi,M)$ is a pseudo(metrical) structure and $T \in \mathcal{T}\,{}^{1}_{2}(E,\pi,M)$ such that its components are skew symmetric in the lover indices, then the local real functions*

$$\rho\overset{\circ}{\Gamma}{}^a_{bc} = \rho\Gamma^a_{bc} + \frac{1}{2}\tilde{g}^{ad}\left(g_{de}\mathbb{T}^e_{bc} - g_{be}\mathbb{T}^e_{dc} + g_{ec}\mathbb{T}^e_{bd}\right), \tag{4.3.4'}$$

are the components of a linear ρ-connection compatible with the (pseudo)metrical structure g, where $\rho\Gamma^a_{bc}$ are the components of linear ρ-connection of Levi-Civita type. Therefore, the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

In addition, the tensor field $\mathbb{T}$ is the ρ-torsion tensor field.

Theorem 4.3.3 *If $g \in \mathcal{T}\,{}^{0}_{2}(E,\pi,M)$ is a pseudo (metrical) structure and $\rho\overset{\circ}{\Gamma}$ is a linear ρ-connection for the vector bundle (E,π,M), then the local real functions*

$$\rho\overset{k}{\Gamma}{}^a_{b\alpha} = \rho\overset{\circ}{\Gamma}{}^a_{b\alpha} + \tfrac{1}{2}\tilde{g}^{ac}g_{cb\overset{\circ}{|}\alpha} \tag{4.3.5}$$

are the components of a linear ρ-connection compatible with the (pseudo) metrical structure g. Therefore, the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

Theorem 4.3.4 *If $g \in \mathcal{T}\,{}^{0}_{2}(E,\pi,M)$ is a pseudo (metrical) structure, $\rho\overset{\circ}{\Gamma}$ is a linear ρ-connection for the vector bundle (E,π,M) and $T = T^d_{c\alpha}s_d \otimes s^c \otimes t^\alpha$, then the local real functions*

$$\rho\Gamma^a_{b\alpha} = \rho\overset{k}{\Gamma}{}^a_{b\alpha} + \tfrac{1}{2}O^{ca}_{bd}T^d_{c\alpha}, \tag{4.3.6}$$

are the components of a linear ρ-connection compatible with (pseudo) metrical structure g, where

$$O_{bd}^{ca} = \frac{1}{2}\left(\delta_b^c \delta_d^a - g_{bd}\tilde{g}^{ca}\right) \tag{4.3.7}$$

is the Obata operator.

Therefore, the vector bundle (E, π, M) becomes ρ-(pseudo)metrizable.

4.4 Lifts of differentiable curves

In this section we extend the notion of lift of a curve c at the total space of a vector bundle using the new notion of *locally invertible* $\mathbf{B^v}$*-morphism.*

4.4.1 The lift of a differentiable curve at the total space of a vector bundle

We consider the following diagram:

$$\begin{array}{ccc} E & & \left(F, [,]_{F,h}, (\rho,\eta)\right) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array} \tag{4.4.1.1}$$

where $(E, \pi, M) \in |\mathbf{B^v}|$ and $\Big((F, \nu, M), [,]_{F,h}, (\rho, \eta)\Big) \in |\mathbf{GLA}|$.

We admit that $(\rho, \eta)\,\Gamma$ is a (ρ, η)-connection for the vector bundle (E, π, M).

Let

$$I \ \xrightarrow{c}\ M$$

be a differentiable curve.

We say that

$$\left(E_{|\operatorname{Im}(\eta\circ h\circ c)}, \pi_{|\operatorname{Im}(\eta\circ h\circ c)}, \operatorname{Im}(\eta\circ h\circ c)\right)$$

is a vector subbundle of the vector bundle (E, π, M).

Definition 4.4.1.1 Let

$$\begin{array}{ccc} I & \xrightarrow{\dot{c}} & E_{|\operatorname{Im}(\eta\circ h\circ c)} \\ t & \longmapsto & y^a(t)\, s_a(\eta\circ h\circ c(t)) \end{array} \tag{4.4.1.2}$$

be a differentiable curve.

If there exists $g \in \mathbf{Man}(E, F)$ such that the following conditions are satisfied:

1. $(g, h) \in \mathbf{B}^v((E, \pi, M), (F, \nu, N))$ and
2. $\rho\circ g\circ \dot{c}(t) = \dfrac{d(\eta\circ h\circ c)^i(t)}{dt}\dfrac{\partial}{\partial x^i}((\eta\circ h\circ c)(t))$, for any $t \in I$, then we will say that $\dot{c}$ *is the (g, h)-lift of the differentiable curve c.*

Remark 4.4.1.1 Condition 2 is equivalent with the following affirmation:

$$\rho_\alpha^i(\eta\circ h\circ c(t))\cdot g_a^\alpha(h\circ c(t))\cdot y^a(t) = \frac{d(\eta\circ h\circ c)^i(t)}{dt},\ i\in\overline{1,m}. \tag{4.4.1.3}$$

Definition 4.4.1.2 If
$$I \overset{\dot{c}}{\to} E_{|\operatorname{Im}(\eta\circ h\circ c)}$$
is a differentiable (g,h)-lift of the differentiable curve c, then the section

(4.4.1.4)
$$\begin{array}{ccc} \operatorname{Im}(\eta\circ h\circ c) & \xrightarrow{u(c,\dot{c})} & E_{|\operatorname{Im}(\eta\circ h\circ c)} \\ \eta\circ h\circ c(t) & \longmapsto & \dot{c}(t) \end{array}$$

will be called the *canonical section associated to the couple* $(c,\dot{c})$.

We will denote by $\left(T^E(c,\dot{c}),\tau,\operatorname{Im}(\eta\circ h\circ c)\right)$ the vector subbundle with minimal dimension such that

(4.4.1.5)
$$u(c,\dot{c})\in\Gamma\left(T^E(c,\dot{c}),\tau,\operatorname{Im}(\eta\circ h\circ c)\right)$$

and will denoted by $\left(S^E(c,\dot{c}),\sigma,\operatorname{Im}(\eta\circ h\circ c)\right)$ the vector subbundle such that
$$T^E(c,\dot{c})\oplus S^E(c,\dot{c})=E_{|\operatorname{Im}(\eta\circ h\circ c)}.$$

Definition 4.4.1.3 If $(g,h)\in\mathbf{B^v}((E,\pi,M),(F,\nu,N))$ has the components
$$g_a^\alpha;a\in\overline{1,r},\ \alpha\in\overline{1,p}$$
such that for any local vector $(n+p)$-chart (V,t_V) of (F,ν,N) there exists the real functions
$$V\ \xrightarrow{\tilde{g}_\alpha^a}\ \mathbb{R}\ ;\ a\in\overline{1,r},\ \alpha\in\overline{1,p}$$
such that
$$\tilde{g}_\alpha^b(\varkappa)\cdot g_a^\alpha(\varkappa)=\delta_a^b,$$
for any $\varkappa\in V$, then we will say that *the* $\mathbf{B^v}$*-morphism* (g,h) *is locally invertible.*

Remark 4.4.2.2 In particular, if $(Id_{TM},Id_M,Id_M)=(\rho,\eta,h)$ and the $\mathbf{B^v}$ morphism (g,Id_M) is locally invertible, then we have the differentiable (g,Id_M)-lift

(4.4.1.6)
$$\begin{array}{ccc} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \tilde{g}_j^i(c(t))\dfrac{dc^j(t)}{dt}\dfrac{\partial}{\partial x^i}(c(t)) \end{array}.$$

Moreover, if $g=Id_{TM}$, then we obtain the usual lift of tangent vectors

(4.4.1.6)′
$$\begin{array}{ccc} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \dfrac{dc^i(t)}{dt}\dfrac{\partial}{\partial x^i}(c(t)) \end{array}.$$

Definition 4.4.1.4 If

(4.4.1.7)
$$I\ \xrightarrow{\dot{c}}\ E_{|\operatorname{Im}(\eta\circ h\circ c)}$$

is a differentiable (g,h)-lift of differentiable curve c, such that its components functions $\left(y^a,\ a\in\overline{1,n}\right)$ are solutions for the differentiable system of equations:

(4.4.1.8)
$$\frac{du^a}{dt}+(\rho,\eta)\Gamma_\alpha^a\circ u(c,\dot{c})\circ(\eta\circ h\circ c)\cdot g_b^\alpha\circ h\circ c\cdot u^b=0,$$

then we will say that *the (g,h)-lift $\dot{c}$ is parallel with respect to the (ρ,η)-connection* $(\rho,\eta)\,\Gamma$.

Remark 4.4.1.3 In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$ and the $\mathbf{B^v}$ morphism (g,Id_M) is locally invertible, then the differentiable (g,Id_{TM})-lift

$$\begin{array}{ccc} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \left(\tilde{g}^i_j\circ c\cdot\dfrac{dc^j}{dt}\right)\dfrac{\partial}{\partial x^i}\left(c\left(t\right)\right), \end{array} \tag{4.4.1.9}$$

is parallel with respect to the connection Γ if the component functions

$$\left(\tilde{g}^i_j\circ c\cdot\frac{dc^j}{dt},\ i\in\overline{1,n}\right)$$

are solutions for the differentiable system of equations

$$\frac{du^i}{dt}+\Gamma^i_k\circ u\left(c,\dot{c}\right)\circ c\cdot g^k_h\circ c\cdot u^h=0, \tag{4.4.1.10}$$

namely

$$\begin{array}{l} \dfrac{d}{dt}\left(\tilde{g}^i_j\left(c\left(t\right)\right)\cdot\dfrac{dc^j\left(t\right)}{dt}\right) \\ \qquad +\Gamma^i_k\left(c\left(t\right),\left(\tilde{g}^i_j\left(c\left(t\right)\right)\cdot\dfrac{dc^j\left(t\right)}{dt}\right)\cdot\dfrac{\partial}{\partial x^i}\left(c\left(t\right)\right)\right)\cdot\dfrac{dc^k\left(t\right)}{dt}=0. \end{array} \tag{4.4.1.10$'$}$$

Moreover, if $g=Id_{TM}$, then the usual lift of tangent vectors $(4.4.1.6)'$ is parallel with respect to the connection Γ if the component functions $\left(\frac{dc^j}{dt},\ j\in\overline{1,n}\right)$ are solutions for the differentiable system of equations

$$\frac{du^i}{dt}+\Gamma^i_k\circ u\left(c,\dot{c}\right)\circ c\cdot u^k=0, \tag{4.4.1.10$''$}$$

namely

$$\frac{d}{dt}\left(\frac{dc^j\left(t\right)}{dt}\right)+\Gamma^i_k\left(c\left(t\right),\frac{dc^j\left(t\right)}{dt}\cdot\frac{\partial}{\partial x^i}\left(c\left(t\right)\right)\right)\cdot\frac{dc^k\left(t\right)}{dt}=0. \tag{4.4.1.10$'''$}$$

4.4.2 The lift of a differentiable curve at the total space of dual vector bundle

We consider the following diagram:

$$\begin{array}{ccc} \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \Big\downarrow{\scriptstyle\overset{*}{\pi}} & & \Big\downarrow{\scriptstyle\nu} \\ M & \xrightarrow{\ \ h\ \ } & N \end{array} \tag{4.4.2.1}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)\in|\mathbf{GLA}|$.

We admit that $(\rho,\eta)\overset{*}{\Gamma}$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

Let

$$I \xrightarrow{c} M$$

be a differentiable curve. We say that

$$\left(\overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)},\overset{*}{\pi}_{|\operatorname{Im}(\eta\circ h\circ c)},\operatorname{Im}(\eta\circ h\circ c)\right)$$

is a vector subbundle of the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

Definition 4.4.2.1 Let

$$\begin{array}{rcl} I & \xrightarrow{\dot{c}} & \overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)} \\ t & \longmapsto & p_a(t)\, s^a(\eta\circ h\circ c(t)) \end{array} \tag{4.4.2.2}$$

be a differentiable curve.

If there exists $g\in \mathbf{Man}\left(\overset{*}{E},F\right)$ such that the following conditions are satisfied:

1. $(g,h)\in \mathbf{B}^v\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),(F,\nu,N)\right)$ and
2. $\rho\circ g\circ \dot{c}(t)=\dfrac{d(\eta\circ h\circ c)^i(t)}{dt}\dfrac{\partial}{\partial x^i}((\eta\circ h\circ c)(t))$, for any $t\in I$,then we will say that $\dot{c}$ *is the* (g,h)*-lift of the differentiable curve* c.

Remark 4.4.2.1 Condition 2 is equivalent with the following affirmation:

$$\rho^i_\alpha(\eta\circ h\circ c(t))\, g^{\alpha a}(h\circ c(t))\, p_a(t)=\frac{d(\eta\circ h\circ c)^i(t)}{dt},\ \ i\in\overline{1,m}. \tag{4.4.2.3}$$

Definition 4.4.2.2 If

$$I \xrightarrow{\dot{c}} \overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)}$$

is a differentiable (g,h)-lift of the differentiable curve c, then the section

$$\begin{array}{rcl} \operatorname{Im}(\eta\circ h\circ c) & \xrightarrow{\overset{*}{u}(c,\dot{c})} & \overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)} \\ \eta\circ h\circ c(t) & \longmapsto & \dot{c}(t) \end{array} \tag{4.4.2.4}$$

will be called the *canonical section associated to the couple* $(c,\dot{c})$.

We will denote by $\left(T^{\overset{*}{E}}(c,\dot{c}),\tau,\operatorname{Im}(\eta\circ h\circ c)\right)$ the vector subbundle with minimal dimension such that

$$\overset{*}{u}(c,\dot{c})\in\Gamma\left(T^{\overset{*}{E}}(c,\dot{c}),\tau,\operatorname{Im}(\eta\circ h\circ c)\right) \tag{4.4.2.5}$$

and will denoted by $\left(S^{\overset{*}{E}}(c,\dot{c}),\sigma,\operatorname{Im}(\eta\circ h\circ c)\right)$ the vector subbundle such that

$$T^{\overset{*}{E}}(c,\dot{c})\oplus S^{\overset{*}{E}}(c,\dot{c})=\overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)}.$$

Definition 4.4.2.3 If $(g,h) \in \mathbf{B^v}\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),(F,\nu,N)\right)$ has the components

$$g^{\alpha a}; a\in\overline{1,r},\ \alpha\in\overline{1,p}$$

such that for any vector local $(n+p)$-chart (V,t_V) of (F,ν,N) there exists the real functions

$$V \xrightarrow{\tilde{g}_{a\alpha}} \mathbb{R}\ ;\ a\in\overline{1,r},\ \alpha\in\overline{1,p}$$

such that

$$\tilde{g}_{a\alpha}(\varkappa)\cdot g^{\alpha b}(\varkappa)=\delta_a^b,\ \forall\varkappa\in V,$$

then we will say that *the* $\mathbf{B^v}$*-morphism* (g,h) *is locally invertible.*

Remark 4.4.2.2 In particular, if $(Id_{TM},Id_M,Id_M)=(\rho,\eta,h)$ and the $\mathbf{B^v}$ morphism (g,Id_M) is locally invertible, then we have the differentiable (g,Id_M)-lift

$$\begin{array}{ccc} I & \xrightarrow{\dot{c}} & \overset{*}{T}M \\ t & \longmapsto & \tilde{g}_{ji}(c(t))\dfrac{dc^j(t)}{dt}dx^i(c(t)) \end{array}. \tag{4.4.2.6}$$

Definition 4.4.2.4 If

$$I \xrightarrow{\dot{c}} \overset{*}{E}_{|\operatorname{Im}(\eta\circ h\circ c)} \tag{4.4.2.7}$$

is a differentiable (g,h)-lift for the curve c such that its components functions $\left(p_b,\ b\in\overline{1,r}\right)$ are solutions for the differentiable system of equations:

$$\frac{du_b}{dt}+(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}\circ\overset{*}{u}(c,\dot{c})\circ(\eta\circ h\circ c)\cdot g^{a\alpha}\circ h\circ c\cdot u_a=0, \tag{4.4.2.8}$$

then we will say that *the* (g,h)*-lift* $\dot{c}$ *is parallel with respect to the* (ρ,η)*-connection* $(\rho,\eta)\overset{*}{\Gamma}$.

Remark 4.4.2.3 In particular, if $(Id_{TM},Id_M,Id_M)=(\rho,\eta,h)$ and the $\mathbf{B^v}$ morphism (g,Id_M) is locally invertible, then the differentiable (g,Id_M)-lift (4.4.2.6) is parallel with respect to the connection Γ if the component functions $\left(\tilde{g}_{ji}\circ c\cdot\dfrac{dc^i}{dt},\ j\in\overline{1,m}\right)$ are solutions for the differentiable system of equations

$$\frac{du_j}{dt}+\Gamma_{jk}\circ\overset{*}{u}(c,\dot{c})\circ c\cdot g^{kh}\circ c\cdot u_h=0, \tag{4.4.2.9}$$

namely

$$\begin{array}{l} \dfrac{d}{dt}\left(\tilde{g}_{ji}\circ c(t)\cdot\dfrac{dc^i(t)}{dt}\right) \\ \qquad +\Gamma_{jk}\left(c(t),\left(\tilde{g}_{ji}\circ c(t)\cdot\dfrac{dc^j(t)}{dt}\right)\cdot dx^i(c(t))\right)\cdot\dfrac{dc^k(t)}{dt}=0, \end{array} \tag{4.4.2.9$'$}$$

4.5 Parallel transport

We consider the following diagram:

$$\begin{array}{ccccc} & & E & \xrightarrow{g} & (F,[,]_F,(\rho,Id_M)) \\ (4.5.1) & & \downarrow\pi & & \downarrow\nu \\ I & \xrightarrow{c} & M & \xrightarrow{Id_M} & M \end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$, $((F,\nu,M),[,]_F,(\rho,Id_M))\in|\mathbf{LA}|$, (g,Id_M) is a $\mathbf{B^v}$-morphism and c is a differentiable curve.

Let $\dot{c}$ be a (g,Id_M)-lift of the curve c.

We admit that $\rho\Gamma$ is a linear ρ-connection for the vector bundle (E,π,M).

Definition 4.5.1 We will called *parallel transport of tensor fields of* (r,s) *type along a curve* c any family

$$\mathcal{P}_c=\left\{P_{t_1,t_2}\in Izo\left(\mathcal{T}_q^p(E,\pi,M)_{c(t_1)},\mathcal{T}_q^p(E,\pi,M)_{c(t_2)}\right),\ t_1,t_2\in I\right\}$$

with the following properties:

1. For any $t_1,t_2\in I$ it exists a unique isomorphism $P_{t_1,t_2}\in\mathcal{P}_c$ such that $(P_{t_1,t_2})^{-1}=P_{t_2,t_1}$.
2. For any $t_1,t_2,t_3\in I$ we have that $P_{t_2,t_3}\circ P_{t_1,t_2}=P_{t_1,t_3}$.

Theorem 4.5.1 *If* $t_0,t\in I$ *and* U *is a local vector* $(m+n)$*-chart such that* $c(t_0),c(t)\in U$, *then it exists an unique isomorphism*

$$P_{t_0,t}\in Izo\left(\mathcal{T}_q^p(E,\pi,M)_{c(t_0)},\mathcal{T}_q^p(E,\pi,M)_{c(t)}\right)$$

such that $(P_{t_0,t})^{-1}=P_{t,t_0}$ *which not depend on the local vector chart used.*

Proof. Let $T_{c(t_0)}\in\mathcal{T}_q^p(E,\pi,M)_{c(t_0)}$ be. We admit that

$$T_{\pi\circ c(t_0)}=\left(T^{a_1,\dots,a_p}_{b_1,\dots,b_q}s_{a_1}\otimes\dots\otimes s_{a_p}\otimes s^{b_1}\otimes\dots\otimes s^{b_q}\right)(c(t_0))$$

and

$$\begin{aligned} P_{t_0,t}\left(T_{c(t_0)}\right) &= T^{a_1,\dots,a_p}_{b_1,\dots,b_q}(c(t_0))\,A^{\tilde{a}_1}_{a_1}(t_0,t)\cdot\ldots\cdot A^{\tilde{a}_p}_{a_p}(t_0,t)\cdot B^{b_1}_{\tilde{b}_1}(t_0,t)\cdot \\ &\ldots\cdot B^{b_q}_{\tilde{b}_q}(t_0,t)\cdot\cdot\left(s_{\tilde{a}_1}\otimes\dots\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes\dots\otimes s^{\tilde{b}_q}\right)(c(t_0)), \end{aligned}$$

where the matrices

$$\left\|A^{\tilde{a}_1}_{a_1}(t_0,t)\right\|,\dots\left\|A^{\tilde{a}_p}_{a_p}(t_0,t)\right\|,\left\|B^{b_1}_{\tilde{b}_1}(t_0,t)\right\|,\dots,\left\|B^{b_q}_{\tilde{b}_q}(t_0,t)\right\|$$

are the matrices used for base transformation. Using the equality

$$\begin{aligned}
0=\frac{d}{dt}\left(T_{b_1,\dots,b_q}^{a_1,\dots,a_p}c\left(t_0\right)\right) &= \frac{d}{dt}\left(T_{b_1,\dots,b_q}^{a_1,\dots,a_p}c\left(t_0\right)A_{a_1}^{\tilde{a}_1}\left(t_0,t\right)\cdot\ldots\cdot A_{a_p}^{\tilde{a}_p}\left(t_0,t\right)\right.\\
&\left.\cdot B_{\tilde{b}_1}^{b_1}\left(t_0,t\right)\ldots\cdot B_{\tilde{b}_q}^{b_q}\left(t_0,t\right)\right)\cdot A_{\tilde{a}_1}^{a_1}\left(t,t_0\right)\cdot\ldots\\
&\cdot A_{\tilde{a}_p}^{a_p}\left(t,t_0\right)\cdot B_{b_1}^{\tilde{b}_1}\left(t,t_0\right)\ldots\cdot B_{b_q}^{\tilde{b}_q}\left(t,t_0\right)+T_{b_1,\dots,b_q}^{a_1,\dots,a_p}c\left(t_0\right)\\
&\cdot A_{a_1}^{\tilde{a}_1}\left(t_0,t\right)\cdot\ldots\cdot A_{a_p}^{\tilde{a}_p}\left(t_0,t\right)\cdot B_{\tilde{b}_1}^{b_1}\left(t_0,t\right)\cdot\ldots\\
&\cdot B_{\tilde{b}_q}^{b_q}\left(t_0,t\right)\cdot\frac{d}{dt}\left(A_{\tilde{a}_1}^{a_1}\left(t,t_0\right)\cdot\ldots\cdot A_{\tilde{a}_p}^{a_p}\left(t,t_0\right)\right.\\
&\left.\cdot B_{b_1}^{\tilde{b}_1}\left(t,t_0\right)\ldots\cdot B_{b_q}^{\tilde{b}_q}\left(t,t_0\right)\right)
\end{aligned}$$

and the notation

$$\tilde{T}_{b_1,\dots,b_q}^{a_1,\dots,a_p}c\left(t\right) = T_{b_1,\dots,b_q}^{a_1,\dots,a_p}\left(\pi\circ c\left(t_0\right)\right)A_{a_1}^{\tilde{a}_1}\left(t_0,t\right)\cdot\ldots\cdot A_{a_p}^{\tilde{a}_p}\left(t_0,t\right)\cdot B_{\tilde{b}_1}^{b_1}\left(t_0,t\right)\ldots\cdot B_{\tilde{b}_q}^{b_q}\left(t_0,t\right)$$

we obtain the equality

$$\begin{aligned}
-\frac{d}{dt}\tilde{T}_{\tilde{b}_1,\dots,\tilde{b}_q}^{\tilde{a}_1,\dots,\tilde{a}_p}c\left(t\right) &= A_{a_1}^{\tilde{a}_1}\left(t_0,t\right)\frac{d}{dt}A_{a}^{a_1}\left(t,t_0\right)T_{\tilde{b}_1,\dots,\tilde{b}_q}^{a\tilde{a}_2,\dots,\tilde{a}_p}c\left(t\right)+\ldots\\
&+A_{a_p}^{\tilde{a}_p}\left(t_0,t\right)\frac{d}{dt}A_{a}^{a_p}\left(t,t_0\right)T_{\tilde{b}_1,\dots,\tilde{b}_q}^{\tilde{a}_1,\dots,\tilde{a}_{p-1}a}c\left(t\right)\\
&+B_{\tilde{b}_1}^{b_1}\left(t_0,t\right)\frac{d}{dt}B_{b_1}^{b}\left(t,t_0\right)T_{b\tilde{b}_2,\dots,\tilde{b}_q}^{\tilde{a}_1,\dots,\tilde{a}_p}c\left(t\right)+\ldots\\
&+B_{\tilde{b}_q}^{b_q}\left(t_0,t\right)\frac{d}{dt}B_{b_q}^{b}\left(t,t_0\right)T_{\tilde{b}_1,\dots,\tilde{b}_{q-1}b}^{\tilde{a}_1,\dots,\tilde{a}_p}c\left(t\right).
\end{aligned}$$

Since the differentiable equations:

$$\begin{aligned}
A_{a_1}^{\tilde{a}_1}\left(t_0,t\right)\frac{d}{dt}A_{a}^{a_1}\left(t,t_0\right) &= \rho\Gamma_{a\alpha}^{\tilde{a}_1}c\left(t\right)g_c^{\alpha}\left(x\left(t\right)\right)y^c\left(t\right)\\
A_{\tilde{a}_1}^{a_1}\left(t_0,t_0\right) &= \delta_{\tilde{a}_1}^{a_1}\\
&\ldots\\
A_{a_p}^{\tilde{a}_p}\left(t_0,t\right)\frac{d}{dt}A_{a}^{a_p}\left(t,t_0\right) &= \rho\Gamma_{a\alpha}^{\tilde{a}_p}c\left(t\right)g_c^{\alpha}\left(x\left(t\right)\right)y^c\left(t\right)\\
A_{\tilde{a}_p}^{a_p}\left(t_0,t_0\right) &= \delta_{\tilde{a}_p}^{a_p}\\
B_{\tilde{b}_1}^{b_1}\left(t_0,t\right)\frac{d}{dt}B_{b_1}^{b}\left(t,t_0\right) &= -\rho\Gamma_{\tilde{b}_1\alpha}^{b}c\left(t\right)g_c^{\alpha}\left(x\left(t\right)\right)y^c\left(t\right)\\
B_{b_1}^{\tilde{b}_1}\left(t_0,t_0\right) &= \delta_{b_1}^{\tilde{b}_1}\\
&\ldots\\
B_{\tilde{b}_q}^{b_q}\left(t_0,t\right)\frac{d}{dt}B_{b_q}^{b}\left(t,t_0\right) &= -\rho\Gamma_{\tilde{b}_q\alpha}^{b}c\left(t\right)g_c^{\alpha}\left(x\left(t\right)\right)y^c\left(t\right)\\
B_{b_q}^{\tilde{b}_q}\left(t_0,t_0\right) &= \delta_{b_q}^{\tilde{b}_q}
\end{aligned}$$

are equivalent with the following differentiable equations

$$
\begin{array}{rl}
\dfrac{d}{dt}A_{\tilde{a}_1}^{a_1}(t,t_0) & = A_a^{a_1}(t,t_0)\,\rho\Gamma_{\tilde{a}_1\alpha}^{a}(\pi\circ c(t))\,g_c^{\alpha}(x(t))\,y^c(t)\\
A_{\tilde{a}_1}^{a_1}(t_0,t_0) & = \delta_{\tilde{a}_1}^{a_1}\\
 & ...\\
\dfrac{d}{dt}A_{\tilde{a}_p}^{a_p}(t,t_0) & = A_a^{a_p}(t,t_0)\,\rho\Gamma_{\tilde{a}_p\alpha}^{a}(\pi\circ c(t))\,g_c^{\alpha}(x(t))\,y^c(t)\\
A_{\tilde{a}_p}^{a_p}(t_0,t_0) & = \delta_{\tilde{a}_p}^{a_p}\\
\dfrac{d}{dt}B_{b_1}^{\tilde{b}_1}(t,t_0) & = -B_{b_1}^{b}(t,t_0)\,\rho\Gamma_{b\alpha}^{\tilde{b}_1}(\pi\circ c(t))\,g_c^{\alpha}(x(t))\,y^c(t)\\
B_{b_1}^{\tilde{b}_1}(t_0,t_0) & = \delta_{b_1}^{\tilde{b}_1}\\
 & ...\\
\dfrac{d}{dt}B_{b_q}^{\tilde{b}_q}(t,t_0) & = -B_{b_q}^{b}(t,t_0)\,\rho\Gamma_{b\alpha}^{\tilde{b}_q}(\pi\circ c(t))\,g_c^{\alpha}(x(t))\,y^c(t)\\
B_{b_q}^{\tilde{b}_q}(t_0,t_0) & = \delta_{b_q}^{\tilde{b}_q}
\end{array}
$$

which has unique solutions which not depend on the local vector chart used, it results the conclusion of the theorem. *q.e.d.*

Corollary 4.5.1 *For any $p,q\in\mathbb{N}$, it exists a parallel transport $\mathcal{P}_c$ between the tensors of (p,q) type.*

This parallel transport will be called the *parallel transport along the curve c associated to linear ρ-connection $\rho\Gamma$.*

Proof. Let $p,q\in\mathbb{N}$ and $t_0,t\in I$ be. Without restricting the generality, we admit that not exists a vector local $m+r$-chart U which contain the points $c(t_0)$ and $c(t)$.

Since I is a conex manifold, it results that it exist a finite numbers of real numbers $t_1,t_2,...,t_r=t$ such that for each $j\in\overline{1,r}$, the points $c(t_{j-1})$ and $c(t_j)$ belongs to the same vector local $m+r$-chart.

Using the previous theorem, we build the linear isomorphisms $P_{t_0,t_1},\ P_{t_1,t_2},\ \dots,P_{t_{r-1},t}$.

The linear isomorphism $P_{t_{r-1},t}\circ...\circ P_{t_1,t_2}\circ P_{t_0,t_1}=P_{t_0,t}$ not depend on the vector local $m+r$-charts used. *q.e.d.*

Remark 4.5.1 Using the notations of the previous theorem we obtain:

$$
\begin{array}{rl}
-\dfrac{d}{dt}\tilde{T}_{\tilde{b}_1,...,\tilde{b}_q}^{\tilde{a}_1,...,\tilde{a}_p}c(t) & = T_{\tilde{b}_1,...,\tilde{b}_q}^{a\tilde{a}_2,...,\tilde{a}_p}c(t)\,\rho\Gamma_{a\alpha}^{\tilde{a}_1}c(t)\,g_c^{\alpha}(x(t))\,y^c(t)+...\\
 & +T_{\tilde{b}_1,...,\tilde{b}_q}^{\tilde{a}_1,...,\tilde{a}_{p-1}a}c(t)\,\rho\Gamma_{a\alpha}^{\tilde{a}_p}c(t)\,g_c^{\alpha}(x(t))\,y^c(t)+\\
 & -T_{b\tilde{b}_2,...,\tilde{b}_q}^{\tilde{a}_1,...,\tilde{a}_p}c(t)\,\rho\Gamma_{\tilde{b}_1\alpha}^{b}c(t)\,g_c^{\alpha}(x(t))\,y^c(t)-...\\
 & -T_{\tilde{b}_1,...,\tilde{b}_{q-1}b}^{\tilde{a}_1,...,\tilde{a}_p}c(t)\,\rho\Gamma_{\tilde{b}_q\alpha}^{b}c(t)\,g_c^{\alpha}(x(t))\,y^c(t).
\end{array}
\tag{4.5.2}
$$

Theorem 4.5.2 *If $(E,\pi,M)=(F,\nu,N)$ and $\mathcal{P}_c$ is the parallel transport along the curve c associated to linear ρ-connection $\rho\Gamma$, then, for any $t\in I$ we obtain:*

$$
\lim_{h\longrightarrow 0}\frac{P_{t+h,t}\left(T_{c(t+h)}\right)-T_{c(t)}}{h}=\left(\rho D_{u(c,\dot{c})}T\right)c(t),
\tag{4.5.3}
$$

for any $T \in \mathcal{T}_q^p(E,\pi,M)$.

Proof. Let be $T \in \mathcal{T}_q^p(E,\pi,M)$. Let $t \in I$ and $h > 0$ be such that $]t-h,t+h[\subset I$.

For any $\tilde{a}_1,...,\tilde{a}_p,\tilde{b}_1,...,\tilde{b}_q \in \overline{1,n}$ we build the following application

$$\begin{array}{ccc} [t,t+h] & \xrightarrow{z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}} & \mathbb{R} \\ \theta & \longmapsto & z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(\theta) \end{array}$$

defined by

$$z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(\theta)=T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t+h)\,A^{\tilde{a}_1}_{a_1}(t+h,\theta)\cdot...\cdot A^{\tilde{a}_p}_{a_p}(t+h,\theta)\cdot B^{b_1}_{\tilde{b}_1}(t+h,\theta)...\cdot B^{b_q}_{\tilde{b}_q}(t+h,\theta)$$

Using the main theorem, it exists a unique real number

$$\xi^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\in]t,t+h[$$

such that

$$z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(t+h)=z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(t)+h\left(z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\right)'\left(\xi^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\right).$$

Since the component by indices ${}^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}$ of a tensor $P_{t+h,t}\left(T_{c(t+h)}\right)$ is $z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(t)$, it results that

$$\begin{aligned}
&\lim_{h\longrightarrow 0}\frac{P_{t+h,t}\left(T_{\pi\circ c(t+h)}\right)-T_{\pi\circ c(t)}}{h}=\\
&=\lim_{h\longrightarrow 0}\frac{z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}(t)-T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)}{h}\cdot\left(s_{\tilde{a}_1}\otimes...\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes...\otimes s^{\tilde{b}_q}\right)c(t)\\
&=\lim_{h\longrightarrow 0}\frac{T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t+h)-T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)}{h}\cdot\left(s_{\tilde{a}_1}\otimes...\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes...\otimes s^{\tilde{b}_q}\right)c(t)\\
&-\lim_{h\longrightarrow 0}\frac{h\left(z^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\right)'\left(\xi^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\right)}{h}\cdot\left(s_{\tilde{a}_1}\otimes...\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes...\otimes s^{\tilde{b}_q}\right)c(t)\\
&=\frac{d}{dt}T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)\cdot\left(s_{\tilde{a}_1}\otimes...\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes...\otimes s^{\tilde{b}_q}\right)c(t)\\
&-\frac{d}{dt}\tilde{T}^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)\cdot\left(s_{\tilde{a}_1}\otimes...\otimes s_{\tilde{a}_p}\otimes s^{\tilde{b}_1}\otimes...\otimes s^{\tilde{b}_q}\right)c(t).
\end{aligned}$$

Using Remark 4.5.1 and the equality

$$\frac{d}{dt}T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)=\frac{dx^i}{dt}\frac{\partial T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)}{\partial x^i}=g^\alpha_c(x(t))\,y^c(t)\,\rho^i_\alpha\frac{\partial T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c(t)}{\partial x^i},$$

it results the conclusion of theorem. *q.e.d.*

Definition 4.5.2 *The tensor field* $T \in \mathcal{T}_q^p(E,\pi,M)$ *is parallel along the curve* c with respect to the linear ρ-connection $\rho\Gamma$ if for any $t_1,t_2\in I$ it results that

$$P_{t_1,t_2}\left(T_{c(t_1)}\right)=T_{c(t_2)}. \tag{4.5.4}$$

Theorem 4.5.3 The tensor field $T \in \mathcal{T}_q^p(E,\pi,M)$ is parallel along the curve c with respect to linear ρ-connection $\rho\Gamma$ if and only if

$$\left(\rho D_{u(c,\dot{c})}T\right)c(t)=0,\forall t\in I. \tag{4.5.5}$$

Corollary 4.5.2 The tensor field $T \in \mathcal{T}^p_q(E,\pi,M)$ is parallel along the curve c with respect to linear ρ-connection $\rho\Gamma$ if and only if

$$\begin{array}{r}\frac{d}{dt}T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}c\left(t\right)+g^{\alpha}_{c}\left(x\left(t\right)\right)y^{c}\left(t\right)\Big(T^{a\tilde{a}_2,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_q}\rho\Gamma^{\tilde{a}_1}_{a\alpha}+...+T^{\tilde{a}_1,...,\tilde{a}_{p-1}a}_{\tilde{b}_1,...,\tilde{b}_q}\rho\Gamma^{\tilde{a}_p}_{a\alpha}\\ -T^{\tilde{a}_1,...,\tilde{a}_p}_{b\tilde{b}_2,...,\tilde{b}_q}\rho\Gamma^{b}_{\tilde{b}_1\alpha}-T^{\tilde{a}_1,...,\tilde{a}_p}_{\tilde{b}_1,...,\tilde{b}_{q-1}b}\rho\Gamma^{b}_{\tilde{b}_q\alpha}\Big)c\left(t\right)=0,\quad\forall t\in I.\end{array}$$

4.6 Formulas of Gauss-Weingarten type

Using the main ideas of the theory of Myller configurations, introduced by R. Miron in [37] and applied to Finsler spaces by O. Constantinescu in [13] and his Ph.D. Thesis, we present the Gauss-Weingarten formulas for generalized Lie algebroids.

Let

$$\left(\left(F,\nu,N\right),[,]_{F,h},(\rho,\eta)\right)$$

be a generalized Lie algebroid given by the diagram:

$$(4.6.1)\qquad \begin{tikzcd} & \left(F,[,]_{F,h},(\rho,\eta)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

The geometry of the couple (M,h) is the geometry of the pull-back vector bundle $(h^*F,h^*\nu,M)$ using the diagram

$$(4.6.2)\qquad \begin{tikzcd} h^*F \arrow[d, "h^*\nu"'] & & \left(h^*F,[,]_{h^*F},\left(\overset{h^*F}{\rho},Id_M\right)\right) \arrow[d, "h^*\nu"] \\ M \arrow[rr, "Id_M"] & & M \end{tikzcd}$$

Let $I\xrightarrow{c}M$ be a differentiable curve and let $M'=Im\left(\eta\circ h\circ c\right)$ be.

Let $I\xrightarrow{\dot{c}}h^*F_{|M'}$ be the (Id_{h^*F},Id_M)-lift of the curve c.

Let $\left\{T_\alpha,\alpha\in\overline{1,p}\right\}$, $\left\{s_a,a\in\overline{1,q}\right\}$ and $\left\{\chi_i,i\in\overline{1,s}\right\}$ be the base for

$$\Gamma\left(h^*F_{|M'},h^*\nu_{|M'},M'\right),\ \Gamma\left(T^{h^*F}\left(c,\dot{c}\right),\tau,M'\right)\text{ and }\Gamma\left(S^{h^*F}\left(c,\dot{c}\right),\sigma,M'\right),$$

respectively.

The dimension of type fibre of the vector bundle $\left(h^*F_{|M'},h^*\nu_{|M'},M'\right)$ is $q+s=p$.

Consequently, for any $a\in\overline{1,q}$ we have

$$(4.6.3)\qquad s_a=\Lambda^\alpha_aT_\alpha$$

and for any $i\in\overline{1,s}$ we have

$$(4.6.4)\qquad \chi_i=\Lambda^\alpha_iT_\alpha.$$

Let $g=g_{\alpha\beta}t^\alpha\otimes t^\beta\in\mathcal{T}{}^{\,0}_{\,2}\left(F,\nu,N\right)$ be a (pseudo)metrical structure.

Remark 4.6.1 The following affirmations are satisfied:

1. The section $\overset{h^*F}{g} = \overset{h^*F}{g}_{\alpha\beta} T^\alpha \otimes T^\beta$ defined by

$$\overset{h^*F}{g}_{\alpha\beta}(x) = g_{\alpha\beta}(h(x)) \tag{4.6.5}$$

is a (pseudo)metrical structure.

2. The section $\overset{T^{h^*F}}{g} = \overset{T^{h^*F}}{g}_{ab} s^a \otimes s^b$ defined by

$$\overset{T^{h^*F}}{g}_{ab}(x) = \Lambda_a^\alpha g_{\alpha\beta}(h(x)) \Lambda_b^\beta \tag{4.6.6}$$

is a (pseudo)metrical structure.

3. The section $\overset{S^{h^*F}}{g} = \overset{S^{h^*F}}{g}_{ij} \chi^i \otimes \chi^j$ defined by

$$\overset{S^{h^*F}}{g}_{ij}(x) = \Lambda_i^\alpha g_{\alpha\beta}(h(x)) \Lambda_j^\beta \tag{4.6.7}$$

is a (pseudo)metrical structure.

Remark 4.6.2 Using the diagram (4.6.2) we can construct the linear ρ-connection of Levi Civita type $\overset{h^*F}{\rho}\Gamma$ of components $\overset{h^*F}{\rho}\Gamma^\alpha_{\beta\gamma}$.

We have the covariant ρ-derivative defined by

$$\overset{h^*F}{\rho} D_z w = z^\gamma \left(\left(\overset{h^*F}{\rho}\right)_\gamma^k \frac{\partial w^\alpha}{\partial x^k} + \overset{h^*F}{\rho}\Gamma^\alpha_{\beta\gamma} w^\beta \right) T_\alpha, \tag{4.6.8}$$

where $\left(\overset{h^*F}{\rho}\right)_\gamma^k$ are the components of the map $\overset{h^*F}{\rho}$.

Definition 4.6.1 If we can defined

$$\overset{h^*F}{\rho} D_{v\oplus 0}(u \oplus 0) = v^c \left(\left(\overset{h^*F}{\rho}\right)_c^k \frac{\partial u^a}{\partial x^k} + \overset{h^*F}{\rho}\Gamma^a_{bc} u^b \right) s_a, \tag{4.6.9}$$

$$\overset{h^*F}{\rho} D_{v\oplus 0}(0 \oplus \xi) = v^c \left(\left(\overset{h^*F}{\rho}\right)_c^k \frac{\partial \xi^i}{\partial x^k} + \overset{h^*F}{\rho}\Gamma^i_{jc} \xi^j \right) \chi_i, \tag{4.6.10}$$

$$\overset{h^*F}{\rho} D_{0\oplus \eta}(u \oplus 0) = \eta^h \left(\overset{h^*F}{\rho}{}_h^k \frac{\partial u^a}{\partial x^k} + \overset{h^*F}{\rho}\Gamma^a_{bh} u^b \right) s_a, \tag{4.6.11}$$

$$\overset{h^*F}{\rho} D_{0\oplus \eta}(0 \oplus \xi) = \eta^h \left(\left(\overset{h^*F}{\rho}\right)_h^k \frac{\partial \xi^i}{\partial x^k} + \overset{h^*F}{\rho}\Gamma^i_{jh} \xi^j \right) \chi_i. \tag{4.6.12}$$

and we can consider the bilinear applications

$$\Gamma\left(T^{h^*F}(c,\dot{c}),\tau,M'\right) \times \Gamma\left(T^{h^*F}(c,\dot{c}),\tau,M'\right) \ \xrightarrow{H}\ \Gamma\left(S^{h^*F}(c,\dot{c}),\sigma,M'\right)$$

and

$$\Gamma\left(S^{h^*F}(c,\dot{c}),\sigma,M'\right) \times \Gamma\left(T^{h^*F}(c,\dot{c}),\tau,M'\right) \ \xrightarrow{A}\ \Gamma\left(T^{h^*F}(c,\dot{c}),\tau,M'\right)$$

which satisfy the following relations

$$\overset{h^*F}{\rho} D_{v^c\Lambda_c^\gamma T_\gamma}\left(u^b \Lambda_b^\beta T_\beta\right) = \overset{h^*F}{\rho} D_{v\oplus 0}(u \oplus 0) \oplus H(u,v), \tag{4.6.13}$$

$$
{}^{h^*F}_{\ \ \rho}D_{v^c\Lambda_c^\gamma T_\gamma}\left(\xi^j\Lambda_j^\beta T_\beta\right)=-A_\xi\left(v\right)\oplus{}^{h^*F}_{\ \ \rho}D_{v\oplus 0}\left(0\oplus\xi\right)\tag{4.6.14}
$$

and

$$
{}^{S^{h^*F}}g\left(H\left(u,v\right),\xi\right)={}^{T^{h^*F}}g\left(A_\xi\left(v\right),u\right),\tag{4.6.15}
$$

then we will say that the *relations* (4.6.13), (4.6.14) *and* (4.6.15) *are formulas of Gauss-Weingarten type associated to differentiable curve c, metrical structure g and bilinear applications H and A.*

The bilinear application H will be called *the second fundamental form of differentiable curve c.*

Remark 4.6.2 Using the base sections, then the formulas of Gauss-Weingarten type become:

$$
\Lambda_c^\gamma\left(\left({}^{h^*F}_{\ \ \rho}\right)_\gamma^k\frac{\partial\Lambda_b^\alpha}{\partial x^k}+{}^{h^*F}_{\ \ \rho}\Gamma_{\beta\gamma}^\alpha\Lambda_b^\beta\right)={}^{h^*F}_{\ \ \rho}\Gamma_{bc}^a\Lambda_a^\alpha+H_{bc}^i\Lambda_i^\alpha,\tag{4.6.13$'$}
$$

$$
\Lambda_c^\gamma\left(\left({}^{h^*F}_{\ \ \rho}\right)_\gamma^h\frac{\partial\Lambda_j^\alpha}{\partial x^h}+{}^{h^*F}_{\ \ \rho}\Gamma_{\beta\gamma}^\alpha\Lambda_j^\beta\right)=-A_{jc}^a\Lambda_a^\alpha+{}^{h^*F}_{\ \ \rho}\Gamma_{jc}^i\Lambda_i^\alpha\tag{4.6.14$'$}
$$

$$
{}^{S^{h^*F}}g_{ij}H_{bc}^i={}^{T^{h^*F}}g_{ab}A_{jc}^a.\tag{4.6.15$'$}
$$

5 The geometry of total space of the Lie algebroid generalized tangent bundle for a vector bundle

5.1 Adapted (ρ,η)-basis and adapted dual (ρ,η)-basis

In the following, we consider the following diagram:

$$
\begin{array}{ccc}
E & & \left(F,\left[,\right]_{F,h},\left(\rho,\eta\right)\right)\\
\downarrow\pi & & \downarrow\nu\\
M & \xrightarrow{h} & N
\end{array}
$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$ is a generalized Lie algebroid.

Let $\left(\rho,\eta\right)\Gamma$ be a (ρ,η)-connection for the vector bundle $\left(E,\pi,M\right)$.

If we put the problem of finding a base for the $\mathcal{F}\left(E\right)$-module

$$
\left(\Gamma\left(H\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),+,\cdot\right)
$$

of the type

$$
\frac{\delta}{\delta\tilde{z}^\alpha}=\tilde{Z}_\alpha^\beta\frac{\partial}{\partial\tilde{z}^\alpha}+Y_\alpha^a\frac{\partial}{\partial\tilde{y}^a},\alpha\in\overline{1,r}
$$

which satisfies the following conditions:

$$
\begin{array}{rcl}
\Gamma\left(\left(\rho,\eta\right)\pi!,Id_E\right)\left(\dfrac{\delta}{\delta\tilde{z}^\alpha}\right) & = & \tilde{T}_\alpha,\\
\Gamma\left(\left(\rho,\eta\right)\Gamma,Id_E\right)\left(\dfrac{\delta}{\delta\tilde{z}^\alpha}\right) & = & 0,
\end{array}\tag{5.1.1}
$$

then we obtain the sections

$$\frac{\delta}{\delta \tilde{z}^{\alpha}}=\frac{\partial}{\partial \tilde{z}^{\alpha}}-(\rho,\eta)\,\Gamma_{\alpha}^{a}\frac{\partial}{\partial \tilde{y}^{a}}. \tag{5.1.2}$$

We observe that their law of change is a tensorial law under a change of vector fiber charts.

Definition 5.1.1 The base

$$\left(\frac{\delta}{\delta \tilde{z}^{\alpha}},\frac{\partial}{\partial \tilde{y}^{a}}\right)\overset{put}{=}\left(\tilde{\delta}_{\alpha},\dot{\tilde{\partial}}_{a}\right)$$

will be called the *adapted* (ρ,η)*-base.*

The following equality holds good

$$\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\tilde{\delta}_{\alpha}\right)=\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\partial_{i}-(\rho,\eta)\,\Gamma_{\alpha}^{a}\dot{\partial}_{a}, \tag{5.1.3}$$

where $\left(\partial_{i},\dot{\partial}_{a}\right)$ is the natural base for the $\mathcal{F}(E)$-module $\left(\Gamma\left(TE,\tau_{E},E\right),+,\cdot\right).$

Moreover, if $\rho\Gamma$ is the ρ-connection associated to the connection Γ, then we obtain

$$\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\tilde{\delta}_{\alpha}\right)=\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\delta_{i}, \tag{5.1.4}$$

where $\left(\delta_{i},\dot{\partial}_{a}\right)$ is the adapted base for the $\mathcal{F}(E)$-module $\left(\Gamma\left(TE,\tau_{E},E\right),+,\cdot\right).$

Theorem 5.1.1 *The following equality holds good*

$$\left[\tilde{\delta}_{\alpha},\tilde{\delta}_{\beta}\right]_{(\rho,\eta)TE}=L_{\alpha\beta}^{\gamma}\circ(h\circ\pi)\,\tilde{\delta}_{\gamma}+(\rho,\eta,h)\,\mathbb{R}^{a}{}_{\alpha\beta}\dot{\tilde{\partial}}_{a}, \tag{5.1.5}$$

where

$$\begin{aligned}(\rho,\eta,h)\,\mathbb{R}^{a}{}_{\alpha\beta}&=\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\tilde{\delta}_{\beta}\right)\left((\rho,\eta)\,\Gamma_{\alpha}^{a}\right)\\&-\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\tilde{\delta}_{\alpha}\right)\left((\rho,\eta)\,\Gamma_{\beta}^{a}\right)+\left(L_{\alpha\beta}^{\gamma}\circ h\circ\pi\right)(\rho,\eta)\,\Gamma_{\gamma}^{a},\end{aligned} \tag{5.1.6}$$

Moreover, we have:

$$\left[\tilde{\delta}_{\alpha},\dot{\tilde{\partial}}_{b}\right]_{(\rho,\eta)TE}=\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\dot{\tilde{\partial}}_{b}\right)\left((\rho,\eta)\,\Gamma_{\alpha}^{a}\right)\dot{\tilde{\partial}}_{a}, \tag{5.1.7}$$

and

$$\Gamma\left(\tilde{\rho},Id_{E}\right)\left[\tilde{\delta}_{\alpha},\tilde{\delta}_{\beta}\right]_{(\rho,\eta)TE}=\left[\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\tilde{\delta}_{\alpha}\right),\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\tilde{\delta}_{\beta}\right)\right]_{TE}. \tag{5.1.8}$$

Let $\left(d\tilde{z}^{\alpha},d\tilde{y}^{b}\right)$ be the natural dual (ρ,η)-base.

If we consider the problem of finding a base for the $\mathcal{F}(E)$-module

$$\left(\Gamma\left(\left(V\left(\rho,\eta\right)TE\right)^{*},\left(\left(\rho,\eta\right)\tau_{E}\right)^{*},E\right),+,\cdot\right)$$

of the type

$$\delta\tilde{y}^{a}=\theta_{\alpha}^{a}d\tilde{z}^{\alpha}+\omega_{b}^{a}d\tilde{y}^{b},\ a\in\overline{1,n}$$

which satisfies the following conditions:

$$\left\langle \delta\tilde{y}^a, \dot{\tilde{\partial}}_a \right\rangle = 1 \wedge \left\langle \delta\tilde{y}^a, \tilde{\delta}_\alpha \right\rangle = 0, \tag{5.1.9}$$

then we obtain the sections

$$\delta\tilde{y}^a = (\rho,\eta)\,\Gamma^a_\alpha d\tilde{z}^\alpha + d\tilde{y}^a, a \in \overline{1,n}. \tag{5.1.10}$$

We observe that their changing rule is tensorial under a change of vector fiber charts.

Definition 5.1.2 The base $(d\tilde{z}^\alpha, \delta\tilde{y}^a)$ will be called the *adapted dual* (ρ,η)*-base.*

5.2 Remarkable Mod-endomorphisms

In the following we consider the diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \left(F, [,]_{F,h}, (\rho,\eta)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E,\pi,M) \in |\mathbf{B^v}|$ and $\left((F,\nu,N), [,]_{F,h}, (\rho,\eta)\right)$ is a generalized Lie algebroid.

Definition 5.2.1 For any **Mod**-endomorphism e of

$$(\Gamma\left((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E\right), +, \cdot)$$

we define the application of Nijenhuis type

$$\Gamma((\rho,\eta)TE,(\rho,\eta)\tau_E, E)^2 \xrightarrow{N_e} \Gamma((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$$

defined by

$$N_e(X,Y) = [eX, eY]_{\rho TE} + e^2 [X,Y]_{\rho TE} - e[eX,Y]_{\rho TE} - e[X,eY]_{\rho TE},$$

for any $X, Y \in \Gamma((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$.

Remark 5.2.1 The vertical and the horizontal vector subbundles are interior differential systems for the Lie algebroid generalized tangent bundle

$$\left(((\rho,\eta)TE,(\rho,\eta)\tau_E,E), [,]_{(\rho,\eta)TE}, (\tilde{\rho}, Id_E)\right).$$

These interior differential systems will be called *vertical* and *horizontal interior differential systems.*

5.2.1 Projectors

Definition 5.2.1.1 Any **Mod**-endomorphism e of

$$\Gamma\left((\rho,\eta)TE,(\rho,\eta\ )\tau_E,E\right)$$

with the property

$$e^2 = e \tag{5.2.1.1}$$

will be called *projector.*

Example 5.2.1.1 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right) & \xrightarrow{\mathcal{V}} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\\ \tilde{Z}^\alpha\tilde{\delta}_\alpha+Y^a\dot{\tilde{\partial}}_a & \longmapsto & Y^a\dot{\tilde{\partial}}_a\end{array}$$

is a projector which will be called *the vertical projector.*

Remark 5.2.1.1 We have $\mathcal{V}\left(\tilde{\delta}_\alpha\right)=0$ and $\mathcal{V}\left(\dot{\tilde{\partial}}_a\right)=\dot{\tilde{\partial}}_a$. Therefore, it follows

$$\mathcal{V}\left(\tilde{\partial}_\alpha\right)=(\rho,\eta)\,\Gamma^a_\alpha\dot{\tilde{\partial}}_a.$$

Theorem 5.2.1.1 *A (ρ,η)-connection for the vector bundle (E,π,M) is characterized by the existence of a* **Mod**-*endomorphism $\mathcal{V}$ of*

$$\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\mathcal{V}\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\right)\subset\Gamma\left(V\left(\rho,\eta\right)TE,(\rho,\eta)\,\tau_E,E\right)\\ \mathcal{V}\left(X\right)=X\Longleftrightarrow X\in\Gamma\left(V\left(\rho,\eta\right)TE,(\rho,\eta)\,\tau_E,E\right)\end{array} \tag{5.2.1.2}$$

Example 5.2.1.2 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right) & \xrightarrow{\mathcal{H}} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\\ \tilde{Z}^\alpha\tilde{\delta}_\alpha+Y^a\dot{\tilde{\partial}}_a & \longmapsto & \tilde{Z}^\alpha\tilde{\delta}_\alpha\end{array}$$

is a projector which will be called the *horizontal projector.*

Remark 5.2.1.2 We have $\mathcal{H}\left(\tilde{\delta}_\alpha\right)=\tilde{\delta}_\alpha$ and $\mathcal{H}(\dot{\tilde{\partial}}_a)=0$. Therefore, we obtain $\mathcal{H}\left(\tilde{\partial}_\alpha\right)=\tilde{\delta}_\alpha$.

Theorem 5.2.1.2 *A (ρ,η)-connection for the vector bundle (E,π,M) is characterized by the existence of a* **Mod**-*endomorphism $\mathcal{H}$ of*

$$\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\mathcal{H}\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\right)\subset\Gamma\left(H\left(\rho,\eta\right)TE,(\rho,\eta)\,\tau_E,E\right)\\ \mathcal{H}\left(X\right)=X\Longleftrightarrow X\in\Gamma\left(H\left(\rho,\eta\right)TE,(\rho,\eta)\,\tau_E,E\right).\end{array} \tag{5.2.1.3}$$

Corollary 5.2.1.1 *A (ρ,η)-connection for the vector bundle (E,π,M) is characterized by the existence of a* **Mod***-endomorphism $\mathcal{H}$ of*

$$\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\mathcal{H}^2=\mathcal{H}\\ Ker\left(\mathcal{H}\right)=\left(\Gamma\left(V\left(\rho,\eta\right)TE,(\rho,\eta)\,\tau_E,E\right),+,\cdot\right).\end{array}\tag{5.2.1.4}$$

Remark 5.2.1.3 For any

$$X\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$$

we obtain the following unique decomposition

$$X=\mathcal{H}X+\mathcal{V}X.$$

Proposition 5.2.1.1 *After some calculations we obtain*

$$N_{\mathcal{V}}\left(X,Y\right)=\mathcal{V}\left[\mathcal{H}X,\mathcal{H}Y\right]_{(\rho,\eta)TE}=N_{\mathcal{H}}\left(X,Y\right),\tag{5.2.1.5}$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$.

Corollary 5.2.1.2 *The horizontal interior differential system*

$$\left(H\left(\rho,\eta\right)TE,(\rho,\eta)\,\tau_E,E\right)$$

is involutive if and only if $N_{\mathcal{V}}=0$ or $N_{\mathcal{H}}=0$.

5.2.2 The almost product structure

Definition 5.2.2.1 Any **Mod**-endomorphism e of

$$\left(\Gamma\left((\rho,\eta)TE,(\rho,\eta\ )\tau_E,E\right),+,\cdot\right)$$

with the property

$$e^2=Id\tag{5.2.2.1}$$

will be called the *almost product structure.*

Example 5.2.2.1 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right) & \xrightarrow{\mathcal{P}} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\\ \tilde{Z}^{\alpha}\tilde{\delta}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a} & \longmapsto & \tilde{Z}^{\alpha}\tilde{\delta}_{\alpha}-Y^{a}\dot{\tilde{\partial}}_{a}\end{array}$$

is an almost product structure.

Remark 5.2.2.1 The previous almost product structure has the properties:

$$\begin{array}{l}\mathcal{P}=2\mathcal{H}-Id;\\ \mathcal{P}=Id-2\mathcal{V};\\ \mathcal{P}=\mathcal{H}-\mathcal{V}.\end{array}\tag{5.2.2.2}$$

Remark 5.2.2.2 We obtain that $\mathcal{P}\left(\tilde{\delta}_\alpha\right) = \tilde{\delta}_\alpha$ and $\mathcal{P}\left(\dot{\tilde{\partial}}_a\right) = -\dot{\tilde{\partial}}_a$. Therefore, it follows

$$\mathcal{P}\left(\tilde{\partial}_\alpha\right) = \tilde{\delta}_\alpha - \rho\Gamma^a_\alpha \dot{\tilde{\partial}}_a.$$

Theorem 5.2.2.1 *A (ρ,η)-connection for the vector bundle (E,π,M) is characterized by the existence of a* **Mod**-*endomorphism $\mathcal{P}$ of*

$$\left(\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),+,\cdot\right)$$

with the following property:

$$\mathcal{P}\left(X\right) = -X \Longleftrightarrow X \in \Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right). \tag{5.2.2.3}$$

Proposition 5.2.2.1 *After some calculations, we obtain*

$$N_{\mathcal{P}}\left(X,Y\right) = 4\mathcal{V}\left[\mathcal{H}X,\mathcal{H}Y\right],$$

for any $X,Y \in \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$.

Corollary 5.2.2.1 *The horizontal interior differential system*

$$\left(H\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$$

is involutive if and only if $N_{\mathcal{P}} = 0$.

5.2.3 The almost tangent structure

Definition 5.2.3.1 Any **Mod**-endomorphism e of

$$(\Gamma((\rho,\eta)TE,(\rho,\eta)\tau_E,E),+,\cdot)$$

with the property

$$e^2 = 0 \tag{5.2.3.1}$$

will be called the *almost tangent structure.*

Example 5.2.3.1 If $(E,\pi,M) = (F,\nu,N)$, $g \in \mathbf{Man}\left(E,E\right)$ such that (g,h) is a $\mathbf{B^v}$-morphism locally invertible, then the **Mod**-endomorphism

$$\begin{array}{ccc} \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right) & \xrightarrow{\mathcal{J}_{(g,h)}} & \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right) \\ \tilde{Z}^a\tilde{\delta}_a + \tilde{Y}^b\dot{\tilde{\partial}}_b & \longmapsto & \left(\tilde{g}^b_a \circ h \circ \pi\right)\tilde{Z}^a\dot{\tilde{\partial}}_b \end{array}$$

is an almost tangent structure which will be called the *almost tangent structure associated to* $\mathbf{B^v}$*-morphism* (g,h). (See: **Definition 4.4.1.3)**

Remark 5.2.3.1 We obtain that

$$\mathcal{J}_{(g,h)}\left(\tilde{\delta}_a\right) = \mathcal{J}_{(g,h)}\left(\tilde{\partial}_a\right) = \left(\tilde{g}^b_a \circ h \circ \pi\right)\dot{\tilde{\partial}}_b \text{ and } \mathcal{J}_{(g,h)}\left(\dot{\tilde{\partial}}_b\right) = 0.$$

Remark 5.2.3.2 The previous almost tangent structure has the following properties:

$$
\begin{array}{rcl}
\mathcal{J}_{(g,h)} \circ \mathcal{P} & = & \mathcal{J}_{(g,h)}; \\
\mathcal{P} \circ \mathcal{J}_{(g,h)} & = & -\mathcal{J}_{(g,h)}; \\
\mathcal{J}_{(g,h)} \circ \mathcal{H} & = & \mathcal{J}_{(g,h)}; \\
\mathcal{H} \circ \mathcal{J}_{(g,h)} & = & 0; \\
\mathcal{J}_{(g,h)} \circ \mathcal{V} & = & 0; \\
\mathcal{V} \circ \mathcal{J}_{(g,h)} & = & \mathcal{J}_{(g,h)}; \\
N_{\mathcal{J}_{(g,h)}} & = & 0.
\end{array} \tag{5.2.3.2}
$$

5.2.4 The almost complex structure

Let us consider in the case $(E, \pi, M) = (F, \nu, N)$.

Definition 5.2.4.1 Any **Mod**-endomorphism e of

$$
(\Gamma((\rho,\eta)\,TE, (\rho,\eta)\tau_E, E), +, \cdot)
$$

with the property

$$
e^2 = -Id \tag{5.3.4.1}
$$

will be called the *almost complex structure*.

Example 5.2.4.1 If (g, h) is a $\mathbf{B^v}$-morphism of (E, π, M) source and target locally invertible, then the **Mod**-endomorphism

$$
\begin{array}{ccc}
\Gamma\left((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E\right) & \xrightarrow{\mathcal{F}\ \ _{(g,h)}} & \Gamma\left((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E\right) \\
\tilde{Z}^a \tilde{\delta}_a + Y^b \dot{\tilde{\partial}}_b & \longmapsto & \left(g_b^a \circ h \circ \pi\right) Y^b \tilde{\delta}_a - \left(\tilde{g}_a^b \circ h \circ \pi\right) \tilde{Z}^a \dot{\tilde{\partial}}_b
\end{array}
$$

is an almost complex structure.

Remark 5.2.4.1 We have

$$
\mathcal{F}_{(g,h)}\left(\tilde{\delta}_a\right) = -\left(\tilde{g}_a^b \circ h \circ \pi\right) \dot{\tilde{\partial}}_b
$$

and

$$
\mathcal{F}_{(g,h)}\left(\dot{\tilde{\partial}}_b\right) = \left(g_b^a \circ h \circ \pi\right) \tilde{\delta}_a.
$$

Therefore, we obtain:

$$
\mathcal{F}_{(g,h)}\left(\tilde{\partial}_c\right) = (\rho,\eta)\,\Gamma_c^a \left(g_b^a \circ h \circ \pi\right) \tilde{\delta}_a - \left(\tilde{g}_c^b \circ h \circ \pi\right) \dot{\tilde{\partial}}_b.
$$

Remark 5.2.4.2 The previous almost complex structure has the following properties:

$$
\begin{array}{rcl}
\mathcal{F}_{(g,h)} \circ \mathcal{J}_{(g,h)} & = & \mathcal{H}; \\
\mathcal{F}_{(g,h)} \circ \mathcal{H} & = & -\mathcal{J}_{(g,h)}; \\
\mathcal{J}_{(g,h)} \circ \mathcal{F}_{(g,h)} & = & \mathcal{V}.
\end{array} \tag{5.2.4.2}
$$

5.2.5 The (ρ,η)-tension endomorphism

Since

$$\frac{\partial\left(\rho,\eta\right)\Gamma_{\alpha'}^{a'}}{\partial y^{b'}}=M_{a}^{a'}\left(\rho_{\alpha}^{i}\frac{\partial M_{b'}^{a}}{\partial x^{i}}+\frac{\partial\left(\rho,\eta\right)\Gamma_{c}^{a}}{\partial y^{b}}M_{b'}^{b}\right)\Lambda_{\alpha'}^{\alpha},$$

it results that

$$\left(\rho,\eta\right)\Gamma_{\alpha'}^{a'}-y^{b'}\frac{\partial\left(\rho,\eta\right)\Gamma_{\alpha'}^{a'}}{\partial y^{b'}}=M_{a}^{a'}\left(\left(\rho,\eta\right)\Gamma_{\alpha}^{a}-y^{b}\frac{\partial\left(\rho,\eta\right)\Gamma_{\alpha}^{a}}{\partial y^{b}}\right)\Lambda_{\alpha'}^{\alpha},$$

Therefore, we can introduce the following

Definition 5.2.5.1 The **Mod**-endomorphism

$$\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)\ \xrightarrow{\left(\rho,\eta\right)\mathbb{H}}\ \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$$

defined by

$$(5.2.5.1)\qquad \begin{array}{rcl}\left(\rho,\eta\right)\mathbb{H}\left(\tilde{\delta}_{\alpha}\right)&=&\left(\left(\rho,\eta\right)\Gamma_{\alpha}^{a}-y^{b}\dfrac{\partial\left(\rho,\eta\right)\Gamma_{\alpha}^{a}}{\partial y^{b}}\right)\dot{\tilde{\partial}}_{a},\\ \left(\rho,\eta\right)\mathbb{H}\left(\dot{\tilde{\partial}}_{a}\right)&=&0_{\left(\rho,\eta\right)TE}\end{array}$$

will be called the (ρ,η)*-tension of* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$.

In particular, if $h=Id_M$ and $(\rho,\eta)=(Id_{TM},Id_M)$, then we obtain the *tension of connection* Γ.

Proposition 5.2.5.1 *We obtain the following equalities*

$$\mathcal{J}_{\left(Id_{E},Id_{M}\right)}\circ\left(\rho,\eta\right)\mathbb{H}=0=\left(\rho,\eta\right)\mathbb{H}\circ\mathcal{J}_{\left(Id_{E},Id_{M}\right)}.$$

5.3 The (ρ,η,h)-torsion and the (ρ,η,h)-curvature of a (ρ,η)-connection

We consider the following diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \left(F,\left[,\right]_{F,h},\left(\rho,\eta\right)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left(\left(F,\nu,N\right),\left[,\right]_{F,h},\left(\rho,\eta\right)\right)$ is a generalized Lie algebroid.

Definition 5.3.1 If $(E,\pi,M)=(F,\nu,N)$, then the $\mathcal{F}(E)$-bilinear application

$$\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)^{2}\ \xrightarrow{\left(\rho,\eta,h\right)\mathbb{T}}\ \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$$

defined by

$$(5.3.1)\qquad \begin{array}{rl}\left(\rho,\eta,h\right)\mathbb{T}\left(\tilde{\delta}_{b},\tilde{\delta}_{c}\right)&=\left(\dfrac{\partial\left(\rho,\eta\right)\Gamma_{c}^{a}}{\partial y^{b}}-\dfrac{\partial\left(\rho,\eta\right)\Gamma_{b}^{a}}{\partial y^{c}}-L_{bc}^{a}\circ h\circ\pi\right)\dot{\tilde{\partial}}_{a};\\ \left(\rho,\eta,h\right)\mathbb{T}\left(\tilde{\delta}_{b},\dot{\tilde{\partial}}_{c}\right)&=0=\left(\rho,\eta,h\right)\mathbb{T}\left(\dot{\tilde{\partial}}_{c},\tilde{\delta}_{b}\right);\\ \left(\rho,\eta,h\right)\mathbb{T}\left(\dot{\tilde{\partial}}_{b},\dot{\tilde{\partial}}_{c}\right)&=0;\end{array}$$

will be called the (ρ,η,h)*-torsion associated to* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$.

In particular, if $h = Id_M$, then we obtain the (ρ,η)*-torsion associated to* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$.

Moreover, if $(\rho,\eta) = (Id_{TM}, Id_M)$, then we obtain the *torsion associated to connection* Γ.

Remark 5.3.1 If $(\rho,\eta,h)\,\mathbb{T}$ is the (ρ,η,h)-torsion associated to (ρ,η)-connection $(\rho,\eta)\,\Gamma$, then

$$(\rho,\eta,h)\,\mathbb{T}(X,Y) = -\,(\rho,\eta,h)\,\mathbb{T}(Y,X), \tag{5.3.2}$$

for any $X,Y \in \Gamma\left((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E\right)$.

Definition 5.3.2 If we consider the notation

$$(\rho,\eta,h)\,\mathbb{T}^a{}_{bc} \overset{put}{=} \frac{\partial\,(\rho,\eta)\,\Gamma^a_c}{\partial y^b} - \frac{\partial\,(\rho,\eta)\,\Gamma^a_b}{\partial y^c} - L^a_{bc}\circ h\circ\pi \tag{5.3.3}$$

then the tensor field

$$(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}\frac{\delta}{\delta\tilde{z}^a}\otimes d\tilde{z}^b\otimes d\tilde{z}^c \tag{5.3.4}$$

will be called the (ρ,η,h)*-torsion tensor field associated to* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$.

Proposition 5.3.1 *We obtain*

$$\mathcal{J}_{(Id_E,Id_M)}\circ(\rho,\eta)\,\mathbb{T} = 0$$

and

$$\begin{array}{rcl}(\rho,\eta,h)\,\mathbb{T}\left(\mathcal{J}_{(Id_E,Id_M)}X,Y\right) & = & (\rho,\eta)\,\mathbb{T}\left(\mathcal{J}_{(Id_E,Id_M)}X,\mathcal{J}_{(Id_E,Id_M)}Y\right)\\ & = & (\rho,\eta)\,\mathbb{T}\left(X,\mathcal{J}_{(Id_E,Id_M)}Y\right),\end{array}$$

for any $X,Y \in \Gamma\left((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E\right)$.

Theorem 5.3.1 *Using the* (ρ,η)*-tension tensor field*

$$(\rho,\eta)\,\mathbb{H}^a_b\frac{\partial}{\partial\tilde{y}^a}\otimes d\tilde{z}^b = \left((\rho,\eta)\,\Gamma^a_b - y^c\frac{\partial\,(\rho,\eta)\,\Gamma^a_b}{\partial y^c}\right)\frac{\partial}{\partial\tilde{y}^a}\otimes d\tilde{z}^b, \tag{5.3.5}$$

and the (ρ,η,h)*-deflection of the* (ρ,η)*-connection* $(\rho,\eta)\,\Gamma$

$$(\rho,\eta,h)\,\mathbb{D}^a_b = -\,(\rho,\eta)\,\Gamma^a_b + y^c\frac{\partial\,(\rho,\eta)\,\Gamma^a_c}{\partial y^b} - y^cL^a_{bc}\circ h\circ\pi, \tag{5.3.6}$$

we obtain that $(\rho,\eta,h)\mathbb{D}^a_b{=}0$ *if and only if* $(\rho,\eta)\mathbb{H}^a_b{=}0$ *and* $(\rho,\eta,h)\mathbb{T}^a{}_{bc}{=}0$.

Proof. If $(\rho,\eta,h)\,\mathbb{D}^a_b{=}0$, then deriving with respect to y^c, we obtain:

$$-\frac{\partial\,(\rho,\eta)\,\Gamma^a_b}{\partial y^c} + \frac{\partial\,(\rho,\eta)\,\Gamma^a_c}{\partial y^b} - L^a_{bc}\circ h\circ\pi = 0 \Longleftrightarrow (\rho,\eta,h)\,\mathbb{T}^a{}_{bc} = 0.$$

The equality $(\rho,\eta,h)\,\mathbb{D}_b^a{=}0$ implies:

$$(1)\qquad (\rho,\eta)\,\Gamma_b^a = y^c\frac{\partial\,(\rho,\eta)\,\Gamma_c^a}{\partial y^b} - y^c L_{bc}^a\circ h\circ\pi.$$

Since

$$\begin{array}{ll} (\rho,\eta)\,\mathbb{H}_b^a & = (\rho,\eta)\,\Gamma_b^a - y^c\dfrac{\partial\,(\rho,\eta)\,\Gamma_b^a}{\partial y^c} = \\ & = y^c\dfrac{\partial\,(\rho,\eta)\,\Gamma_c^a}{\partial y^b} - y^c L_{bc}^a\circ h\circ\pi - y^c\dfrac{\partial\,(\rho,\eta)\,\Gamma_b^a}{\partial y^c} = y^c\,(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}\end{array}$$

it results the equality $(\rho,\eta)\,\mathbb{H}_b^a{=}0$.

Conservely, if $(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}{=}0$, then, multiplying with y^c, we obtain:

$$(2)\qquad \frac{\partial\,(\rho,\eta)\,\Gamma_c^a}{\partial y^b}y^c - \frac{\partial\,(\rho,\eta)\,\Gamma_b^a}{\partial y^c}y^c - y^c L_{bc}^a\circ h\circ\pi = 0.$$

The equality $(\rho,\eta)\,\mathbb{H}_b^a{=}0$ is equivalent with:

$$(3)\qquad (\rho,\eta)\,\Gamma_b^a = y^c\frac{\partial\,(\rho,\eta)\,\Gamma_b^a}{\partial y^c}.$$

Using (2) and (3), it results the equality $(\rho,\eta,h)\,\mathbb{D}_b^a = 0$. *q.e.d.*

Definition 5.3.3 The $\mathcal{F}(E)$-bilinear application

$$\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)^2 \ \xrightarrow{(\rho,\eta,h)\mathbb{R}}\ \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$$

defined by

$$(5.3.7)\qquad \begin{array}{l} (\rho,\eta,h)\,\mathbb{R}\left(\tilde{\delta}_\alpha,\tilde{\delta}_\beta\right) = (\rho,\eta,h)\,\mathbb{R}^a{}_{\alpha\beta}\dot{\tilde{\partial}}_a;\\ (\rho,\eta,h)\,\mathbb{R}\left(\tilde{\delta}_\alpha,\dot{\tilde{\partial}}_b\right) = 0 = (\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_b,\tilde{\delta}_\alpha\right);\\ (\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_a,\dot{\tilde{\partial}}_b\right) = 0;\end{array}$$

will be called the (ρ,η,h)-*curvature associated to* (ρ,η)-*connection* $(\rho,\eta)\,\Gamma$.

In particular, if $h = Id_M$, then we obtain the (ρ,η)-*curvature associated to* (ρ,η)-*connection* $(\rho,\eta)\,\Gamma$.

Moreover, if $(\rho,\eta) = (Id_{TM}, Id_M)$, then we obtain the *curvature associated to connection* Γ.

Remark 5.3.2 If $(\rho,\eta,h)\,\mathbb{R}$ is the (ρ,η,h)-curvature associated to (ρ,η)-connection $(\rho,\eta)\,\Gamma$, then

$$(5.3.8)\qquad (\rho,\eta,h)\,\mathbb{R}\,(X,Y) = -\,(\rho,\eta,h)\,\mathbb{R}\,(Y,X)\,,$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$.

Definition 5.3.4 The tensor field

$$(5.3.9) \qquad (\rho,\eta,h)\,\mathbb{R}^a{}_{\alpha\beta}\frac{\partial}{\partial\tilde{y}^a}\otimes d\tilde{z}^\alpha\otimes d\tilde{z}^\beta$$

will be called the *(ρ,η,h)-curvature tensor field associated to the (ρ,η)-connection $(\rho,\eta)\,\Gamma$.*

Using equality (5.1.5), we obtain

Remark 5.3.3 The horizontal interior differential system $(H\,(\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ is involutive if and only if the (ρ,η,h)-curvature tensor field associated to the (ρ,η)-connection $(\rho,\eta)\,\Gamma$ is null.

Theorem 5.3.2 *If $\mathcal{F}$ is the almost complex structure presented in Example 5.2.4.1, then $(\rho,\eta,h)\,\mathbb{T}=0$ and $(\rho,\eta,h)\,\mathbb{R}=0$ if and only if $N_{\mathcal{F}}=0$.*

Proof. After some calculations, we obtain the relations:

$$\begin{aligned}
N_{\mathcal{F}}\left(\tilde{\delta}_b,\tilde{\delta}_c\right) &= (\rho,\eta,h)\,\mathbb{T}^a{}_{bc}\tilde{\delta}_a-(\rho,\eta,h)\,\mathbb{R}^a{}_{bc}\dot{\tilde{\partial}}_a,\\
N_{\mathcal{F}}\left(\tilde{\delta}_b,\dot{\tilde{\partial}}_c\right) &= (\rho,\eta,h)\,\mathbb{R}^a{}_{bc}\tilde{\delta}_a-(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}\dot{\tilde{\partial}}_a,\\
N_{\mathcal{F}}\left(\dot{\tilde{\partial}}_b,\dot{\tilde{\partial}}_c\right) &= -\,(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}\tilde{\delta}_a-(\rho,\eta,h)\,\mathbb{R}^a{}_{bc}\dot{\tilde{\partial}}_a.
\end{aligned}$$

Obviously, $(\rho,\eta,h)\,\mathbb{T}=0$ and $(\rho,\eta,h)\,\mathbb{R}=0$ imply $N_{\mathcal{F}}=0$.

Conservely, if $N_{\mathcal{F}}=0$, then we have the equalities:

$$\begin{aligned}
(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}\tilde{\delta}_a-(\rho,\eta,h)\,\mathbb{R}^a{}_{bc}\dot{\tilde{\partial}}_a&=0,\\
(\rho,\eta,h)\,\mathbb{R}^a{}_{bc}\tilde{\delta}_a-(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}\dot{\tilde{\partial}}_a&=0,\\
-\,(\rho,\eta,h)\,\mathbb{T}^a{}_{bc}\tilde{\delta}_a-(\rho,\eta,h)\,\mathbb{R}^a{}_{bc}\dot{\tilde{\partial}}_a&=0.
\end{aligned}$$

q.e.d.

5.4 Tensor d-fields. Distinguished linear (ρ,η)-connections

We consider the following diagram:

$$\begin{array}{ccc}
E & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\
\downarrow\pi & & \downarrow\nu\\
M & \xrightarrow{h} & N
\end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let

$$\left(\mathcal{T}{}^{\,p,r}_{\,q,s}\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right),+,\cdot\right)$$

be the $\mathcal{F}(E)$-module of tensor fields by $\binom{p,r}{q,s}$-type from the generalized tangent bundle

$$\left(H\,(\rho,\eta)\,TE\oplus V\,(\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right).$$

An arbitrarily tensor field T is written as

$$T = T^{\alpha_1 \dots \alpha_p a_1 \dots a_r}_{\beta_1 \dots \beta_q b_1 \dots b_s} \tilde{\delta}_{\alpha_1} \otimes \dots \otimes \tilde{\delta}_{\alpha_p} \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes$$
$$\dot{\tilde{\partial}}_{a_1} \otimes \dots \otimes \dot{\tilde{\partial}}_{a_r} \otimes \delta\tilde{y}^{b_1} \otimes \dots \otimes \delta\tilde{y}^{b_s}.$$

Let

$$\left(\mathcal{T}\ \left((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E\right), +, \cdot, \otimes\right)$$

be the tensor fields algebra of generalized tangent bundle $((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E)$.

If $T_1 \in \mathcal{T}^{p_1,r_1}_{q_1,s_1}((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$ and $T_2 \in \mathcal{T}^{p_2,r_2}_{q_2,s_2}((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$, then the components of product tensor field $T_1 \otimes T_2$ are the products of local components of T_1 and T_2.

Therefore, we obtain $T_1 \otimes T_2 \in \mathcal{T}^{p_1+p_2, r_1+r_2}_{q_1+q_2, s_1+s_2}\left((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E\right).$

Let $\mathcal{DT}((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$ be the family of tensor fields

$$T \in \mathcal{T}((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$$

for which there exists

$$T_1 \in \mathcal{T}^{p,0}_{q,0}((\rho,\eta)TE, (\rho,\eta)\tau_E, E) \text{ and } T_2 \in \mathcal{T}^{0,r}_{0,s}((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$$

such that $T = T_1 + T_2$.

The $\mathcal{F}(E)$-module $\left(\mathcal{DT}\left((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E\right), +, \cdot\right)$ will be called the *module of distinguished tensor fields* or the *module of tensor d-fields.*

Remark 5.4.1 The elements of

$$\Gamma\left((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E\right)$$

respectively

$$\Gamma(((\rho,\eta)TE)^*, ((\rho,\eta)\tau_E)^*, E)$$

are tensor d-fields.

Definition 5.4.1 Let (E,π,M) be a vector bundle endowed with a (ρ,η)-connection $(\rho,\eta)\,\Gamma$ and let

$$(X,T) \xrightarrow{(\rho,\eta)D} (\rho,\eta)\, D_X T \tag{5.4.1}$$

be a covariant (ρ,η)-derivative for the tensor algebra of generalized tangent bundle

$$\left((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E\right)$$

which preserves the horizontal and vertical IDS by parallelism.

The real local functions

$$\left((\rho,\eta)\, H^{\alpha}_{\beta\gamma}, (\rho,\eta)\, H^{a}_{b\gamma}, (\rho,\eta)\, V^{\alpha}_{\beta c}, (\rho,\eta)\, V^{a}_{bc}\right)$$

defined by the following equalities:

$$\begin{array}{ll} (\rho,\eta)\, D_{\tilde{\delta}_\gamma}\tilde{\delta}_\beta = (\rho,\eta)\, H^{\alpha}_{\beta\gamma}\tilde{\delta}_\alpha, & (\rho,\eta)\, D_{\tilde{\delta}_\gamma}\dot{\tilde{\partial}}_b = (\rho,\eta)\, H^{a}_{b\gamma}\dot{\tilde{\partial}}_a \\ (\rho,\eta)\, D_{\dot{\tilde{\partial}}_c}\tilde{\delta}_\beta = (\rho,\eta)\, V^{\alpha}_{\beta c}\tilde{\delta}_\alpha, & (\rho,\eta)\, D_{\dot{\tilde{\partial}}_c}\dot{\tilde{\partial}}_b = (\rho,\eta)\, V^{a}_{bc}\dot{\tilde{\partial}}_a \end{array} \tag{5.4.2}$$

are the components of a linear (ρ,η)-connection

$$((\rho,\eta)\,H,(\rho,\eta)\,V)$$

for the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ which will be called the *distinguished linear (ρ,η)-connection.*

If $h=Id_M$, then the distinguished linear (Id_{TM},Id_M)-connection will be called the *distinguished linear connection.*

The components of a distinguished linear connection (H,V) will be denoted

$$\left(H^i_{jk},H^a_{bk},V^i_{jc},V^a_{bc}\right).$$

Theorem 5.4.1 *If $((\rho,\eta)H,(\rho,\eta)V)$ is a distinguished linear (ρ,η)- connection for the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$, then its components satisfy the change relations:*

$$\begin{aligned}(\rho,\eta)\,H^{\acute{\alpha}}_{\acute{\beta}\acute{\gamma}} &= \Lambda^{\acute{\alpha}}_{\alpha}\circ h\circ\pi\cdot\Big[\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_{\gamma}\right)\left(\Lambda^{\alpha}_{\acute{\beta}}\circ h\circ\pi\right)+\\ &\quad+(\rho,\eta)\,H^{\alpha}_{\beta\gamma}\cdot\Lambda^{\beta}_{\acute{\beta}}\circ h\circ\pi\Big]\cdot\Lambda^{\gamma}_{\acute{\gamma}}\circ h\circ\pi,\\ (\rho,\eta)\,H^{\acute{a}}_{\acute{b}\acute{\gamma}} &= M^{\acute{a}}_{a}\circ\pi\cdot\Big[\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_{\gamma}\right)\left(M^{a}_{\acute{b}}\circ\pi\right)+\\ &\quad+(\rho,\eta)\,H^{a}_{b\gamma}\cdot M^{b}_{\acute{b}}\circ\pi\Big]\cdot\Lambda^{\gamma}_{\acute{\gamma}}\circ h\circ\pi,\\ (\rho,\eta)\,V^{\acute{\alpha}}_{\acute{\beta}\acute{c}} &= \Lambda^{\acute{\alpha}}_{\alpha}\circ h\circ\pi\cdot(\rho,\eta)\,V^{\alpha}_{\beta c}\cdot\Lambda^{\beta}_{\acute{\beta}}\circ h\circ\pi\cdot M^{c}_{\acute{c}}\circ\pi,\\ (\rho,\eta)\,V^{\acute{a}}_{\acute{b}\acute{c}} &= M^{\acute{a}}_{a}\circ\pi\cdot(\rho,\eta)\,V^{a}_{bc}\cdot M^{b}_{\acute{b}}\circ\pi\cdot M^{c}_{\acute{c}}\circ\pi.\end{aligned}\tag{5.4.3}$$

The components of a distinguished linear connection (H,V) verify the change relations:

$$\begin{aligned}H^{\acute{i}}_{\acute{j}\acute{k}} &= \tfrac{\partial x^{\acute{i}}}{\partial x^{i}}\circ\pi\cdot\left[\frac{\delta}{\delta x^k}\left(\frac{\partial x^i}{\partial x^{\acute{j}}}\circ\pi\right)+H^i_{jk}\cdot\frac{\partial x^j}{\partial x^{\acute{j}}}\circ\pi\right]\cdot\frac{\partial x^k}{\partial x^{\acute{k}}}\circ\pi,\\ H^{\acute{a}}_{\acute{b}\acute{k}} &= M^{\acute{a}}_{a}\circ\pi\cdot\left[\frac{\delta}{\delta x^k}\left(M^a_{\acute{b}}\circ\pi\right)+H^a_{bk}\cdot M^b_{\acute{b}}\circ\pi\right]\cdot\frac{\partial x^k}{\partial x^{\acute{k}}}\circ\pi,\\ V^{\acute{i}}_{\acute{j}\acute{c}} &= \frac{\partial x^{\acute{i}}}{\partial x^{i}}\circ\pi\cdot V^i_{jc}\frac{\partial x^j}{\partial x^{\acute{j}}}\circ\pi\cdot M^c_{\acute{c}}\circ\pi,\\ V^{\acute{a}}_{\acute{b}\acute{c}} &= M^{\acute{a}}_{a}\circ\pi\cdot V^a_{bc}\cdot M^b_{\acute{b}}\circ\pi\cdot M^c_{\acute{c}}\circ\pi.\end{aligned}\tag{5.4.3'}$$

Example 5.4.1 If (E,π,M) is a vector bundle endowed with the (ρ,η)-connection $(\rho,\eta)\,\Gamma$, then the local real functions

$$\left(\frac{\partial\,(\rho,\eta)\,\Gamma^a_{\gamma}}{\partial y^b},\frac{\partial\,(\rho,\eta)\,\Gamma^a_{\gamma}}{\partial y^b},0,0\right)$$

are the components of a distinguished linear (ρ,η)-connection for $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$, which will by called the *Berwald linear (ρ,η)-connection.*

The Berwald linear (Id_{TM},Id_M)-connection will be called the *Berwald linear connection.*

Theorem 5.4.2 *If the generalized tangent bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *is endowed with a distinguished linear* (ρ,η)*-connection* $((\rho,\eta)H,(\rho,\eta)V)$*, then, for any*

$$X=\tilde{Z}^{\alpha}\tilde{\delta}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a}\in\Gamma((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$$

and for any

$$T\in\mathcal{T}\,{}^{pr}_{qs}((\rho,\eta)TE,(\rho,\eta)\tau_E,E),$$

we obtain the formula:

$$\begin{array}{l}(\rho,\eta)\,D_X\left(T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}\tilde{\delta}_{\alpha_1}\otimes\ldots\otimes\tilde{\delta}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes\right.\\ \qquad\left.\otimes d\tilde{z}^{\beta_q}\otimes\dot{\tilde{\partial}}_{a_1}\otimes\ldots\otimes\dot{\tilde{\partial}}_{a_r}\otimes\delta\tilde{y}^{b_1}\otimes\ldots\otimes\delta\tilde{y}^{b_s}\right)=\\ \qquad=\tilde{Z}^{\gamma}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s|\gamma}\tilde{\delta}_{\alpha_1}\otimes\ldots\otimes\tilde{\delta}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes d\tilde{z}^{\beta_q}\otimes\dot{\tilde{\partial}}_{a_1}\otimes\ldots\otimes\\ \qquad\otimes\dot{\tilde{\partial}}_{a_r}\otimes\delta\tilde{y}^{b_1}\otimes\ldots\otimes\delta\tilde{y}^{b_s}+Y^{c}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}\mid_c\tilde{\delta}_{\alpha_1}\otimes\ldots\otimes\\ \qquad\otimes\tilde{\delta}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes d\tilde{z}^{\beta_q}\otimes\dot{\tilde{\partial}}_{a_1}\otimes\ldots\otimes\dot{\tilde{\partial}}_{a_r}\otimes\delta\tilde{y}^{b_1}\otimes\ldots\otimes\delta\tilde{y}^{b_s},\end{array}$$

where

$$\begin{array}{l}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s|\gamma}=\Gamma\left(\tilde{\rho},Id_E\right)\left(\tilde{\delta}_{\gamma}\right)T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}\\ \quad+(\rho,\eta)\,H^{\alpha_1}_{\alpha\gamma}T^{\alpha\alpha_2\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}+\ldots+(\rho,\eta)\,H^{\alpha_p}_{\alpha\gamma}T^{\alpha_1\ldots\alpha_{p-1}\alpha a_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}\\ \quad-(\rho,\eta)\,H^{\beta}_{\beta_1\gamma}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta\beta_2\ldots\beta_qb_1\ldots b_s}-\ldots-(\rho,\eta)\,H^{\beta}_{\beta_q\gamma}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_{q-1}\beta b_1\ldots b_s}\\ \quad+(\rho,\eta)\,H^{a_1}_{a\gamma}T^{\alpha_1\ldots\alpha_paa_2\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}+\ldots+(\rho,\eta)\,H^{a_r}_{a\gamma}T^{\alpha_1\ldots\alpha_pa_1\ldots a_{r-1}a}_{\beta_1\ldots\beta_qb_1\ldots b_s}\\ \quad-(\rho,\eta)\,H^{b}_{b_1\gamma}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qbb_2\ldots b_s}-\ldots-(\rho,\eta)\,H^{b}_{b_s\gamma}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_{s-1}b}\end{array}$$

and

$$\begin{array}{l}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}\mid_c=\Gamma\left(\tilde{\rho},Id_E\right)\left(\dot{\tilde{\partial}}_c\right)T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}+\\ \quad+(\rho,\eta)\,V^{\alpha_1}_{\alpha c}T^{\alpha\alpha_2\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}+\ldots+(\rho,\eta)\,V^{\alpha_p}_{\alpha c}T^{\alpha_1\ldots\alpha_{p-1}\alpha a_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}-\\ \quad-(\rho,\eta)\,V^{\beta}_{\beta_1c}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta\beta_2\ldots\beta_qb_1\ldots b_s}-\ldots-(\rho,\eta)\,V^{\beta}_{\beta_qc}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_{q-1}\beta b_1\ldots b_s}+\\ \quad+(\rho,\eta)\,V^{a_1}_{ac}T^{\alpha_1\ldots\alpha_paa_2\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_s}+\ldots+(\rho,\eta)\,V^{a_r}_{ac}T^{\alpha_1\ldots\alpha_pa_1\ldots a_{r-1}a}_{\beta_1\ldots\beta_qb_1\ldots b_s}-\\ \quad-(\rho,\eta)\,V^{b}_{b_1c}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qbb_2\ldots b_s}-\ldots-(\rho,\eta)\,V^{b}_{b_sc}T^{\alpha_1\ldots\alpha_pa_1\ldots a_r}_{\beta_1\ldots\beta_qb_1\ldots b_{s-1}b}.\end{array}$$

Definition 5.4.2 We assume that $(E,\pi,M)=(F,\nu,N)$.

If $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M) and

$$\left((\rho,\eta)\,H^{a}_{bc},(\rho,\eta)\,\tilde{H}^{a}_{bc},(\rho,\eta)\,V^{a}_{bc},(\rho,\eta)\,\tilde{V}^{a}_{bc}\right)$$

are the components of a distinguished linear (ρ,η)-connection for the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ such that

$$(\rho,\eta)\,H^{a}_{bc}=(\rho,\eta)\,\tilde{H}^{a}_{bc}\text{ and }(\rho,\eta)\,V^{a}_{bc}=(\rho,\eta)\,\tilde{V}^{a}_{bc},$$

then we will say that *the generalized tangent bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *is endowed with a normal distinguished linear* (ρ,η)*-connection on components* $((\rho,\eta)\,H^a_{bc},(\rho,\eta)\,V^a_{bc})$.

The components of a normal distinguished linear (Id_{TM},Id_M)-connection (H,V) will be denoted $\left(H^i_{jk},V^i_{jk}\right)$.

5.5 The lift of accelerations for a differentiable curve

We consider the following diagram:

$$
\begin{tikzcd}
E \arrow[d, "\pi"'] & (F,[,]_{F,h},(\rho,\eta)) \arrow[d, "\nu"] \\
M \arrow[r, "h"] & N
\end{tikzcd}
\tag{5.5.1}
$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)\in|\mathbf{GLA}|$.

Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M).

We admit that $((\rho,\eta)\,H,(\rho,\eta)\,V)$ is a distinguished linear (ρ,η)-connection for the vector bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$.

Let $g\in\mathbf{Man}\,(E,F)$ be such that (g,h) is a $\mathbf{B^v}$-morphism of (E,π,M) source and (F,ν,N) target.

Let

$$
\begin{array}{rcl}
I & \xrightarrow{\dot{c}} & E_{|\,\mathrm{Im}(\eta\circ h\circ c)}\\
t & \longmapsto & y^a\,(t)\,s_a\,(\eta\circ h\circ c\,(t))
\end{array}
\tag{5.5.2}
$$

be the (g,h)-lift of differentiable curve $I\ \xrightarrow{c}\ M$.

Definition 5.5.1 The differentiable curve

$$
\begin{array}{rcl}
I & \xrightarrow{\ddot{c}} & (\rho,\eta)\,TE_{|\,\mathrm{Im}\,\dot{c}}\\
t & \longmapsto & (g^\alpha_a\circ h\circ c\,(t)\cdot y^a\,(t))\,\dfrac{\partial}{\partial\tilde{z}^\alpha}\,(\dot{c}\,(t))+\dfrac{dy^a\,(t)}{dt}\dfrac{\partial}{\partial\tilde{y}^a}\,(\dot{c}\,(t))
\end{array}
\tag{5.5.3}
$$

will be called the *differentiable* (g,h)*-lift of accelerations of the differentiable curve* c.

The section

$$
\begin{array}{rcl}
\mathrm{Im}\,(\dot{c}) & \xrightarrow{u(c,\dot{c},\ddot{c})} & (\rho,\eta)\,TE_{|\,\mathrm{Im}(\dot{c})}\\
\dot{c}\,(t) & \longmapsto & \left(g^\alpha_b\circ h\circ c\,(t)\cdot y^b\,(t)\right)\dfrac{\partial}{\partial\tilde{z}^\alpha}\,(\dot{c}\,(t))+\dfrac{dy^a\,(t)}{dt}\dfrac{\partial}{\partial\tilde{y}^a}\,(\dot{c}\,(t))
\end{array}
\tag{5.5.4}
$$

will be called the *canonical section associated to the triple* $(c,\dot{c},\ddot{c})$.

Remark 5.5.1 For any $t\in I$, we obtain:

$$
\begin{array}{c}
u\,(c,\dot{c},\ddot{c})\,(\dot{c}\,(t))=\left(g^\alpha_b\circ h\circ c\,(t)\,y^b\,(t)\right)\dfrac{\delta}{\delta\tilde{z}^\alpha}\,(\dot{c}\,(t))+\dfrac{dy^a\,(t)}{dt}\dfrac{\partial}{\partial\tilde{y}^a}\,(\dot{c}\,(t))\\
+\,(\rho,\eta)\,\Gamma^a_\alpha\circ u\,(c,\dot{c})\circ\eta\circ h\circ c\,(t)\cdot\left(g^\alpha_b\circ h\circ c\,(t)\,y^b\,(t)\right)\dfrac{\partial}{\partial\tilde{y}^a}\,(\dot{c}\,(t))\,.
\end{array}
\tag{5.5.5}
$$

We observe easily that $u(c,\dot{c},\ddot{c})(\dot{c}(t)) \in H(\rho,\eta)TE_{|\operatorname{Im}(\dot{c})}$ if and only if the components functions $\left(y^a,\ a \in \overline{1,n}\right)$ are solutions for the differentiable equations

$$\frac{du^a}{dt} + (\rho,\eta)\Gamma^a_\alpha \circ u(c,\dot{c}) \circ \eta \circ h \circ c \cdot (g^\alpha_b \circ h \circ c) \cdot u^b = 0,\ a \in \overline{1,r}. \tag{5.5.6}$$

Remark 5.5.2 In particular, if $(\rho,\eta,h) = (Id_{TM}, Id_M, Id_M)$ and (g, Id_M) is locally invertible, then, using the (g, Id_M)-lift

$$\begin{array}{rcl} I & \stackrel{\dot{c}}{\longrightarrow} & TM \\ t & \longmapsto & \tilde{g}^i_j \dfrac{dc^j(t)}{dt} \dfrac{\partial}{\partial x^i}(c(t)), \end{array} \tag{5.5.7}$$

the differentiable (g, Id_M)-lift of accelerations for the differentiable curve c is

$$\begin{array}{rcl} I & \stackrel{\ddot{c}}{\longrightarrow} & (Id_{TM}, Id_M)TTM_{|\operatorname{Im}(\dot{c})} \\ t & \longmapsto & \dfrac{dc^i(t)}{dt}\dfrac{\partial}{\partial \tilde{z}^i}(\dot{c}(t)) + \tilde{g}^i_j(c(t))\dfrac{dc^j(t)}{dt}\dfrac{\partial}{\partial \tilde{y}^i}(\dot{c}(t)). \end{array} \tag{5.5.8}$$

Definition 5.5.2 If the component functions

$$\left((g^\alpha_a \circ h \circ c)\, y^a,\ a \in \overline{1,r}\right)$$

are solutions for the differentiable system of equations

$$\frac{dz^\alpha}{dt} + (\rho,\eta)H^\alpha_{\beta\gamma} \circ u(c,\dot{c}) \circ \eta \circ h \circ c \cdot z^\beta \cdot z^\gamma = 0,\ \alpha \in \overline{1,p}, \tag{5.5.9}$$

then the differentiable curve $\dot{c}$ will be called *horizontal parallel with respect to the distinguished linear (ρ,η)-connection $((\rho,\eta)H, (\rho,\eta)V)$.*

If the component functions $\left(y^a,\ a \in \overline{1,n}\right)$ are solutions for the differentiable system of equations

$$\frac{du^a}{dt} + (\rho,\eta)V^a_{bc} \circ u(c,\dot{c}) \circ \eta \circ h \circ c \cdot u^b \cdot u^c = 0,\ a \in \overline{1,r}, \tag{5.5.10}$$

then the differentiable curve $\dot{c}$ will be called *vertical parallel with respect to the distinguished linear (ρ,η)-connection $((\rho,\eta)H, (\rho,\eta)V)$.*

Remark 5.5.3 In particular, if $(\rho,\eta,h) = (Id_{TM}, Id_M, Id_M)$ and (g, Id_M) is locally invertible, then the (g, Id_M)-lift of tangent vectors (5.5.7) is horizontal parallel with respect to the distinguished linear connection (H,V) if the component functions $\left(\dfrac{dc^i}{dt},\ i \in \overline{1,m}\right)$ are solutions for the differentiable system of equations

$$\frac{dz^i}{dt} + H^i_{jk} \circ u(c,\dot{c}) \circ c \cdot z^j \cdot z^k = 0,\ i \in \overline{1,m}. \tag{5.5.12}$$

Moreover, the (g, Id_M)-lift of tangent vectors (5.5.7) is vertical parallel with respect to the distinguished linear connection (H,V) if the component functions

$$\left(\tilde{g}^i_j \circ c \cdot \frac{dc^j(t)}{dt},\ i \in \overline{1,m}\right)$$

are solutions for the differentiable system of equations

$$\frac{du^i}{dt} + V^i_{jk} \circ u(c,\dot{c}) \circ c \cdot u^j \cdot u^k = 0,\ i \in \overline{1,m}. \tag{5.5.13}$$

5.6 The (ρ,η,h)-torsion and the (ρ,η,h)-curvature of a distinguished linear (ρ,η)-connection

We consider the following diagram:

$$\begin{array}{ccc} E & & (F,[,]_{F,h},(\rho,\eta)) \\ \pi\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $((F,\nu,M),[,]_{F.h},(\rho,\eta))\in|\mathbf{GLA}|$. Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M) and let $((\rho,\eta)\,H,(\rho,\eta)\,V)$ be a distinguished linear (ρ,η)-connection for the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$.

Definition 5.6.1 The application

$$\begin{array}{ccc} \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)^2 & \xrightarrow{(\rho,\eta,h)\mathbb{T}} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right) \\ (X,Y) & \longmapsto & (\rho,\eta)\,\mathbb{T}\,(X,Y) \end{array}$$

defined by

$$(\rho,\eta,h)\,\mathbb{T}\,(X,Y)=(\rho,\eta)\,D_XY-(\rho,\eta)\,D_YX-[X,Y]_{(\rho,\eta)TE}\,, \tag{5.6.1}$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$, will be called the *$(\rho,\eta,h)$-torsion associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$.*

The applications

$$\mathcal{H}\,(\rho,\eta,h)\,\mathbb{T}\,(\mathcal{H}\,(\cdot)\,,\mathcal{H}\,(\cdot))\,,\ \mathcal{V}\,(\rho,\eta,h)\,\mathbb{T}\,(\mathcal{H}\,(\cdot)\,,\mathcal{H}\,(\cdot))\,,...,\mathcal{V}\,(\rho,\eta,h)\,\mathbb{T}\,(\mathcal{V}\,(\cdot)\,,\mathcal{V}\,(\cdot))$$

are called $\mathcal{H}\,(\mathcal{HH})\,,\ \mathcal{V}\,(\mathcal{HH})\,,...,\mathcal{V}\,(\mathcal{VV})$ *(ρ,η,h)-torsions associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$.*

Proposition 5.6.1 *The (ρ,η,h)-torsion $(\rho,\eta,h)\,\mathbb{T}$ associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$, is $\mathbb{R}$-bilinear and antisymmetric in the lower indices.*

Using the notations:

$$\begin{array}{c} \mathcal{H}\,(\rho,\eta,h)\,\mathbb{T}\left(\tilde{\delta}_\gamma,\tilde{\delta}_\beta\right)=(\rho,\eta,h)\,\mathbb{T}^\alpha{}_{\beta\gamma}\tilde{\delta}_\alpha,\\ \mathcal{V}\,(\rho,\eta,h)\,\mathbb{T}\left(\tilde{\delta}_\gamma,\tilde{\delta}_\beta\right)=(\rho,\eta,h)\,\mathbb{T}^a{}_{\beta\gamma}\dot{\tilde{\partial}}_a,\\ \mathcal{H}\,(\rho,\eta,h)\,\mathbb{T}\left(\dot{\tilde{\partial}}_c,\tilde{\delta}_\beta\right)=(\rho,\eta,h)\,\mathbb{P}^\alpha{}_{\beta c}\tilde{\delta}_\alpha,\\ \mathcal{V}\,(\rho,\eta,h)\,\mathbb{T}\left(\dot{\tilde{\partial}}_c,\tilde{\delta}_\beta\right)=(\rho,\eta,h)\,\mathbb{P}^a{}_{\beta c}\dot{\tilde{\partial}}_a,\\ \mathcal{V}\,(\rho,\eta,h)\,\mathbb{T}\left(\dot{\tilde{\partial}}_c,\dot{\tilde{\partial}}_b\right)=(\rho,\eta,h)\,\mathbb{S}^a{}_{bc}\dot{\tilde{\partial}}_a. \end{array} \tag{5.6.2}$$

we can easily prove the following

Theorem 5.6.1 *The (ρ,η,h)-torsion $(\rho,\eta,h)\,\mathbb{T}$ associated to the distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$, is characterized by the tensor fields with local components:*

$$
\begin{aligned}
(\rho,\eta,h)\,\mathbb{T}^{\alpha}{}_{\beta\gamma} &= (\rho,\eta)\,H^{\alpha}_{\beta\gamma}-(\rho,\eta)\,H^{\alpha}_{\gamma\beta}-L^{\alpha}_{\beta\gamma}\circ h\circ\pi,\\
(\rho,\eta,h)\,\mathbb{T}^{a}{}_{\beta\gamma} &= (\rho,\eta,h)\,\mathbb{R}^{a}{}_{\beta\gamma},\\
(\rho,\eta,h)\,\mathbb{P}^{\alpha}{}_{\beta c} &= (\rho,\eta)\,V^{\alpha}_{\beta c},\\
(\rho,\eta,h)\,\mathbb{P}^{a}{}_{\beta c} &= \frac{\partial}{\partial y^{c}}\left((\rho,\eta)\,\Gamma^{a}_{\beta}\right)-(\rho,\eta)\,H^{a}_{c\beta},\\
(\rho,\eta,h)\,\mathbb{S}^{a}{}_{bc} &= (\rho,\eta)\,V^{a}_{bc}-(\rho,\eta)\,V^{a}_{cb}.
\end{aligned}
\tag{5.6.3}
$$

In particular, when $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, we regain the local components of torsion associated to distinguished linear connection (H,V):

$$
\begin{aligned}
\mathbb{T}^{i}{}_{jk} &= H^{i}_{jk}-H^{i}_{kj}, & \mathbb{T}^{a}{}_{jk} &= \mathbb{R}^{a}{}_{jk},\\
\mathbb{P}^{i}{}_{jc} &= V^{i}_{jc}, & \mathbb{P}^{a}{}_{jc} &= \frac{\partial\Gamma^{a}_{j}}{\partial y^{c}}-H^{a}_{cj},\\
\mathbb{S}^{a}{}_{bc} &= V^{a}_{bc}-V^{a}_{cb}.
\end{aligned}
\tag{5.6.3$'$}
$$

Definition 5.6.2 The application

$$
\begin{array}{ccc}
\left(\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\right)^3 & \xrightarrow{(\rho,\eta,h)\mathbb{R}} & \Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)\\
((Y,Z),X) & \longmapsto & (\rho,\eta,h)\,\mathbb{R}\,(Y,Z)\,,X
\end{array}
$$

defined by:

$$
\begin{aligned}
(\rho,\eta,h)\,\mathbb{R}\,(Y,Z)\,X &= (\rho,\eta)\,D_Y\left((\rho,\eta)\,D_Z X\right)\\
&\quad-(\rho,\eta)\,D_Z\left((\rho,\eta)\,D_Y X\right)-(\rho,\eta)\,D_{[Y,Z]_{(\rho,\eta)TE}}X,
\end{aligned}
\tag{18.4}
$$

for any $X,Y,Z\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$, will be called the *$(\rho,\eta,h)$-curvature associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$.*

Proposition 5.6.2 *The (ρ,η,h)-curvature $(\rho,\eta,h)\,\mathbb{R}$ associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$, is $\mathbb{R}$-linear in each argument and antisymmetric in the first two arguments.*

Using the notations:

$$
\begin{aligned}
(\rho,\eta,h)\,\mathbb{R}\left(\tilde{\delta}_{\varepsilon},\tilde{\delta}_{\gamma}\right)\tilde{\delta}_{\beta} &= (\rho,\eta,h)\,\mathbb{R}^{\alpha}{}_{\beta\ \gamma\varepsilon}\tilde{\delta}_{\alpha},\\
(\rho,\eta,h)\,\mathbb{R}\left(\tilde{\delta}_{\varepsilon},\tilde{\delta}_{\gamma}\right)\dot{\tilde{\partial}}_{b} &= (\rho,\eta,h)\,\mathbb{R}^{a}{}_{b\ \gamma\varepsilon}\dot{\tilde{\partial}}_{a},\\
(\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_{c},\tilde{\delta}_{\gamma}\right)\tilde{\delta}_{\varepsilon} &= (\rho,\eta,h)\,\mathbb{P}^{\alpha}{}_{\varepsilon\ \gamma c}\tilde{\delta}_{\alpha},\\
(\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_{c},\tilde{\delta}_{\gamma}\right)\dot{\tilde{\partial}}_{b} &= (\rho,\eta,h)\,\mathbb{P}^{a}{}_{b\ \gamma c}\dot{\tilde{\partial}}_{a},\\
(\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_{c},\dot{\tilde{\partial}}_{b}\right)\tilde{\delta}_{\beta} &= (\rho,\eta,h)\,\mathbb{S}^{\alpha}{}_{\beta\ bc}\tilde{\delta}_{\alpha},\\
(\rho,\eta,h)\,\mathbb{R}\left(\dot{\tilde{\partial}}_{d},\dot{\tilde{\partial}}_{c}\right)\dot{\tilde{\partial}}_{b} &= (\rho,\eta,h)\,\mathbb{S}^{\alpha}{}_{b\ cd}\dot{\tilde{\partial}}_{a}.
\end{aligned}
\tag{5.6.5}
$$

we can easily prove the following

Theorem 5.6.2 *The (ρ,η,h)-curvature $(\rho,\eta,h)\,\mathbb{R}$ associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$, is characterized by the tensor fields with local components:*

$$(5.6.6)\quad \left\{\begin{array}{ll} (\rho,\eta,h)\mathbb{R}^{\alpha}{}_{\beta\ \gamma\varepsilon} & =\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\varepsilon}\right)(\rho,\eta)H^{\alpha}_{\beta\gamma}-\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\gamma}\right)(\rho,\eta)H^{\alpha}_{\beta\varepsilon}\\ & +(\rho,\eta)H^{\alpha}_{\theta\varepsilon}(\rho,\eta)H^{\theta}_{\beta\gamma}-(\rho,\eta)H^{\alpha}_{\theta\gamma}(\rho,\eta)H^{\theta}_{\beta\varepsilon}\\ & -(\rho,\eta,h)\mathbb{R}^{c}{}_{\gamma\varepsilon}(\rho,\eta)H^{\alpha}_{\beta c}-L^{\theta}_{\gamma\varepsilon}\circ h\circ\pi(\rho,\eta)H^{\alpha}_{\beta\theta},\\ (\rho,\eta,h)\mathbb{R}^{a}{}_{b\ \gamma\varepsilon} & =\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\varepsilon}\right)(\rho,\eta)H^{a}_{b\gamma}-\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\gamma}\right)(\rho,\eta)H^{a}_{b\varepsilon}\\ & +(\rho,\eta)H^{a}_{d\varepsilon}(\rho,\eta)H^{d}_{b\gamma}-(\rho,\eta)H^{a}_{d\gamma}(\rho,\eta)H^{d}_{b\varepsilon}\\ & -(\rho,\eta,h)\mathbb{R}^{c}{}_{\varepsilon\gamma}(\rho,\eta)V^{a}_{bc}-L^{\theta}_{\gamma\varepsilon}\circ h\circ\pi(\rho,\eta)V^{a}_{b\theta}, \end{array}\right.$$

$$(5.6.7)\quad \left\{\begin{array}{ll} (\rho,\eta,h)\mathbb{P}^{\alpha}{}_{\varepsilon\ \gamma c} & =\Gamma(\tilde{\rho},Id_E)(\dot{\tilde{\partial}}_{c})(\rho,\eta)H^{\alpha}_{\varepsilon\gamma}-\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\gamma}\right)(\rho,\eta)V^{\alpha}_{\varepsilon c}\\ & =(\rho,\eta)V^{\alpha}_{\theta c}(\rho,\eta)H^{\theta}_{\varepsilon\gamma}-(\rho,\eta)H^{\alpha}_{\theta\gamma}(\rho,\eta)V^{\theta}_{\varepsilon c}\\ & +\dfrac{\partial}{\partial y^{c}}\left((\rho,\eta)\,\Gamma^{d}_{\gamma}\right)(\rho,\eta)\,V^{\alpha}_{\varepsilon d},\\ (\rho,\eta,h)\mathbb{P}^{a}{}_{b\ \gamma c} & =\Gamma(\tilde{\rho},Id_E)\left(\dot{\tilde{\partial}}_{c}\right)(\rho,\eta)H^{a}_{b\gamma}-\\ & -\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\gamma}\right)(\rho,\eta)V^{a}_{bc}+(\rho,\eta)V^{a}_{dc}(\rho,\eta)H^{d}_{b\gamma}-\\ & -(\rho,\eta)H^{a}_{d\gamma}(\rho,\eta)V^{d}_{bc}+\dfrac{\partial}{\partial y^{c}}(\left(\rho,\eta)\Gamma^{d}_{\gamma}\right)(\rho,\eta)V^{a}_{bd}, \end{array}\right.$$

$$(5.6.8)\quad \left\{\begin{array}{ll} (\rho,\eta,h)\,\mathbb{S}^{\alpha}{}_{\beta\ bc} & =\Gamma\left(\tilde{\rho},Id_E\right)\left(\dot{\tilde{\partial}}_{c}\right)(\rho,\eta)\,V^{\alpha}_{\beta b}\\ & -\Gamma\left(\tilde{\rho},Id_E\right)\left(\dot{\tilde{\partial}}_{b}\right)(\rho,\eta)\,V^{\alpha}_{\beta c}+(\rho,\eta)\,V^{\alpha}_{\theta c}\,(\rho,\eta)\,V^{\theta}_{\beta b}\\ & -(\rho,\eta)\,V^{\alpha}_{\theta b}\,(\rho,\eta)\,V^{\theta}_{\beta c},\\ (\rho,\eta,h)\,\mathbb{S}^{a}{}_{b\ cd} & =\Gamma\left(\tilde{\rho},Id_E\right)\left(\dot{\tilde{\partial}}_{d}\right)(\rho,\eta)\,V^{a}_{bc}\\ & -\Gamma\left(\tilde{\rho},Id_E\right)\left(\dot{\tilde{\partial}}_{c}\right)(\rho,\eta)\,V^{a}_{bd}+(\rho,\eta)\,V^{a}_{ed}\,(\rho,\eta)\,V^{e}_{bc}\\ & -(\rho,\eta)\,V^{a}_{ec}\,(\rho,\eta)\,V^{e}_{bd}. \end{array}\right.$$

In particular, when $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, we see the local components of the curvature associated to distinguished linear connection (H,V) in the followings:

$$(5.6.6')\quad \begin{array}{l} \mathbb{R}^{i}{}_{j\ kl}=\delta_{l}\left(H^{i}_{jk}\right)-\delta_{k}\left(H^{i}_{jl}\right)+H^{i}_{hl}H^{h}_{jk}-H^{i}_{hk}H^{h}_{jl}-\mathbb{R}^{c}{}_{kl}H^{i}_{jc},\\ \mathbb{R}^{a}{}_{b\ kl}=\delta_{l}\left(H^{a}_{bk}\right)-\delta_{k}\left(H^{a}_{bl}\right)+H^{a}_{dl}H^{d}_{bk}-H^{a}_{dk}H^{d}_{bl}-\mathbb{R}^{c}{}_{lk}V^{a}_{bc}, \end{array}$$

$$(5.6.7')\quad \begin{aligned}\mathbb{P}^{i}{}_{l\ kc} &= \frac{\partial}{\partial y^{c}}\left(H^{i}_{lk}\right)-\delta_{k}\left(V^{i}_{lc}\right)+V^{i}_{hc}H^{h}_{lk}-H^{i}_{hk}V^{h}_{lc}+\frac{\partial}{\partial y^{c}}\left(\Gamma^{d}_{k}\right)V^{i}_{ld},\\ \mathbb{P}^{a}{}_{b\ kc} &= \frac{\partial}{\partial y^{c}}\left(H^{a}_{bk}\right)-\delta_{k}\left(V^{a}_{bc}\right)+V^{a}_{dc}H^{d}_{bk}-H^{a}_{dk}V^{d}_{bc}+\frac{\partial}{\partial y^{c}}\left(\Gamma^{d}_{k}\right)V^{a}_{bd},\end{aligned}$$

$$(5.6.8')\quad \begin{aligned}\mathbb{S}^{i}{}_{j\ bc} &= \frac{\partial}{\partial y^{c}}\left(V^{i}_{jb}\right)-\frac{\partial}{\partial y^{b}}\left(V^{i}_{jc}\right)+V^{i}_{hc}V^{h}_{jb}-V^{i}_{hb}V^{h}_{jc},\\ \mathbb{S}^{a}{}_{b\ cd} &= \frac{\partial}{\partial y^{d}}\left(V^{a}_{bc}\right)-\frac{\partial}{\partial y^{c}}\left(V^{a}_{bd}\right)+V^{a}_{ed}V^{e}_{bc}-V^{a}_{ec}V^{e}_{bd}.\end{aligned}$$

Definition 5.6.3 The tensor field

$$(5.6.9)\quad \begin{aligned}&\mathbf{Ric}\left(\left(\rho,\eta\right)H,\left(\rho,\eta\right)V\right)=\\ &\quad=\left(\rho,\eta,h\right)\mathbb{R}_{\ \alpha\ \beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+\left(\rho,\eta,h\right)\mathbb{P}_{\ \alpha\ b}d\tilde{z}^{\alpha}\otimes\delta\tilde{y}^{b}\\ &\quad+\left(\rho,\eta,h\right)\mathbb{P}_{\ a\ \beta}\delta\tilde{y}^{a}\otimes d\tilde{z}^{\beta}+\left(\rho,\eta,h\right)\mathbb{S}_{\ a\ b}\delta\tilde{y}^{a}\otimes\delta\tilde{y}^{b},\end{aligned}$$

$$(5.6.10)\quad \begin{array}{ll}\left(\rho,\eta,h\right)\mathbb{R}_{\ \alpha\ \beta}=\left(\rho,\eta,h\right)\mathbb{R}^{\gamma}{}_{\alpha\ \beta\gamma} & \left(\rho,\eta,h\right)\mathbb{P}_{\ \alpha\ b}=\left(\rho,\eta,h\right)\mathbb{P}^{\beta}{}_{\alpha\ \beta b}\\ \left(\rho,\eta,h\right)\mathbb{P}_{\ a\ \beta}=\left(\rho,\eta,h\right)\mathbb{P}^{c}{}_{a\ \beta c} & \left(\rho,\eta,h\right)\mathbb{S}_{\ a\ b}=\left(\rho,\eta,h\right)\mathbb{S}^{c}{}_{a\ cb}\end{array},$$

will be called *the Ricci tensor field associated to distinguished linear* (ρ,η)*-connection* $\left(\left(\rho,\eta\right)H,\left(\rho,\eta\right)V\right)$.

This tensor field will be used to write the Einstein equations in Subsection 5.10.

5.7 Formulas of Ricci type. Identities of Cartan and Bianchi type

We consider the following diagram:

$$\begin{array}{ccc} E & & \left(F,[,]_{F,h},\left(\rho,\eta\right)\right)\\ \pi\downarrow & & \downarrow\nu\\ M & \xrightarrow{h} & N\end{array}$$

where $\left(E,\pi,M\right)\in\left|\mathbf{B^{v}}\right|$ and $\left(\left(F,\nu,M\right),[,]_{F.h},\left(\rho,\eta\right)\right)\in\left|\mathbf{GLA}\right|$. Let $\left(\rho,\eta\right)\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M) and let $\left(\left(\rho,\eta\right)H,\left(\rho,\eta\right)V\right)$ be a distinguished linear (ρ,η)-connection for the generalized tangent bundle $\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$.

Theorem 5.7.1 *Using the definition of* (ρ,η,h)*-curvature associated to the distinguished linear* (ρ,η)*-connection* $\left(\left(\rho,\eta\right)H,\left(\rho,\eta\right)V\right)$*, it results the following formulas:*

$$(\mathcal{R}_1)\quad \left\{\begin{array}{l}\left(\rho,\eta\right)D_{\mathcal{H}X}\left(\rho,\eta\right)D_{\mathcal{H}Y}\mathcal{H}Z-\left(\rho,\eta\right)D_{\mathcal{H}Y}\left(\rho,\eta\right)D_{\mathcal{H}X}\mathcal{H}Z\\ =\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{H}X,\mathcal{H}Y\right)\mathcal{H}Z+\left(\rho,\eta\right)D_{\mathcal{H}[\mathcal{H}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{H}Z\\ +\left(\rho,\eta\right)D_{\mathcal{V}[\mathcal{H}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{H}Z,\\ \left(\rho,\eta\right)D_{\mathcal{V}X}\left(\rho,\eta\right)D_{\mathcal{H}Y}\mathcal{H}Z-\left(\rho,\eta\right)D_{\mathcal{H}Y}\left(\rho,\eta\right)D_{\mathcal{V}X}\mathcal{H}Z\\ =\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{V}X,\mathcal{H}Y\right)\mathcal{H}Z+\left(\rho,\eta\right)D_{\mathcal{H}[\mathcal{V}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{H}Z\\ +\left(\rho,\eta\right)D_{\mathcal{V}[\mathcal{V}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{H}Z,\\ \left(\rho,\eta\right)D_{\mathcal{V}X}\left(\rho,\eta\right)D_{\mathcal{V}Y}\mathcal{H}Z-\left(\rho,\eta\right)D_{\mathcal{V}Y}\left(\rho,\eta\right)D_{\mathcal{V}X}\mathcal{H}Z\\ =\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{V}X,\mathcal{V}Y\right)\mathcal{H}Z+\left(\rho,\eta\right)D_{\mathcal{V}[\mathcal{V}X,\mathcal{V}Y]_{(\rho,\eta)TE}}\mathcal{H}Z,\end{array}\right.$$

and

$$(\mathcal{R}_2)\quad \left\{\begin{array}{l}(\rho,\eta)\, D_{\mathcal{H}X}\,(\rho,\eta)\, D_{\mathcal{H}Y}\mathcal{V}Z-(\rho,\eta)\, D_{\mathcal{H}Y}\,(\rho,\eta)\, D_{\mathcal{H}X}\mathcal{V}Z\\ =(\rho,\eta,h)\,\mathbb{R}\,(\mathcal{H}X,\mathcal{H}Y)\,\mathcal{V}Z+(\rho,\eta)\, D_{\mathcal{H}[\mathcal{H}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{V}Z\\ +(\rho,\eta)\, D_{\mathcal{V}[\mathcal{H}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{V}Z,\\ (\rho,\eta)\, D_{\mathcal{V}X}\,(\rho,\eta)\, D_{\mathcal{H}Y}\mathcal{V}Z-(\rho,\eta)\, D_{\mathcal{H}Y}\,(\rho,\eta)\, D_{\mathcal{V}X}\mathcal{V}Z\\ =(\rho,\eta,h)\,\mathbb{R}\,(\mathcal{V}X,\mathcal{H}Y)\,\mathcal{V}Z+(\rho,\eta)\, D_{h[\mathcal{V}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{V}Z\\ +(\rho,\eta)\, D_{\mathcal{V}[\mathcal{V}X,\mathcal{H}Y]_{(\rho,\eta)TE}}\mathcal{V}Z,\\ (\rho,\eta)\, D_{\mathcal{V}X}\,(\rho,\eta)\, D_{\mathcal{V}Y}\mathcal{V}Z-(\rho,\eta)\, D_{\mathcal{V}Y}\,(\rho,\eta)\, D_{\mathcal{V}X}\mathcal{V}Z\\ =(\rho,\eta,h)\,\mathbb{R}\,(\mathcal{V}X,\mathcal{V}Y)\,\mathcal{V}Z+(\rho,\eta)\, D_{\mathcal{V}[\mathcal{V}X,\mathcal{V}Y]_{(\rho,\eta)TE}}\mathcal{V}Z.\end{array}\right.$$

Using the previous theorem, the horizontal and vertical sections of adapted base and an arbitrary section

$$Z^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y^{a}\frac{\partial}{\partial\tilde{y}^{a}}\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_{E},E\right),$$

we propose the following

Theorem 5.7.2 *We obtain the following formulas of Ricci type:*

$$(\mathcal{R}_1)\quad \left\{\begin{array}{ll}\tilde{Z}^{\alpha}{}_{|\gamma|\beta}-\tilde{Z}^{\alpha}{}_{|\beta|\gamma} & =(\rho,\eta,h)\,\mathbb{R}^{\alpha}_{\theta\ \gamma\beta}\tilde{Z}^{\theta}+\left(L^{\theta}_{\beta\gamma}\circ h\circ\pi\right)\tilde{Z}^{\alpha}{}_{|\theta}\\ & \quad +(\rho,\eta,h)\,\mathbb{T}^{a}{}_{\beta\gamma}\tilde{Z}^{\alpha}|_{a}+(\rho,\eta,h)\,\mathbb{T}^{\theta}{}_{\beta\gamma}\tilde{Z}^{\alpha}{}_{|\theta},\\ \tilde{Z}^{\alpha}{}_{|\gamma}|_{b}-\tilde{Z}^{\alpha}|_{b|\gamma} & =(\rho,\eta,h)\,\mathbb{P}^{\alpha}_{\theta\ \gamma b}\tilde{Z}^{\theta}-(\rho,\eta,h)\,\mathbb{P}^{a}{}_{\gamma b}\tilde{Z}^{\alpha}|_{a}\\ & \quad -(\rho,\eta)\,\mathbb{H}^{a}_{b\gamma}\tilde{Z}^{\alpha}|_{a},\\ \tilde{Z}^{\alpha}|_{c}|_{b}-\tilde{Z}^{\alpha}|_{b}|_{c} & =(\rho,\eta,h)\,\mathbb{S}^{\alpha}_{\theta\ cb}\tilde{Z}^{\theta}+(\rho,\eta,h)\,\mathbb{S}^{a}{}_{bc}\tilde{Z}^{\alpha}|_{a},\end{array}\right.$$

and

$$(\mathcal{R}_2)\quad \left\{\begin{array}{ll}Y^{a}{}_{|\gamma|\beta}-Y^{a}{}_{|\beta|\gamma} & =(\rho,\eta,h)\,\mathbb{R}^{a}_{c\ \gamma\beta}Y^{c}+\left(L^{\theta}_{\beta\gamma}\circ h\circ\pi\right)Y^{c}{}_{|\theta}\\ & \quad +(\rho,\eta)\,\mathbb{T}^{b}{}_{\beta\gamma}Y^{a}|_{b}+(\rho,\eta,h)\,\mathbb{T}^{\theta}{}_{\beta\gamma}Y^{a}{}_{|\theta},\\ Y^{a}{}_{|\gamma}|_{b}-Y^{a}|_{b|\gamma} & =(\rho,\eta,h)\,\mathbb{P}^{a}_{\theta\ \gamma b}Y^{\theta}-(\rho,\eta,h)\,\mathbb{P}^{c}{}_{\gamma b}Y^{a}|_{c}\\ & \quad -(\rho,\eta)\,\mathbb{H}^{c}_{b\gamma}Y^{a}|_{c},\\ Y^{a}|_{c}|_{b}-Y^{a}|_{b}|_{c} & =(\rho,\eta,h)\,\mathbb{S}^{a}_{d\ cb}Y^{d}+(\rho,\eta,h)\,\mathbb{S}^{d}{}_{bc}Y^{a}|_{d}.\end{array}\right.$$

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_{M},id_{M})$ *and the Lie bracket* $[,]_{TM}$ *is the usual Lie bracket, then the formulas of Ricci type* $(\mathcal{R}_1)$ *and* $(\mathcal{R}_2)$ *become:*

$$(\mathcal{R}_1)'\quad \left\{\begin{array}{ll}\tilde{Z}^{i}{}_{|k|j}-\tilde{Z}^{i}{}_{|j|k} & =\mathbb{R}^{i}_{h\ kj}\tilde{Z}^{h}+\mathbb{T}^{a}{}_{jk}\tilde{Z}^{i}|_{a}+\mathbb{T}^{h}{}_{jk}\tilde{Z}^{i}{}_{|h},\\ \tilde{Z}^{i}{}_{|k}|_{b}-\tilde{Z}^{i}|_{b|k} & =\mathbb{P}^{i}_{h\ kb}\tilde{Z}^{h}-\mathbb{P}^{a}{}_{kb}\tilde{Z}^{i}|_{a}-\mathbb{H}^{a}_{bk}\tilde{Z}^{i}|_{a},\\ \tilde{Z}^{i}|_{c}|_{b}-\tilde{Z}^{i}|_{b}|_{c} & =\mathbb{S}^{i}_{h\ cb}\tilde{Z}^{h}+\mathbb{S}^{a}{}_{bc}\tilde{Z}^{i}|_{a},\end{array}\right.$$

and

$$(\mathcal{R}_2)'\quad \left\{\begin{array}{ll}Y^{a}{}_{|k|j}-Y^{a}{}_{|j|k} & =\mathbb{R}^{a}_{c\ kj}Y^{c}+\mathbb{T}^{b}{}_{jk}Y^{a}|_{b}+\mathbb{T}^{h}{}_{jk}Y^{a}{}_{|h},\\ Y^{a}{}_{|k}|_{b}-Y^{a}|_{b|k} & =\mathbb{P}^{a}_{h\ kb}Y^{h}-\mathbb{P}^{c}{}_{kb}Y^{a}|_{c}-\mathbb{H}^{c}_{bk}Y^{a}|_{c},\\ Y^{a}|_{c}|_{b}-Y^{a}|_{b}|_{c} & =\mathbb{S}^{a}_{d\ cb}Y^{d}+\mathbb{S}^{d}{}_{bc}Y^{a}|_{d}.\end{array}\right.$$

Using the 1-forms associated to distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$

$$\begin{aligned}(\rho,\eta)\,\omega^{\alpha}_{\beta}&=(\rho,\eta)\,H^{\alpha}_{\beta\gamma}d\tilde{z}^{\gamma}+(\rho,\eta)\,V^{\alpha}_{\beta c}\delta\tilde{y}^{c},\\(\rho,\eta)\,\omega^{a}_{b}&=(\rho,\eta)\,H^{a}_{b\gamma}d\tilde{z}^{\gamma}+(\rho,\eta)\,V^{a}_{bc}\delta\tilde{y}^{c},\end{aligned}\tag{5.7.1}$$

the torsion 2-forms

$$\left\{\begin{aligned}(\rho,\eta,h)\,\mathbb{T}^{\alpha}&=\frac{1}{2}(\rho,\eta,h)\,\mathbb{T}^{\alpha}{}_{\beta\gamma}d\tilde{z}^{\beta}\wedge d\tilde{z}^{\gamma}+(\rho,\eta,h)\,\mathbb{P}^{\alpha}{}_{\beta c}d\tilde{z}^{\beta}\wedge\delta\tilde{y}^{c},\\(\rho,\eta,h)\,\mathbb{T}^{a}&=\frac{1}{2}(\rho,\eta,h)\,\mathbb{T}^{a}{}_{\beta\gamma}d\tilde{z}^{\beta}\wedge d\tilde{z}^{\gamma}+(\rho,\eta,h)\,\mathbb{P}^{a}{}_{\beta c}d\tilde{z}^{\beta}\wedge\delta\tilde{y}^{c}\\&\quad+\frac{1}{2}(\rho,\eta,h)\,\mathbb{S}^{a}{}_{bc}\delta\tilde{y}^{b}\wedge\delta\tilde{y}^{c}\end{aligned}\right.\tag{5.7.2}$$

and the curvature 2-forms

$$\left\{\begin{aligned}(\rho,\eta,h)\,\mathbb{R}^{\alpha}{}_{\beta}&=\frac{1}{2}(\rho,\eta,h)\,\mathbb{R}^{\alpha}{}_{\beta\ \gamma\theta}d\tilde{z}^{\gamma}\wedge d\tilde{z}^{\theta}+(\rho,\eta,h)\,\mathbb{P}^{\alpha}{}_{\beta\ \gamma c}d\tilde{z}^{\gamma}\wedge\delta\tilde{y}^{c}\\&\quad+\frac{1}{2}(\rho,\eta,h)\,\mathbb{S}^{\alpha}{}_{\beta\ cd}\delta\tilde{y}^{c}\wedge\delta\tilde{y}^{d},\\(\rho,\eta,h)\,\mathbb{R}^{a}{}_{b}&=\frac{1}{2}(\rho,\eta,h)\,\mathbb{R}^{a}{}_{b\ \gamma\theta}d\tilde{z}^{\gamma}\wedge d\tilde{z}^{\theta}+(\rho,\eta,h)\,\mathbb{P}^{a}{}_{b\ \gamma c}d\tilde{z}^{\gamma}\wedge\delta\tilde{y}^{c}\\&\quad+\frac{1}{2}(\rho,\eta,h)\,\mathbb{S}^{a}{}_{b\ cd}\delta\tilde{y}^{c}\wedge\delta\tilde{y}^{d},\end{aligned}\right.\tag{5.7.3}$$

we obtain the following

Theorem 5.7.3 *We obtain the following identities of Cartan type:*

$$\begin{aligned}(\rho,\eta,h)\,\mathbb{T}^{\alpha}&=d^{(\rho,\eta)TE}\left(d\tilde{z}^{\alpha}\right)+(\rho,\eta)\,\omega^{\alpha}_{\beta}\wedge d\tilde{z}^{\beta},\\(\rho,\eta,h)\,\mathbb{T}^{a}&=d^{(\rho,\eta)TE}\left(\delta\tilde{y}^{a}\right)+(\rho,\eta)\,\omega^{a}_{b}\wedge\delta\tilde{y}^{b},\end{aligned}\tag{$\mathcal{C}_1$}$$

$$\begin{aligned}(\rho,\eta,h)\,\mathbb{R}^{\alpha}{}_{\beta}&=d^{(\rho,\eta)TE}\left((\rho,\eta)\,\omega^{\alpha}_{\beta}\right)+(\rho,\eta)\,\omega^{\alpha}_{\gamma}\wedge(\rho,\eta)\,\omega^{\gamma}_{\beta},\\(\rho,\eta,h)\,\mathbb{R}^{a}{}_{b}&=d^{(\rho,\eta)TE}\left((\rho,\eta)\,\omega^{a}_{b}\right)+(\rho,\eta)\,\omega^{a}_{c}\wedge(\rho,\eta)\,\omega^{c}_{b}.\end{aligned}\tag{$\mathcal{C}_2$}$$

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$ and the Lie bracket $[,]_{TM}$ is the usual Lie bracket, then the identities of Cartan type $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ become:

$$\begin{aligned}\mathbb{T}^{i}&=d^{(Id_{TE},Id_E)TE}\left(d\tilde{z}^{i}\right)+\omega^{i}_{j}\wedge d\tilde{z}^{j},\\\mathbb{T}^{a}&=d^{(Id_{TE},Id_E)TE}\left(\delta\tilde{y}^{a}\right)+\omega^{a}_{b}\wedge\delta\tilde{y}^{b},\end{aligned}\tag{$\mathcal{C}_1'$}$$

and

$$\begin{aligned}\mathbb{R}^{i}{}_{j}&=d^{(Id_{TE},Id_E)TE}\left(\omega^{i}_{j}\right)+\omega^{i}_{k}\wedge\omega^{k}_{j},\\\mathbb{R}^{a}{}_{b}&=d^{(Id_{TE},Id_E)TE}\left(\omega^{a}_{b}\right)+\omega^{a}_{c}\wedge\omega^{c}_{b}.\end{aligned}\tag{$\mathcal{C}_2'$}$$

Remark 5.7.1 For any $X,Y,Z\in\Gamma\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$, the following identities

$$\begin{aligned}\mathcal{V}\,(\rho,\eta,h)\,\mathbb{R}\,(X,Y)\,\mathcal{H}Z&=0,\\\mathcal{H}\,(\rho,\eta,h)\,\mathbb{R}\,(X,Y)\,\mathcal{V}Z&=0,\end{aligned}\tag{5.7.4}$$

$$
(5.7.5) \qquad \begin{aligned} \mathcal{V}D_X\left(\left(\rho,\eta,h\right)\mathbb{R}\left(Y,Z\right)\mathcal{H}U\right) &= 0,\\ \mathcal{H}D_X\left(\left(\rho,\eta,h\right)\mathbb{R}\left(Y,Z\right)\mathcal{V}U\right) &= 0,\end{aligned}
$$

and

$$
(5.7.6) \qquad \left(\rho,\eta,h\right)\mathbb{R}\left(X,Y\right)Z=\mathcal{H}\left(\rho,\eta,h\right)\mathbb{R}\left(X,Y\right)\mathcal{H}Z+\mathcal{V}\left(\rho,\eta,h\right)\mathbb{R}\left(X,Y\right)\mathcal{V}Z.
$$

hold. Using the formulas of Bianchi type from Theorem 4.2.3 and Remark 5.7.1, we obtain the following

Theorem 5.7.4 *The identities of Bianchi type:*

$$
(\mathcal{B}_1) \qquad \left\{\begin{aligned} &\sum_{cyclic(X,Y,Z)}\{\mathcal{H}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\mathbb{T}\left(Y,Z\right)\right)-\mathcal{H}\left(\rho,\eta,h\right)\mathbb{R}\left(X,Y\right)Z\\ &\qquad +\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)\\ &\qquad +\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)\}=0,\\ &\sum_{cyclic(X,Y,Z)}\{\mathcal{V}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\mathbb{T}\left(Y,Z\right)\right)-\mathcal{V}\left(\rho,\eta,h\right)\mathbb{R}\left(X,Y\right)Z\\ &\qquad +\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)\\ &\qquad +\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)\}=0.\end{aligned}\right.
$$

and

$$
(\mathcal{B}_2) \qquad \left\{\begin{aligned} &\sum_{cyclic(X,Y,Z,U)}\{\mathcal{H}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\mathbb{R}\left(Y,Z\right)U\right)\\ &\qquad -\mathcal{H}\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)U\\ &\qquad -\mathcal{H}\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)U\}=0,\\ &\sum_{cyclic(X,Y,Z,U)}\{\mathcal{V}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\mathbb{R}\left(Y,Z\right)U\right)\\ &\qquad -\mathcal{V}\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{H}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)U\\ &\qquad -\mathcal{V}\left(\rho,\eta,h\right)\mathbb{R}\left(\mathcal{V}\left(\rho,\eta,h\right)\mathbb{T}\left(X,Y\right),Z\right)U\}=0,\end{aligned}\right.
$$

hold good for any $X,Y,Z\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right).$

Corollary 5.7.1 *Using the following sections* $(\delta_\theta,\delta_\gamma,\delta_\beta)$, *the identities* $(\mathcal{B}_1)$ *become:*

$$
(\mathcal{B}_1)' \qquad \left\{\begin{aligned} &\sum_{cyclic(\beta,\gamma,\theta)}\Big\{\left(\rho,\eta,h\right)\mathbb{T}^\alpha{}_{\beta\gamma_{|\theta}}-\left(\rho,\eta,h\right)\mathbb{R}^\alpha{}_{\beta\ \gamma\theta}\\ &\qquad +\left(\rho,\eta,h\right)\mathbb{T}^\lambda{}_{\gamma\theta}\left(\rho,\eta,h\right)\mathbb{T}^\alpha{}_{\beta\gamma}+\left(\rho,\eta,h\right)\mathbb{T}^a{}_{\gamma\theta}\left(\rho,\eta,h\right)\mathbb{T}^\alpha{}_{\beta a}\Big\}=0,\\ &\sum_{cyclic(\beta,\gamma,\theta)}\Big\{\left(\rho,\eta,h\right)\mathbb{T}^a{}_{\beta\gamma_{|\theta}}+\left(\rho,\eta,h\right)\mathbb{T}^\alpha{}_{\gamma\theta}\left(\rho,\eta,h\right)\mathbb{P}^a{}_{\beta\alpha}\\ &\qquad +\left(\rho,\eta,h\right)\mathbb{P}^b{}_{\gamma\theta}\left(\rho,\eta,h\right)\mathbb{P}^a{}_{b\beta}\Big\}=0,\end{aligned}\right.
$$

and using the following sections $(\delta_\lambda, \delta_\theta, \delta_\gamma, \delta_\beta)$, *the identities* $(\mathcal{B}_2)$ *become:*

$$(\mathcal{B}_2)' \qquad \left\{ \begin{array}{l} \sum\limits_{cyclic(\beta,\gamma,\theta,\lambda)} \Big\{ (\rho,\eta,h)\,\mathbb{R}^{\alpha}{}_{\beta\ \gamma\theta_{|\lambda}} - (\rho,\eta,h)\,\mathbb{T}^{\mu}{}_{\theta\lambda}\,(\rho,\eta,h)\,\mathbb{R}^{\alpha}{}_{\beta\ \gamma\mu} \\ \qquad\qquad - (\rho,\eta,h)\,\mathbb{T}^{a}{}_{\theta\lambda}\,(\rho,\eta,h)\,\mathbb{P}^{\alpha}{}_{\beta\ \gamma a} \Big\} = 0, \\ \sum\limits_{cyclic(\beta,\gamma,\theta,\lambda)} \Big\{ (\rho,\eta,h)\,\mathbb{R}^{a}{}_{b\ \gamma\theta_{|\lambda}} - (\rho,\eta,h)\,\mathbb{T}^{\mu}{}_{\theta\lambda}\,(\rho,\eta,h)\,\mathbb{R}^{a}{}_{\beta\ \gamma\mu} \\ \qquad\qquad - (\rho,\eta,h)\,\mathbb{T}^{a}{}_{\theta\lambda}\,(\rho,\eta,h)\,\mathbb{P}^{\alpha}{}_{\beta\ \gamma a} \Big\} = 0. \end{array} \right.$$

Using another base of sections, we shall obtain new identities of Bianchi type necessary in the applications.

5.8 The (ρ, η)-(pseudo)metrizability

We consider the following diagram:

$$\begin{array}{ccc} E & & \left(F, [,]_{F,h}, (\rho,\eta)\right) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\ \ h\ \ } & N \end{array}$$

where $(E, \pi, M) \in |\mathbf{B^v}|$ and $\left((F, \nu, M), [,]_{F,h}, (\rho, \eta)\right) \in |\mathbf{GLA}|$. Let $(\rho, \eta)\,\Gamma$ be a (ρ, η)-connection for the vector bundle (E, π, M) and let $((\rho, \eta)\,H, (\rho, \eta)\,V)$ be a distinguished linear (ρ, η)-connection for the generalized tangent bundle $((\rho, \eta)\,TE, (\rho, \eta)\,\tau_E, E)$.

Definition 5.8.1 A tensor d-field

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab} \delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b} \in \mathcal{DT}^{00}_{22}\left((\rho, \eta)\,TE, (\rho, \eta)\,\tau_E, E\right)$$

will be called *pseudometrical structure* if its components are symmetric and the matrices $\|g_{\alpha\beta}(u_x)\|$and $\|g_{ab}(u_x)\|$ are nondegenerate, for any point $u_x \in E$.

Moreover, if the matrices $\|g_{\alpha\beta}(u_x)\|$ and $\|g_{ab}(u_x)\|$ has constant signature, then the tensor d-field G will be called *metrical structure.*

Let

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab} \delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b}$$

be a (pseudo)metrical structure. If $\alpha, \beta \in \overline{1,p}$ and $a, b \in \overline{1,r}$, then for any vector local $(m+r)$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, we consider the real functions

$$\pi^{-1}(U) \ \xrightarrow{\ \tilde{g}^{\beta\alpha}\ } \ \mathbb{R}$$

and

$$\pi^{-1}(U) \ \xrightarrow{\ \tilde{g}^{ba}\ } \ \mathbb{R}$$

such that

$$\left\|\tilde{g}^{\beta\alpha}(u_x)\right\| = \left\|g_{\alpha\beta}(u_x)\right\|^{-1}$$

and

$$\left\|\tilde{g}^{ba}\left(u_x\right)\right\| = \left\|g_{ab}\left(u_x\right)\right\|^{-1},$$

for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 5.8.2 We will say that the *(pseudo)metrical structure*

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab}\delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b}$$

is Riemannian (pseudo)metrical structure if around each point $x \in M$ it exists a local vector $m+r$-chart (U, s_U) and a local m-chart (U, ξ_U) such that $g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,y)$ and $g_{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,y)$ depends only on x, for any $u_x \in \pi^{-1}(U)$.

If only the condition is verified:

"$g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,y)$ depends only on x, for any $u_x \in \pi^{-1}(U)$" respectively "$g_{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,y)$ depends only on x, for any $u_x \in \pi^{-1}(U)$", then we will say that the *(pseudo)metrical structure G is a Riemannian $\mathcal{H}$-(pseudo)metrical structure* respectively a *Riemannian $\mathcal{V}$-(pseudo)metrical structure.*

Definition 5.8.3 We will say that the *(pseudo)metrical structure*

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab}\delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b}$$

is a locally Minkowski structure if around each point $x \in M$ there exists a local vector $m+r$-chart (U, s_U) and a local m-chart (U, ξ_U) such that $g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,y)$ and $g_{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,y)$ depends only on y, *for any $u_x \in \pi^{-1}(U)$.*

If only the condition is verified:

"$g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,y)$ depends only on y, for any $u_x \in \pi^{-1}(U)$" respectively "$g_{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,y)$ depends only on y, for any $u_x \in \pi^{-1}(U)$", then we will say that *the (pseudo)metrical structure G is a (pseudo)metrical structure $\mathcal{H}$-locally Minkowski* and *$\mathcal{V}$-locally Minkowski, respectively.*

Definition 5.8.4 The generalized tangent bundle $((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$ will be called (ρ,η)-*(pseudo)metrizable* if there exists a (pseudo)metrical structure

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab}\delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b}$$

and a distinguished linear (ρ,η)-connection

$$\left((\rho,\eta)H, (\rho,\eta)V\right)$$

such that

$$(\rho,\eta) D_{\tilde{X}} G = 0,\ \forall \tilde{X} \in \Gamma\left((\rho,\eta)TE, (\rho,\eta)\tau_E, E\right). \tag{5.8.1}$$

Condition (5.8.1) is equivalent with the following equalities:

$$g_{\alpha\beta|\gamma} = 0,\ g_{ab|\gamma} = 0,\ g_{\alpha\beta}\,|_c = 0\,,\ g_{ab}\,|_c = 0. \tag{5.8.2}$$

If $g_{\alpha\beta|\gamma}=0$ and $g_{ab|\gamma}=0$, then we will say that *the vector bundle $((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$ is $\mathcal{H}$-(ρ,η)-(pseudo)metrizable.*

If $g_{\alpha\beta}|_c=0$ and $g_{ab}|_c=0$, then we will say that *the vector bundle $((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$ is $\mathcal{V}$-(ρ,η)-(pseudo)metrizable.*

Theorem 5.8.1 *If* $\left((\rho,\eta)\overset{0}{H},(\rho,\eta)\overset{0}{V}\right)$ *is a distinguished linear* (ρ,η)*-connection for the generalized tangent bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *and* $G=g_{\alpha\beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+g_{ab}\delta\tilde{y}^{a}\otimes\delta\tilde{y}^{b}$ *is a (pseudo)metrical structure, then the following real local functions:*

$$\begin{array}{ll} (\rho,\eta)\,H^{\alpha}_{\beta\gamma} & =\frac{1}{2}\tilde{g}^{\alpha\varepsilon}\left(\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\gamma}\right)g_{\varepsilon\beta}\right.\\ & +\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\beta}\right)g_{\varepsilon\gamma}-\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\varepsilon}\right)g_{\beta\gamma}\\ & \left.+g_{\theta\varepsilon}L^{\theta}_{\gamma\beta}\circ h\circ\pi-g_{\beta\theta}L^{\theta}_{\gamma\varepsilon}\circ h\circ\pi-g_{\theta\gamma}L^{\theta}_{\beta\varepsilon}\circ h\circ\pi\right),\\ (\rho,\eta)\,H^{a}_{b\gamma} & =(\rho,\eta)\,\overset{0}{H}{}^{a}_{b\gamma}+\frac{1}{2}\tilde{g}^{ac}g_{bc\overset{0}{|}\gamma},\\ (\rho,\eta)\,V^{\alpha}_{\beta c} & =(\rho,\eta)\,\overset{0}{V}{}^{\alpha}_{\beta c}+\frac{1}{2}\tilde{g}^{\alpha\varepsilon}g_{\beta\varepsilon\overset{0}{|}c},\\ (\rho,\eta)\,V^{a}_{bc} & =\frac{1}{2}\tilde{g}^{ae}\left(\Gamma(\tilde{\rho},Id_E)\left(\dot{\tilde{\partial}}_{c}\right)g_{eb}\right.\\ & \left.+\Gamma(\tilde{\rho},Id_E)\left(\dot{\tilde{\partial}}_{b}\right)g_{ec}-\Gamma(\tilde{\rho},Id_E)\left(\dot{\tilde{\partial}}_{e}\right)g_{bc}\right)\end{array}\tag{5.8.3}$$

are components of a distinguished linear (ρ,η)*-connection such that the generalized tangent bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *becomes* (ρ,η)*-(pseudo)metrizable.*

Corollary 5.8.1 *If the distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\overset{0}{H},(\rho,\eta)\overset{0}{V}\right)$ *coincides with the Berwald linear* (ρ,η)*-connection, then the local real functions:*

$$\begin{array}{ll} (\rho,\eta)\,\overset{c}{H}{}^{\alpha}_{\beta\gamma} & =\frac{1}{2}\tilde{g}^{\alpha\varepsilon}\left(\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\gamma}\right)g_{\varepsilon\beta}+\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\beta}\right)g_{\varepsilon\gamma}\right.\\ & -\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_{\varepsilon}\right)g_{\beta\gamma}+g_{\theta\varepsilon}L^{\theta}_{\gamma\beta}\circ h\circ\pi\\ & \left.-g_{\beta\theta}L^{\theta}_{\gamma\varepsilon}\circ h\circ\pi-g_{\theta\gamma}L^{\theta}_{\beta\varepsilon}\circ h\circ\pi\right),\\ (\rho,\eta)\,\overset{c}{H}{}^{a}_{b\gamma} & =\dfrac{\partial(\rho,\eta)\Gamma^{a}_{\gamma}}{\partial y^{b}}+\frac{1}{2}\tilde{g}^{ac}g_{bc\overset{0}{|}\gamma},\\ (\rho,\eta)\,\overset{c}{V}{}^{\alpha}_{\beta c} & =\frac{1}{2}\tilde{g}^{\alpha\varepsilon}\dfrac{\partial g_{\beta\varepsilon}}{\partial y^{c}},\\ (\rho,\eta)\,\overset{c}{V}{}^{a}_{bc} & =\frac{1}{2}\tilde{g}^{ae}\left(\dfrac{\partial g_{e\beta}}{\partial y^{c}}+\dfrac{\partial g_{ec}}{\partial y^{b}}-\dfrac{\partial g_{bc}}{\partial y^{e}}\right)\end{array}\tag{5.8.4}$$

are the components of a distinguished linear (ρ,η)*-connection such that the generalized tangent bundle* $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ *becomes* (ρ,η)*-(pseudo)metrizable.*

Moreover, if the (pseudo)metrical structure G *is* $\mathcal{H}$*- and* $\mathcal{V}$*-Riemannian, then the local real functions:*

$$
\begin{aligned}
&(\rho,\eta)\overset{c}{H}{}^{\alpha}_{\beta\gamma}=\frac{1}{2}\tilde{g}^{\alpha\varepsilon}\left(\rho^{k}_{\gamma}\circ h\circ\pi\frac{\partial g_{\varepsilon\beta}}{\partial x^{k}}+\rho^{j}_{\beta}\circ h\circ\pi\frac{\partial g_{\varepsilon\gamma}}{\partial x^{j}}-\rho^{e}_{\varepsilon}\circ h\circ\pi\frac{\partial g_{\beta\gamma}}{\partial x^{e}}+\right.\\
&\qquad\left.+g_{\theta\varepsilon}L^{\theta}_{\gamma\beta}\circ h\circ\pi-g_{\beta\theta}L^{\theta}_{\gamma\varepsilon}\circ h\circ\pi-g_{\theta\gamma}L^{\theta}_{\beta\varepsilon}\circ h\circ\pi\right),\\
&(\rho,\eta)\overset{c}{H}{}^{a}_{b\gamma}=\frac{\partial\left(\rho,\eta\right)\Gamma^{a}_{\gamma}}{\partial y^{b}}+\frac{1}{2}\tilde{g}^{ac}\left(\rho^{i}_{\gamma}\circ h\circ\pi\frac{\partial g_{bc}}{\partial x^{i}}-\frac{\partial\rho\Gamma^{e}_{\gamma}}{\partial y^{b}}g_{ec}-\frac{\partial\rho\Gamma^{e}_{\gamma}}{\partial y^{c}}g_{eb}\right),\\
&(\rho,\eta)\overset{c}{V}{}^{\alpha}_{\beta c}=0,\ \ (\rho,\eta)\overset{c}{V}{}^{a}_{bc}=0
\end{aligned}
\tag{5.8.5}
$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ becomes (ρ,η)-(pseudo)metrizable.

Theorem 5.8.2 *Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M). Let*

$$\left((\rho,\eta)\,\overset{0}{H},(\rho,\eta)\,\overset{0}{V}\right)$$

be a distinguished linear (ρ,η)-connection for $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ and let

$$G=g_{\alpha\beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+g_{ab}\delta\tilde{y}^{a}\otimes\delta\tilde{y}^{b}$$

be a (pseudo)metrical structure.

Let

$$
\begin{aligned}
O^{\alpha\varepsilon}_{\beta\gamma}&=\frac{1}{2}\left(\delta^{\alpha}_{\beta}\delta^{\varepsilon}_{\gamma}-g_{\beta\gamma}\tilde{g}^{\alpha\varepsilon}\right),\quad O^{*\alpha\varepsilon}_{\beta\gamma}=\frac{1}{2}\left(\delta^{\alpha}_{\beta}\delta^{\varepsilon}_{\gamma}+g_{\beta\gamma}\tilde{g}^{\alpha\varepsilon}\right),\\
O^{ae}_{bc}&=\frac{1}{2}\left(\delta^{a}_{b}\delta^{e}_{c}-g_{bc}\tilde{g}^{ae}\right),\qquad O^{*ae}_{bc}=\frac{1}{2}\left(\delta^{a}_{b}\delta^{e}_{c}+g_{bc}\tilde{g}^{ae}\right),
\end{aligned}
\tag{5.8.6}
$$

be the Obata operators.

If the real local functions $X^{\alpha}_{\beta\gamma},X^{\alpha}_{\beta c},Y^{a}_{b\gamma},Y^{a}_{bc}$ are components of tensor fields, then the local real functions are given in the following:

$$
\begin{aligned}
(\rho,\eta)\,H^{\alpha}_{\beta\gamma}&=(\rho,\eta)\,\overset{c}{H}{}^{\alpha}_{\beta\gamma}+O^{\alpha\varepsilon}_{\gamma\eta}X^{\eta}_{\varepsilon\beta},\\
(\rho,\eta)\,H^{a}_{b\gamma}&=(\rho,\eta)\,\overset{c}{H}{}^{a}_{b\gamma}+O^{ae}_{bd}Y^{d}_{e\gamma},\\
(\rho,\eta)\,V^{\alpha}_{\beta c}&=(\rho,\eta)\,\overset{c}{V}{}^{\alpha}_{\beta c}+O^{*\alpha\varepsilon}_{\beta\eta}X^{\eta}_{\varepsilon c},\\
(\rho,\eta)\,V^{a}_{bc}&=(\rho,\eta)\,\overset{c}{V}{}^{a}_{bc}+O^{*ae}_{bd}Y^{d}_{ec},
\end{aligned}
\tag{5.8.7}
$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ becomes (ρ,η)-(pseudo)metrizable.

Theorem 5.8.3 *Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M).*

If

$$\left((\rho,\eta)\,\overset{0}{H},(\rho,\eta)\,\overset{0}{V}\right)$$

is a distinguished linear (ρ,η)-connection for the generalized tangent bundle

$$((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$$

and

$$G=g_{\alpha\beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+g_{ab}\delta\tilde{y}^{a}\otimes\delta\tilde{y}^{b}$$

is a (pseudo)metrical structure, then the real local functions:

$$(5.8.8)\qquad \begin{array}{l} (\rho,\eta)\, H^{\alpha}_{\beta\gamma} = (\rho,\eta)\, \overset{0}{H}{}^{\alpha}_{\beta\gamma} + \dfrac{1}{2}\tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta\overset{0}{|}\gamma}, \\ (\rho,\eta)\, H^{a}_{b\gamma} = (\rho,\eta)\, \overset{0}{H}{}^{a}_{b\gamma} + \dfrac{1}{2}\tilde{g}^{ae} g_{eb\overset{0}{|}\gamma}, \\ (\rho,\eta)\, V^{\alpha}_{\beta c} = (\rho,\eta)\, \overset{0}{V}{}^{\alpha}_{\beta c} + \dfrac{1}{2}\tilde{g}^{\alpha\varepsilon} g_{\varepsilon\beta} \overset{0}{|}_{c}, \\ (\rho,\eta)\, V^{a}_{bc} = (\rho,\eta)\, \overset{0}{V}{}^{a}_{bc} + \dfrac{1}{2}\tilde{g}^{ae} g_{eb} \overset{0}{|}_{c} \end{array}$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle $((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E)$ becomes (ρ,η)-(pseudo)metrizable.

5.9 Generalized Lagrange (ρ,η)-spaces, Lagrange (ρ,η)-spaces and Finsler (ρ,η)-spaces

We consider the following diagram:

$$\begin{array}{ccc} E & & (F,[,]_{F,h},(\rho,\eta)) \\ \pi\downarrow & & \downarrow\nu \\ M & \xrightarrow{\ h\ } & N \end{array}$$

such that $(E,\pi,M) = (F,\nu,N)$ and the generalized tangent bundle

$$((\rho,\eta)\, TE, (\rho,\eta)\, \tau_E, E)$$

is (ρ,η)-(pseudo)metrizable.

Definition 5.9.1 A smooth *Lagrange fundamental function* on the vector bundle (E,π,M) is a mapping

$$E \ \xrightarrow{\ L\ }\ \mathbb{R}$$

which satisfies the following conditions:

1. $L\circ u \in C^{\infty}(M)$, for any $u \in \Gamma(E,\pi,M)\setminus\{0\}$;
2. $L\circ 0 \in C^{0}(M)$, where 0 means the null section of (E,π,M).

Let L be a Lagrangian defined on the total space of the vector bundle (E,π,M).

If (U,s_U) is a local vector $(m+r)$-chart for (E,π,M), then we obtain the following real functions defined on $\pi^{-1}(U)$:

$$(5.9.1)\qquad \begin{array}{ll} L_i \overset{put}{=} \dfrac{\partial L}{\partial x^i} \overset{put}{=} \dfrac{\partial}{\partial x^i}(L) & L_{ib} \overset{put}{=} \dfrac{\partial^2 L}{\partial x^i \partial y^b} \overset{put}{=} \dfrac{\partial}{\partial x^i}\left(\dfrac{\partial}{\partial y^b}(L)\right) \\ L_a \overset{put}{=} \dfrac{\partial L}{\partial y^a} \overset{put}{=} \dfrac{\partial}{\partial y^a}(L) & L_{ab} \overset{put}{=} \dfrac{\partial^2 L}{\partial y^a \partial y^b} \overset{put}{=} \dfrac{\partial}{\partial y^a}\left(\dfrac{\partial}{\partial y^b}(L)\right). \end{array}$$

Definition 5.9.2 If for any local vector $m+r$-chart (U,s_U) of (E,π,M), we have:

$$(5.9.2)\qquad rank\, \|L_{ab}(u_x)\| = r,$$

for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$, then we will say that *the Lagrangian L is regular.*

Proposition 5.9.1 *If the Lagrangian L is regular, then for any local vector $m+r$-chart (U, s_U) of (E, π, M), we obtain the real functions $\tilde{L}^{ab}$ locally defined by*

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\tilde{L}^{ab}} & \mathbb{R} \\
u_x & \longmapsto & \tilde{L}^{ab}(u_x)
\end{array}, \tag{5.9.3}
$$

where $\left\| \tilde{L}^{ab}(u_x) \right\| = \| L_{ab}(u_x) \|^{-1}$, *for any* $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 5.9.3 A smooth *Finsler fundamental function* on the vector bundle (E, π, M) is a mapping

$$
E \xrightarrow{F} \mathbb{R}_+
$$

which satisfies the following conditions:

1. $F \circ u \in C^\infty(M)$, for any $u \in \Gamma(E, \pi, M) \setminus \{0\}$;
2. $F \circ 0 \in C^0(M)$, where 0 means the null section of (E, π, M);
3. F is positively 1-homogenous on the fibres of vector bundle (E, π, M);
4. For any local vector $m+r$-chart (U, s_U) of (E, π, M), the hessian:

$$
\left\| F^2_{ab}(u_x) \right\| \tag{5.9.4}
$$

is positively define for any $u_x \in \pi^{-1}(U) \setminus \{0_x\}$.

Definition 5.9.4 If the (pseudo)metrical structure G is determined by a (pseudo)metrical structure

$$
g \in \mathcal{T}^{\,0}_{\,2}(V(\rho, \eta) TE, (\rho, \eta), \tau_E, E),
$$

then the (ρ, η)-(pseudo)metrizable vector bundle

$$
((\rho, \eta) TE, (\rho, \eta) \tau_E, E)
$$

will be called the *generalized Lagrange (ρ, η)-space.*

In particular, if the (pseudo)metrical structure g is determined with the help of a Lagrange fundamental function or Finsler fundamental function, then the (ρ, η)-(pseudo)metrizable vector bundle

$$
((\rho, \eta) TE, (\rho, \eta) \tau_E, E)
$$

will be called the *Lagrange (ρ, η)-space* or the *Finsler (ρ, η)-space, respectively.*

The generalized Lagrange (Id_{TM}, Id_M)-space, the Lagrange (Id_{TM}, Id_M)-space, and the Finsler (Id_{TM}, Id_M)-space will be called the *generalized Lagrange space, Lagrange space*, *Finsler space*.

Definition 5.9.5 The normal distinguished linear (ρ, η)-connections of a Lagrange or Finsler (ρ, η)-space will be called *Lagrange* or *Finsler linear (ρ, η)-connections.*

The Lagrange and Finsler linear (Id_{TM}, Id_M)-connections will be called *Lagrange* and *Finsler linear connections*, respectively.

Theorem 5.9.1 *If the (pseudo)metrical structure G is determined by a (pseudo)metrical structure*

$$g\in\mathcal{T}\,{}^{0}_{2}\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right),\tau_{E},E\right),$$

then, the real local functions:

$$\begin{aligned}\left(\rho,\eta\right)H_{bc}^{a}&=\frac{1}{2}\tilde{g}^{ae}\left(\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\delta_{b}\right)g_{ec}+\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\delta_{c}\right)g_{be}-\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\delta_{e}\right)g_{bc}\right.\\&\left.-g_{cd}L_{be}^{d}\circ\left(h\circ\pi\right)+g_{bd}L_{ec}^{d}\circ\left(h\circ\pi\right)-g_{ed}L_{bc}^{d}\circ\left(h\circ\pi\right)\right),\\\left(\rho,\eta\right)V_{bc}^{a}&=\frac{1}{2}\tilde{g}^{ae}\left(\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\dot{\tilde{\partial}}_{c}\right)g_{eb}\right.\\&\left.+\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\dot{\tilde{\partial}}_{b}\right)g_{ec}-\Gamma\left(\tilde{\rho},Id_{E}\right)\left(\dot{\tilde{\partial}}_{e}\right)g_{bc}\right)\end{aligned}\tag{5.9.5}$$

are the components of a normal distinguished linear (ρ,η)-connection with (ρ,η)-$\mathcal{H}\,(\mathcal{HH})$ and (ρ,η)-$\mathcal{V}\,(\mathcal{VV})$ torsions free such that the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ derives generalized Lagrange (ρ,η)-space.

This normal distinguished linear (ρ,η)-connection will be called *generalized linear (ρ,η)-connection of Levi-Civita type.*

If the *(pseudo)metrical structure* g is determined with the help of a Lagrange and Finsler fundamental function, then the Lagrange and Finsler linear (ρ,η)-connections will be called *canonical Lagrange* and *Finsler linear (ρ,η)-connection, respectively.*

The canonical Lagrange and Finsler linear (Id_{TM},Id_M)-connection will be called the *canonical Lagrange* and *Finsler linear connection* respectively.

Theorem 5.9.2 *Let $((\rho,\eta)\,H,(\rho,\eta)\,V)$ be the normal distinguished linear (ρ,η)-connection presented in the previous theorem.*

If

$$\mathbb{T}_{bc}^{a}\tilde{\delta}_{a}\otimes d\tilde{z}^{b}\otimes d\tilde{z}^{c}\in\mathcal{T}_{20}^{10}\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$$

and

$$\mathbb{S}_{bc}^{a}\dot{\tilde{\partial}}_{a}\otimes\delta\tilde{y}^{b}\otimes\delta\tilde{y}^{c}\in\mathcal{T}_{02}^{01}\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$$

such that they satisfy the conditions:

$$\mathbb{T}_{bc}^{a}=-\mathbb{T}_{cb}^{a},\ \mathbb{S}_{bc}^{a}=-\mathbb{S}_{cb}^{a},\ \forall b,c\in\overline{1,n},$$

then the following real local functions:

$$\begin{aligned}\left(\rho,\eta\right)\tilde{H}_{bc}^{a}&=\left(\rho,\eta\right)H_{bc}^{a}+\frac{1}{2}\tilde{g}^{ae}\left(g_{ed}\mathbb{T}_{bc}^{d}-g_{bd}\mathbb{T}_{ec}^{d}+g_{cd}\mathbb{T}_{be}^{d}\right),\\\left(\rho,\eta\right)\tilde{V}_{bc}^{a}&=\left(\rho,\eta\right)V_{bc}^{a}+\frac{1}{2}\tilde{g}^{ae}\left(g_{ed}\mathbb{S}_{bc}^{d}-g_{bd}\mathbb{S}_{ec}^{d}+g_{cd}\mathbb{S}_{be}^{d}\right)\end{aligned}\tag{5.9.6}$$

are the components of a normal distinguished linear (ρ,η)-connection with (ρ,η)-$\mathcal{H}\,(\mathcal{HH})$ and (ρ,η)-$\mathcal{V}\,(\mathcal{VV})$ torsions a priori given such that the generalized tangent bundle $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ derives generalized Lagrange (ρ,η)-space.

Moreover, we obtain:

(5.9.87

$$\begin{aligned}\mathbb{T}_{bc}^{a}&=\left(\rho,\eta\right)H_{bc}^{a}-\left(\rho,\eta\right)H_{cb}^{a}-L_{bc}^{a}\circ h\circ\pi,\\\mathbb{S}_{bc}^{a}&=\left(\rho,\eta\right)V_{bc}^{a}-\left(\rho,\eta\right)V_{cb}^{a}.\end{aligned}$$

5.10 Einstein equations

We shall consider a metric structure

$$G = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g_{ab} \delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b}$$

and a distinguished linear (ρ, η)-connection $((\rho, \eta) H, (\rho, \eta) V)$ compatible with the structure metric G having $\mathcal{H}(\mathcal{HH})$ and $\mathcal{V}(\mathcal{VV})$-torsions prescribed.

Definition 5.10.1 If $(\rho, \eta, h) \mathbb{R}\ _{\alpha\ \beta}$ and $(\rho, \eta, h) \mathbb{S}\ _{a\ b}$ are the components of tensor Ricci associated to distinguished linear (ρ, η)-connection

$$((\rho, \eta) H, (\rho, \eta) V),$$

then the scalar

$$(\rho, \eta, h) \mathbb{R} = (\rho, \eta, h) \mathbb{R}\ _{\alpha\ \beta} \tilde{g}^{\alpha\beta} + (\rho, \eta, h) \mathbb{S}\ _{a\ b} \tilde{g}^{ab} \tag{5.10.1}$$

will be called the *scalar of curvature of distinguished linear (ρ, η)-connection $((\rho, \eta)H, (\rho, \eta)V)$.*

Definition 5.10.2 The tensor field

$$\begin{aligned} (\rho, \eta, h) \mathbb{T} &= (\rho, \eta, h) \mathbb{T}\ _{\alpha\ \beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + (\rho, \eta, h) \mathbb{T}\ _{\alpha\ b} d\tilde{z}^{\alpha} \otimes \delta\tilde{y}^{b} \\ &\quad + (\rho, \eta, h) \mathbb{T}\ _{a\ \beta} \delta\tilde{y}^{a} \otimes d\tilde{z}^{\alpha} + (\rho, \eta, h) \mathbb{T}\ _{a\ b} \delta\tilde{y}^{a} \otimes \delta\tilde{y}^{b} \end{aligned} \tag{5.10.2}$$

such that its components verify the following conditions:

$$\begin{aligned} \varkappa (\rho, \eta, h) \mathbb{T}\ _{\alpha\ \beta} &= (\rho, \eta, h) \mathbb{R}\ _{\alpha\ \beta} - \frac{1}{2} (\rho, \eta, h) \mathbb{R} \cdot g_{\alpha\beta}, \\ -\varkappa (\rho, \eta, h) \mathbb{T}\ _{\alpha\ b} &= (\rho, \eta, h) \mathbb{P}\ _{\alpha\ b}, \\ \varkappa (\rho, \eta, h) \mathbb{T}\ _{a\ \beta} &= (\rho, \eta, h) \mathbb{P}\ _{a\ \beta}, \\ \varkappa (\rho, \eta, h) \mathbb{T}\ _{a\ b} &= (\rho, \eta, h) \mathbb{S}\ _{a\ b} - \frac{1}{2} (\rho, \eta, h) \mathbb{R} \cdot g_{ab}, \end{aligned} \tag{5.10.3}$$

where $\varkappa$ is a constant, will be called *the energy-momentum tensor field associated to distinguished linear (ρ, η)-connection $((\rho, \eta) H, (\rho, \eta) V)$ and metrical structure G.*

The equations (5.10.3) will be called *the Einstein equations associated to distinguished linear (ρ, η)-connection $((\rho, \eta) H, (\rho, \eta) V)$ and metrical structure G.*

Formally, the Einstein equations will be written

$$\mathbf{Ric}\,((\rho, \eta) H, (\rho, \eta) V) - \frac{1}{2} (\rho, \eta, h) \mathbb{R} \cdot G = \varkappa \cdot (\rho, \eta, h) \mathbb{T}. \tag{5.10.3'}$$

5.11 Mechanical systems

Using the diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \left(E, [,]_{E,h}, (\rho, \eta)\right) \arrow[d, "\pi"] \\ M \arrow[r, "h"] & M \end{tikzcd} \tag{5.11.1}$$

where $\left(\left(E,\pi,M\right),\left[,\right]_{E,h},\left(\rho,\eta\right)\right)$ is a generalized Lie algebroid, we build the generalized tangent bundle

$$(((\rho,\eta)TE,(\rho,\eta)\tau_E,E),[,]_{(\rho,\eta)TE},(\tilde{\rho},Id_E)).$$

Definition 5.11.1 A triple

$$\left(\left(E,\pi,M\right),F_e,\left(\rho,\eta\right)\Gamma\right), \tag{5.11.2}$$

where

$$F_e=F^a\frac{\partial}{\partial\tilde{y}^a}\in\Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right) \tag{5.11.3}$$

is an external force and $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection, will be called the *mechanical (ρ,η)-system.*

A mechanical (ρ,η)-system

$$\left(\left(E,\pi,M\right),F_e,\left(\rho,\eta\right)\Gamma\right)$$

endowed with a (pseudo)metrical structure G determined with the help of a (pseudo)metrical structure

$$g=g_{ab}d\tilde{y}^a\otimes d\tilde{y}^b\in\mathcal{T}\,{}^{00}_{02}\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$$

will be denoted

$$\left(\left(E,\pi,M\right),F_e,\left(\rho,\eta\right)\Gamma,G\right). \tag{5.11.4}$$

and will be called *generalized Lagrange mechanical (ρ,η)-system.*

Any mechanical (Id_{TM},Id_M)-system and any generalized Lagrange mechanical (Id_{TM},Id_M)-system will be called *mechanical system* and *generalized Lagrange mechanical system,* respectively.

Definition 5.11.2 If L (respectively F) is a smooth Lagrange (respectively Finsler function), then we put the triples

$$\left(\left(E,\pi,M\right),F_e,L\right)\quad(\text{ respectively }(E,F_e,F))$$

where $F_e=F^a\dfrac{\partial}{\partial\tilde{y}^a}\in\Gamma\left(V\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right)$ is an external force. These are called *Lagrange mechanical (ρ,η)-system* (*Finsler mechanical (ρ,η)-system, respectively*).

Any Lagrange mechanical (Id_{TM},Id_M)-system and any Finsler mechanical (Id_{TM},Id_M)-system will be called *Lagrange mechanical system* and *Finsler mechanical system*, respectively.

5.11.1 (ρ,η)-semisprays and (ρ,η)-sprays for mechanical (ρ,η)-systems

Let $\left(\left(E,\pi,M\right),F_e,(\rho,\eta)\Gamma\right)$ be an arbitrary mechanical (ρ,η)-system.

Definition 5.11.1.1 The vertical section

$$\mathbb{C}{=}y^a\dot{\tilde{\partial}}_a, \tag{5.11.1.1}$$

will be called the *Liouville section.*

Definition 5.11.1.2 The section $S\in\Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right)$ will be called (ρ,η)-*semispray* if there exists an almost tangent structure e such that

$$e\left(S\right)=\mathbb{C}. \tag{5.11.1.2}$$

Let $g\in\mathbf{Man}\left(E,E\right)$ be such that (g,h) is a locally invertible $\mathbf{B^v}$-morphism of (E,π,M) source and (E,π,M) target.

Theorem 5.11.1.1 *The section*

$$S=\left(g_b^a\circ h\circ\pi\right)y^b\frac{\partial}{\partial\tilde{z}^a}-2\left(G^a-\frac{1}{4}F^a\right)\frac{\partial}{\partial\tilde{y}^a} \tag{5.11.1.3}$$

is a (ρ,η)-semispray such that the real local functions $G^a,\ a\in\overline{1,n}$, satisfy the following conditions

$$\begin{array}{cl}(\rho,\eta)\,\Gamma_c^a & =\tilde{g}_c^e\circ h\circ\pi\dfrac{\partial\left(G^a-\frac{1}{4}F^a\right)}{\partial y^e}\\ & -\dfrac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi,\ a,b\in\overline{1,r}.\end{array} \tag{5.11.1.4}$$

In addition, we remark that the local real functions

$$\begin{array}{cl}(\rho,\eta)\,\mathring{\Gamma}_c^a & \overset{put}{=}\tilde{g}_c^e\circ h\circ\pi\frac{\partial G^a}{\partial y^e}\\ & -\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi,\ a,b\in\overline{1,r}\end{array} \tag{5.11.1.5}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\,\mathring{\Gamma}$ for the vector bundle (E,π,M).

The (ρ,η)-semispray S will be called *the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,(\rho,\eta)\,\Gamma)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).*

Proof. We consider the **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right) & \xrightarrow{\mathbb{P}} & \Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right)\\ X & \longmapsto & \mathcal{J}_{(g,h)}\left[S,X\right]_{(\rho,\eta)TE}-\left[S,\mathcal{J}_{(g,h)}X\right]_{(\rho,\eta)TE}.\end{array}$$

Let $X=\tilde{Z}^a\tilde{\partial}_a+Y^a\dot{\tilde{\partial}}_a$ be an arbitrary section. Since

$$\begin{array}{cl}\left[S,X\right]_{(\rho,\eta)TE} & =\left[\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_a,\tilde{Z}^b\tilde{\partial}_b\right]_{(\rho,\eta)TE}+\left[\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_a,Y^b\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE}\\ & -\left[2\left(G^a-\dfrac{1}{4}F^a\right)\dot{\tilde{\partial}}_a,\tilde{Z}^b\tilde{\partial}_b\right]_{(\rho,\eta)TE}-\left[2\left(G^a-\dfrac{1}{4}F^a\right)\dot{\tilde{\partial}}_a,Y^b\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE}\end{array}$$

and

$$\begin{array}{cl}\left[\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_a,\tilde{Z}^b\tilde{\partial}_b\right]_{(\rho,\eta)TE} & =\left(g_e^a\circ h\circ\pi\cdot y^e\right)\rho_a^i\circ h\circ\pi\dfrac{\partial\tilde{Z}^c}{\partial x^i}\tilde{\partial}_c\\ & -\tilde{Z}^b\rho_b^j\circ h\circ\pi\dfrac{\partial\left(g_e^c\circ h\circ\pi\cdot y^e\right)}{\partial x^j}\tilde{\partial}_c\\ & +\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{Z}^bL_{ab}^c\tilde{\partial}_c,\end{array}$$

$$\left[\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_a, Y^b\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE} = \left(g_e^a\circ h\circ\pi\cdot y^e\right)\rho_a^i\circ h\circ\pi\frac{\partial Y^c}{\partial x^i}\dot{\tilde{\partial}}_c - Y^b\frac{\partial\left(g_e^c\circ h\circ\pi\cdot y^e\right)}{\partial y^b}\tilde{\partial}_c,$$

$$\left[2\left(G^a-\frac{1}{4}F^a\right)\dot{\tilde{\partial}}_a, \tilde{Z}^b\tilde{\partial}_b\right]_{(\rho,\eta)TE} = 2\left(G^a-\frac{1}{4}F^a\right)\frac{\partial\tilde{Z}^c}{\partial y^a}\tilde{\partial}_c - 2\tilde{Z}^b\rho_b^j\circ h\circ\pi\frac{\partial\left(G^c-\frac{1}{4}F^c\right)}{\partial x^j}\dot{\tilde{\partial}}_c,$$

$$\left[2\left(G^a-\tfrac{1}{4}F^a\right)\dot{\tilde{\partial}}_a, Y^b\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE} = 2\left(G^a-\tfrac{1}{4}F^a\right)\tfrac{\partial Y^c}{\partial y^a}\dot{\tilde{\partial}}_c - 2Y^b\frac{\partial\left(G^c-\frac{1}{4}F^c\right)}{\partial y^b}\dot{\tilde{\partial}}_c,$$

it results that

$$(P_1)\qquad \begin{aligned}\mathcal{J}_{(g,h)}\left[S,X\right]_{(\rho,\eta)TE} &= \left(g_e^a\circ h\circ\pi\cdot y^e\right)\rho_a^i\circ h\circ\pi\frac{\partial\tilde{Z}^c}{\partial x^i}\cdot\tilde{g}_c^d\circ h\circ\pi\dot{\tilde{\partial}}_d\\ &-\tilde{Z}^b\rho_b^j\circ h\circ\pi\frac{\partial\left(g_e^c\circ h\circ\pi\cdot y^e\right)}{\partial x^j}\cdot\tilde{g}_c^d\circ h\circ\pi\dot{\tilde{\partial}}_d\\ &+\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{Z}^bL_{ab}^c\cdot\tilde{g}_c^d\circ h\circ\pi\dot{\tilde{\partial}}_d\\ &-Y^b\frac{\partial\left(g_e^c\circ h\circ\pi\cdot y^e\right)}{\partial y^b}\cdot\tilde{g}_c^d\circ h\circ\pi\dot{\tilde{\partial}}_d\\ &-2\left(G^a-\frac{1}{4}F^a\right)\frac{\partial\tilde{Z}^c}{\partial y^a}\cdot\tilde{g}_c^d\circ h\circ\pi\dot{\tilde{\partial}}_d.\end{aligned}$$

Since

$$\begin{aligned}\left[S,\mathcal{J}_{(g,h)}X\right]_{(\rho,\eta)TE} &= \left[\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_a, \tilde{g}_b^c\circ h\circ\pi\tilde{Z}^b\dot{\tilde{\partial}}_c\right]_{(\rho,\eta)TE}\\ &-\left[2\left(G^a-\frac{1}{4}F^a\right)\dot{\tilde{\partial}}_a, \tilde{g}_b^c\circ h\circ\pi\tilde{Z}^b\dot{\tilde{\partial}}_c\right]_{(\rho,\eta)TE}\end{aligned}$$

and

$$\left[\left(g_e^a\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_a, \tilde{g}_b^c\circ h\circ\pi\tilde{Z}^b\dot{\tilde{\partial}}_c\right]_{(\rho,\eta)TE} = \left(g_e^a\circ h\circ\pi\cdot y^e\right)\rho_a^i\circ h\circ\pi\frac{\partial\tilde{g}_b^d\circ h\circ\pi\tilde{Z}^b}{\partial x^i}\dot{\tilde{\partial}}_d - \tilde{g}_b^c\circ h\circ\pi\cdot\tilde{Z}^b\frac{\partial\left(g_e^d\circ h\circ\pi\cdot y^e\right)}{\partial y^c}\tilde{\partial}_d,$$

$$\left[2\left(G^a-\frac{1}{4}F^a\right)\dot{\tilde{\partial}}_a, \tilde{g}_b^c\circ h\circ\pi\tilde{Z}^b\dot{\tilde{\partial}}_c\right]_{(\rho,\eta)TE} = 2\left(G^a-\frac{1}{4}F^a\right)\frac{\partial\left(\tilde{g}_b^d\circ h\circ\pi\cdot\tilde{Z}^b\right)}{\partial y^a}\dot{\tilde{\partial}}_d - 2\tilde{g}_b^c\circ h\circ\pi\cdot\tilde{Z}^b\frac{\partial\left(G^d-\frac{1}{4}F^d\right)}{\partial y^c}\dot{\tilde{\partial}}_d$$

it results that

$$(P_2)\qquad \begin{aligned}\left[S,\mathcal{J}_{(g,h)}X\right]_{(\rho,\eta)TE} &= \left(g_e^a\circ h\circ\pi\cdot y^e\right)\rho_a^i\circ h\circ\pi\frac{\partial\tilde{g}_b^d\circ h\circ\pi\tilde{Z}^b}{\partial x^i}\dot{\tilde{\partial}}_d\\ &-\tilde{g}_b^c\circ h\circ\pi\cdot\tilde{Z}^b\frac{\partial\left(g_e^d\circ h\circ\pi\cdot y^e\right)}{\partial y^c}\tilde{\partial}_d\\ &-2\left(G^a-\frac{1}{4}F^a\right)\frac{\partial\left(\tilde{g}_b^d\circ h\circ\pi\cdot\tilde{Z}^b\right)}{\partial y^a}\dot{\tilde{\partial}}_d\\ &+2\tilde{g}_b^c\circ h\circ\pi\cdot\tilde{Z}^b\frac{\partial\left(G^d-\frac{1}{4}F^d\right)}{\partial y^c}\dot{\tilde{\partial}}_d.\end{aligned}$$

We remark that

$$\begin{aligned}\left(g_e^a\circ h\circ\pi\cdot y^e\right)\rho_a^i\circ h\circ\pi\frac{\partial\tilde{g}_b^d\circ h\circ\pi\tilde{Z}^b}{\partial x^i} &= g_e^a\circ h\circ\pi\cdot y^e\rho_a^i\circ h\circ\pi\frac{\partial\tilde{Z}^c}{\partial x^i}\cdot\tilde{g}_c^d\circ h\circ\pi\\ &-\tilde{Z}^b\rho_b^j\circ h\circ\pi\frac{\partial\left(g_e^c\circ h\circ\pi\cdot y^e\right)}{\partial x^j}\cdot\tilde{g}_c^d\circ h\circ\pi,\end{aligned}$$

$$Y^b = Y^b\frac{\partial\left(g_e^c\circ h\circ\pi\cdot y^e\right)}{\partial y^b}\cdot\tilde{g}_c^d\circ h\circ\pi$$

and

$$\tilde{Z}^d = \tilde{g}_b^c\circ h\circ\pi\cdot\tilde{Z}^b\frac{\partial\left(g_e^d\circ h\circ\pi\cdot y^e\right)}{\partial y^c}.$$

Using equalities (P_1) and (P_2), we obtain:

$$\begin{aligned}&\mathbb{P}\left(\tilde{Z}^a\tilde{\partial}_a+Y^a\dot{\tilde{\partial}}_a\right)=\tilde{Z}^a\tilde{\partial}_a+\\ &+\left(-Y^a-2\tilde{g}_b^c\circ h\circ\pi\frac{\partial\left(G^a-\frac{1}{4}F^a\right)}{\partial y^c}\tilde{Z}^b+\left(g_e^d\circ h\circ\pi\cdot y^e\right)\tilde{Z}^bL_{db}^c\circ h\circ\pi\cdot\tilde{g}_c^a\circ h\circ\pi\right)\dot{\tilde{\partial}}_a.\end{aligned}$$

After some calculations, it results that $\mathbb{P}$ is an almost product structure.

Using the equality

$$\mathbb{P}=Id-2\left(\rho,\eta\right)\Gamma,$$

we obtain that

$$\begin{aligned}&\left(\rho,\eta\right)\Gamma\left(\tilde{Z}^a\tilde{\partial}_a+Y^a\dot{\partial}_a\right)=\\ &=\left(Y^a+\tilde{g}_b^c\circ h\circ\pi\frac{\partial\left(G^a-\frac{1}{4}F^a\right)}{\partial y^c}\tilde{Z}^b-\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)\tilde{Z}^bL_{db}^c\circ h\circ\pi\cdot\tilde{g}_c^a\circ h\circ\pi\right)\dot{\tilde{\partial}}_a\end{aligned}$$

Since

$$\left(\rho,\eta\right)\Gamma\left(\tilde{Z}^a\tilde{\partial}_a+Y^a\dot{\tilde{\partial}}_a\right)=\left(Y^a+\left(\rho,\eta\right)\Gamma_b^a\tilde{Z}^b\right)\dot{\tilde{\partial}}_a$$

it results that relations (5.11.1.4) are satisfied. In addition, since

$$\left(\rho,\eta\right)\mathring{\Gamma}_c^a=\left(\rho,\eta\right)\Gamma_c^a+\frac{1}{4}\tilde{g}_c^d\circ h\circ\pi\frac{\partial F^a}{\partial y^d}$$

and

$$
\begin{aligned}
(\rho,\eta)\,\overset{\circ}{\Gamma}{}^{\acute{a}}_{\acute{c}} &= (\rho,\eta)\,\Gamma^{\acute{a}}_{\acute{c}} + \frac{1}{2}\tilde{g}^{\acute{b}}_{\acute{c}}\circ h\circ\pi\frac{\partial F^{\acute{a}}}{\partial y^{\acute{b}}}\\
&= M^{\acute{a}}_{a}\circ\pi\left(\rho^{i}_{c}\circ h\circ\pi\cdot\frac{\partial M^{a}_{\acute{b}}}{\partial x^{i}}y^{\acute{b}} + (\rho,\eta)\,\Gamma^{a}_{c}\right)M^{c}_{\acute{c}}\circ h\circ\pi\\
&+M^{\acute{a}}_{a}\circ\pi\left(\frac{1}{4}\tilde{g}^{b}_{c}\circ h\circ\pi\cdot\frac{\partial F^{a}}{\partial y^{b}}\right)M^{c}_{\acute{c}}\circ h\circ\pi\\
&= M^{\acute{a}}_{a}\circ\pi\left(\rho^{i}_{c}\circ h\circ\pi\cdot\frac{\partial M^{a}_{\acute{b}}}{\partial x^{i}}y^{\acute{b}} + \left((\rho,\eta)\,\Gamma^{a}_{c} + \frac{1}{4}\tilde{g}^{b}_{c}\circ h\circ\pi\cdot\frac{\partial F^{a}}{\partial y^{b}}\right)\right)M^{c}_{\acute{c}}\circ h\circ\pi\\
&= M^{\acute{a}}_{a}\circ\pi\left(\rho^{i}_{c}\circ h\circ\pi\cdot\frac{\partial M^{a}_{\acute{b}}}{\partial x^{i}}y^{\acute{b}} + (\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{c}\right)M^{c}_{\acute{c}}\circ h\circ\pi
\end{aligned}
$$

it results the conclusion of the theorem. *q.e.d.*

Remarks

1. If $(\rho,\eta) = (Id_{TM}, Id_M)$, $(g,h) = (Id_E, Id_M)$, and $F_e \neq 0$, then we obtain the canonical semispray associated to connection Γ which is not the same canonical semispray presented by I. Bucataru and R. Miron in [7].
2. If $(\rho,\eta) = (Id_{TM}, Id_M)$, $(g,h) = (Id_E, Id_M)$, and $F_e = 0$, then we obtain the canonical semispray associated to connection Γ which is not the classical canonical semispray associated to connection Γ.

Using *Theorem 5.11.1.1*, we obtain the following:

Theorem 5.11.1.2 *The following properties hold good:*

1° *Since* $\overset{\circ}{\tilde{\delta}}_c = \tilde{\partial}_c - (\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{c}\dot{\tilde{\partial}}_a,\ c\in\overline{1,r}$, *it results that*

$$\overset{\circ}{\tilde{\delta}}_c = \tilde{\delta}_c - \frac{1}{4}\tilde{g}^{b}_{c}\circ h\circ\pi\cdot\frac{\partial F^{a}}{\partial y^{b}}\dot{\tilde{\partial}}_a,\ c\in\overline{1,r}. \tag{5.11.1.6}$$

2° *Since* $\overset{\circ}{\delta}\tilde{y}^a = (\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{c}d\tilde{z}^c + d\tilde{y}^a$, *it results that*

$$\overset{\circ}{\delta}\tilde{y}^a = \delta\tilde{y}^a + \frac{1}{4}\tilde{g}^{b}_{c}\circ h\circ\pi\frac{\partial F^{a}}{\partial y^{b}}d\tilde{z}^c,\ a\in\overline{1,r}. \tag{5.11.1.7}$$

Theorem 5.11.1.3 *The real local functions*

$$\left(\frac{\partial\,(\rho,\eta)\,\Gamma^{a}_{c}}{\partial y^{b}}, \frac{\partial\,(\rho,\eta)\,\Gamma^{a}_{c}}{\partial y^{b}}, 0,\ 0\right),\ a,b,c\in\overline{1,r}, \tag{5.11.1.8}$$

and

$$\left(\frac{\partial\,(\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{c}}{\partial y^{b}}, \frac{\partial\,(\rho,\eta)\,\overset{\circ}{\Gamma}{}^{a}_{c}}{\partial y^{b}}, 0,\ 0\right),\ a,b,c\in\overline{1,r}, \tag{5.11.1.8$'$}$$

respectively, are the coefficients to a normal Berwald linear (ρ,η)-connection for the generalized tangent bundle $((\rho,\eta)\,TE, (\rho,\eta)\,\tau_E, E)$.

Theorem 5.11.1.4 *The tensor of integrability of the* (ρ,η)*-connection* $(\rho,\eta)\overset{\circ}{\Gamma}$ *is as follows:*

$$\begin{array}{l}(\rho,\eta,h)\,\overset{\circ}{\mathbb{R}}{}^a_{cd}=(\rho,\eta,h)\,\mathbb{R}^a_{cd}+\dfrac{1}{4}\left(\tilde{g}^e_d\circ h\circ\pi\dfrac{\partial F^a}{\partial y^e}_{|c}-\tilde{g}^e_c\circ h\circ\pi\dfrac{\partial F^a}{\partial y^e}_{|d}\right)\\ +\dfrac{1}{16}\left(\tilde{g}^e_d\circ h\circ\pi\dfrac{\partial F^b}{\partial y^e}\tilde{g}^f_c\circ h\circ\pi\dfrac{\partial^2F^a}{\partial y^b\partial y^f}-\tilde{g}^f_c\circ h\circ\pi\dfrac{\partial F^b}{\partial y^f}\tilde{g}^e_d\circ h\circ\pi\dfrac{\partial^2F^a}{\partial y^b\partial y^e}\right)\\ +\dfrac{1}{4}\left(L^f_{cd}\circ h\circ\pi\right)\left(\tilde{g}^e_f\circ h\circ\pi\right)\dfrac{\partial F^a}{\partial y^e},\end{array}\tag{5.11.1.9}$$

where $_{|c}$ *is the h-covariant derivation with respect to the normal Berwald linear* (ρ,η)*-connection* (5.11.1.8).

Proof. Since

$$\begin{array}{rl}(\rho,\eta,h)\,\overset{\circ}{\mathbb{R}}{}^a_{cd}= & \Gamma(\tilde{\rho},Id_E)\left(\overset{\circ}{\tilde{\delta}}_c\right)\left((\rho,\eta)\,\overset{\circ}{\Gamma}{}^a_d\right)-\Gamma(\tilde{\rho},Id_E)\left(\overset{\circ}{\tilde{\delta}}_d\right)\left((\rho,\eta)\,\overset{\circ}{\Gamma}{}^a_c\right)\\ & +L^e_{cd}\circ h\circ(h\circ\pi)\,(\rho,\eta)\,\overset{\circ}{\Gamma}{}^a_e,\end{array}$$

and

$$\begin{array}{rl}\Gamma(\tilde{\rho},Id_E)\left(\overset{\circ}{\tilde{\delta}}_c\right)\left((\rho,\eta)\,\overset{\circ}{\Gamma}{}^a_d\right) & =\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_c\right)\left((\rho,\eta)\,\Gamma^a_d\right)\\ & +\dfrac{1}{4}\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_c\right)\left(\tilde{g}^e_d\circ h\circ\pi\dfrac{\partial F^a}{\partial y^e}\right)\\ & -\dfrac{1}{4}\tilde{g}^e_c\circ h\circ\pi\dfrac{\partial F^f}{\partial y^e}\dfrac{\partial}{\partial y^f}\left((\rho,\eta)\,\Gamma^a_d\right)\\ & -\dfrac{1}{16}\tilde{g}^e_c\circ h\circ\pi\dfrac{\partial F^f}{\partial y^e}\dfrac{\partial}{\partial y^f}\left(\tilde{g}^e_d\circ h\circ\pi\dfrac{\partial F^a}{\partial y^e}\right),\end{array}$$

$$\begin{array}{rl}\Gamma(\tilde{\rho},Id_E)\left(\overset{\circ}{\tilde{\delta}}_d\right)\left((\rho,\eta)\,\overset{\circ}{\Gamma}{}^a_c\right) & =\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_d\right)\left((\rho,\eta)\,\Gamma^a_c\right)\\ & +\dfrac{1}{4}\Gamma(\tilde{\rho},Id_E)\left(\tilde{\delta}_d\right)\left(\tilde{g}^e_c\circ h\circ\pi\dfrac{\partial F^a}{\partial y^e}\right)\\ & -\dfrac{1}{4}\tilde{g}^e_d\circ h\circ\pi\dfrac{\partial F^f}{\partial y^e}\dfrac{\partial}{\partial y^f}\left((\rho,\eta)\,\Gamma^a_c\right)\\ & -\dfrac{1}{16}\tilde{g}^e_d\circ h\circ\pi\dfrac{\partial F^f}{\partial y^e}\dfrac{\partial}{\partial y^f}\left(\tilde{g}^e_c\circ h\circ\pi\dfrac{\partial F^a}{\partial y^e}\right),\\ L^e_{cd}\circ h\circ\pi\cdot(\rho,\eta)\,\overset{\circ}{\Gamma}{}^a_e & =L^e_{cd}\circ h\circ\pi\cdot(\rho,\eta)\,\Gamma^a_e\\ & +L^e_{cd}\circ h\circ\pi\cdot\left(\tilde{g}^f_e\circ h\circ\pi\dfrac{\partial F^a}{\partial y^f}\right)\end{array}$$

it results the conclusion of the theorem. *q.e.d.*

Theorem 5.11.1.5 *Let*

$$\mathbb{T}^a_{bc}\delta_a\otimes d\tilde{z}^b\otimes d\tilde{z}^c\in\mathcal{T}^{10}_{20}\left((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E\right)$$

and

$$\mathbb{S}_{bc}^{a}\dot{\tilde{\partial}}_{a}\otimes\delta\tilde{y}^{b}\otimes\delta\tilde{y}^{c}\in\mathcal{T}_{02}^{01}\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$$

such that they verify the following conditions:

$$\mathbb{T}_{bc}^{a}=-\mathbb{T}_{cb}^{a},\ \mathbb{S}_{bc}^{a}=-\mathbb{S}_{cb}^{a},\ \forall b,c\in\overline{1,r}.$$

If $\left(\left(\rho,\eta\right)\widetilde{H},\left(\rho,\eta\right)\widetilde{V}\right)$ *is the distinguished linear* (ρ,η)*-connection presented in the Theorem 5.9.2, then the local real functions:*

$$\begin{aligned}\left(\rho,\eta\right)\mathring{H}_{bc}^{a}&=\left(\rho,\eta\right)\widetilde{H}_{bc}^{a}+\frac{1}{8}\tilde{g}^{ae}\left(-\tilde{g}_{c}^{f}\circ h\circ\pi\frac{\partial F^{d}}{\partial y^{f}}\frac{\partial g_{be}}{\partial y^{d}}\right.\\&\left.\quad+\tilde{g}_{e}^{f}\circ h\circ\pi\frac{\partial F^{d}}{\partial y^{f}}\frac{\partial g_{bc}}{\partial y^{d}}-\tilde{g}_{b}^{f}\circ h\circ\pi\frac{\partial F^{d}}{\partial y^{f}}\frac{\partial g_{ec}}{\partial y^{d}}\right),\\\left(\rho,\eta\right)\mathring{V}_{bc}^{a}&=\left(\rho,\eta\right)\widetilde{V}_{bc}^{a}\end{aligned}\tag{5.11.1.10}$$

are the components of a normal distinguished linear (ρ,η)*-connection with* (ρ,η)*-*$\mathcal{H}\left(\mathcal{HH}\right)$ *and* (ρ,η)*-*$\mathcal{V}\left(\mathcal{VV}\right)$ *torsions a priori given such that the generalized tangent bundle* $\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right)$ *derives generalized Lagrange* (ρ,η)*-space.*

In addition, we have:

$$\begin{aligned}\left(\rho,\eta,h\right)\mathring{\mathbb{T}}_{bc}^{a}&=\mathbb{T}_{bc}^{a}\\\left(\rho,\eta,h\right)\mathring{\mathbb{S}}_{bc}^{a}&=\mathbb{S}_{bc}^{a}.\end{aligned}\tag{5.11.1.11}$$

Proposition 5.11.1.1 *If* S *is the canonical* (ρ,η)*-semispray associated to the mechanical* (ρ,η)*-system* $\left(\left(E,\pi,M\right),F_{e},\left(\rho,\eta\right)\Gamma\right)$ *and from* $\mathbf{B^{v}}$*-morphism* (g,h)*, then*

$$2G^{\acute{a}}=2G^{a}M_{a}^{\acute{a}}\circ h\circ\pi-\left(g_{b}^{a}\circ h\circ\pi\right)y^{b}\left(\rho_{a}^{i}\circ h\circ\pi\right)\frac{\partial y^{\acute{a}}}{\partial x^{i}}.\tag{5.11.1.12}$$

Proof. Since the Jacobian matrix of coordinates transformation is

$$\left\Vert\begin{array}{cc}M_{a}^{\acute{a}}\circ h\circ\pi&0\\\rho_{a}^{i}\circ\left(h\circ\pi\right)\frac{\partial M_{a}^{\acute{a}}\circ\pi}{\partial x^{i}}y^{a}&M_{a}^{\acute{a}}\circ\pi\end{array}\right\Vert=\left\Vert\begin{array}{cc}M_{a}^{\acute{a}}\circ h\circ\pi&0\\\rho_{a}^{i}\circ\left(h\circ\pi\right)\frac{\partial y^{\acute{a}}}{\partial x^{i}}&M_{a}^{\acute{a}}\circ\pi\end{array}\right\Vert$$

and

$$\left\Vert\begin{array}{cc}M_{a}^{\acute{a}}\circ h\circ\pi&0\\\rho_{a}^{i}\circ\left(h\circ\pi\right)\frac{\partial y^{\acute{a}}}{\partial x^{i}}&M_{a}^{\acute{a}}\circ\pi\end{array}\right\Vert\cdot\left(\begin{array}{c}\left(g_{b}^{a}\circ h\circ\pi\right)y^{b}\\-2\left(G^{a}-\frac{1}{4}F^{a}\right)\end{array}\right)=\left(\begin{array}{c}\left(g_{\acute{b}}^{\acute{a}}\circ h\circ\pi\right)y^{\acute{b}}\\-2\left(G^{\acute{a}}-\frac{1}{4}F^{\acute{a}}\right)\end{array}\right),$$

the conclusion results immediately.

In the following, we consider a differentiable curve $I\overset{c}{\rightarrow}M$ and its (g,h)-lift $I\overset{\dot{c}}{\rightarrow}E$.
q.e.d.

Definition 5.11.1.3 The curve $\dot{c}$ is a integral curve of the (ρ,η)-semispray S of the mechanical (ρ,η)-system $\left(\left(E,\pi,M\right),F_{e},\left(\rho,\eta\right)\Gamma\right)$, if it is verifies the following equality:

$$\frac{d\dot{c}\left(t\right)}{dt}=\Gamma\left(\tilde{\rho},Id_{E}\right)S\left(\dot{c}\left(t\right)\right).\tag{5.11.1.13}$$

Theorem 5.11.1.6 *All (g,h)-lifts solutions of the equations:*

$$\frac{dy^a(t)}{dt}+2G^a\circ u(c,\dot{c})(x(t))=\frac{1}{2}F^a\circ u(c,\dot{c})(x(t)),\ a\in\overline{1,r}, \tag{5.11.1.14}$$

where $x(t)=(\eta\circ h\circ c)(t)$, are integral curves of the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,(\rho,\eta)\Gamma)$ and from locally invertible $\mathbf{B^v}$*-morphism (g,h).*

Proof. Since the equality

$$\frac{d\dot{c}(t)}{dt}=\Gamma(\tilde{\rho},Id_E)S(\dot{c}(t))$$

is equivalent to

$$\begin{array}{l}\frac{d}{dt}((\eta\circ h\circ c)^i(t),y^a(t))\\ \qquad=\left(\rho_a^i\circ\eta\circ h\circ c(t)g_b^a\circ h\circ c(t)y^b(t),-2\left(G^a-\frac{1}{4}F^a\right)((\eta\circ h\circ c)^i(t),y^a(t))\right),\end{array}$$

it results

$$\begin{array}{l}\frac{dy^a(t)}{dt}+2G^a\left(x^i(t),y^a(t)\right)=\frac{1}{2}F^a\left(x^i(t),y^a(t)\right),\quad a\in\overline{1,n},\\ \frac{dx^i(t)}{dt}=\rho_a^i\circ\eta\circ h\circ c(t)\,g_b^a\circ h\circ c(t)\,y^b(t),\end{array}$$

where $x^i(t)=(\eta\circ h\circ c)^i(t)$. *q.e.d.*

Definition 5.11.1.4 If S is a (ρ,η)-semispray, then the vector field

$$[\mathbb{C},S]_{(\rho,\eta)TE}-S \tag{5.11.1.15}$$

will be called the *derivation of (ρ,η)-semispray S.*

The (ρ,η)-semispray S will be called *(ρ,η)-spray* if the following conditions are verified:

1. $S\circ 0\in C^1$, where 0 is the null section;
2. Its derivation is the null vector field.

The (ρ,η)-semispray S will be called *quadratic (ρ,η)-spray* if there are verified the following conditions:

1. $S\circ 0\in C^2$, where 0 is the null section;
2. Its derivation is the null vector field.

In particular, if $(\rho,\eta)=(id_{TM},Id_M)$ and $(g,h)=(Id_E,Id_M)$, then we obtain the *spray* and the *quadratic spray* which is similar with the classical spray and quadratic spray.

Theorem 5.11.1.7 *If S is the canonical (ρ,η)-spray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,(\rho,\eta)\Gamma)$ and from locally invertible* $\mathbf{B^v}$*-morphism (g,h), then*

$$\begin{array}{l}2\left(G^a-\frac{1}{4}F^a\right)=(\rho,\eta)\Gamma_c^a\left(g_f^c\circ h\circ\pi\cdot y^f\right)\\ \qquad+\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\tilde{g}_b^a\circ h\circ\pi\left(g_f^c\circ h\circ\pi\cdot y^f\right),\ a\in\overline{1,r}.\end{array} \tag{5.11.1.16}$$

Then, we obtain the spray

$$
\begin{aligned}
S & = \left(g_b^a\circ h\circ\pi\right) y^b\frac{\partial}{\partial\tilde{z}^a}+\left(\rho,\eta\right)\Gamma_c^a\left(g_f^c\circ h\circ\pi\cdot y^f\right)\frac{\partial}{\partial\tilde{y}^a}\\
& +\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi\left(g_f^c\circ h\circ\pi\cdot y^f\right)\frac{\partial}{\partial\tilde{y}^a}.
\end{aligned}
\tag{5.11.1.17}
$$

This (ρ,η)-spray will be called the canonical (ρ,η)-spray associated to mechanical system $\left(\left(E,\pi,M\right),F_e,\left(\rho,\eta\right)\Gamma\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).

In particular, if $(\rho,\eta)=(id_{TM},Id_M)$ and $(g,h)=(Id_E,Id_M)$, then we get the canonical spray associated to connection Γ which is similar with the classical canonical spray associated to connection Γ.

Proof. Since

$$
[\mathbb{C},S]_{(\rho,\eta)TE}=\left[y^a\dot{\tilde{\partial}}_a,\left(g_e^b\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_b\right]_{(\rho,\eta)TE}-2\left[y^a\dot{\tilde{\partial}}_a,\left(G^b-\frac{1}{4}F^b\right)\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE},
$$

$$
\begin{aligned}
\left[y^a\dot{\tilde{\partial}}_a,\left(g_e^b\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_b\right]_{(\rho,\eta)TE} & = y^a\frac{\partial\left(g_e^b\circ h\circ\pi\cdot y^e\right)}{\partial y^a}\tilde{\partial}_b-\left(g_e^b\circ h\circ\pi\cdot y^e\right)\rho_\beta^j\circ h\circ\pi\frac{\partial y^a}{\partial x^i}\\
& = y^a g_e^b\circ h\circ\pi\cdot\delta_a^e\tilde{\partial}_b-0=\left(g_e^b\circ h\circ\pi\cdot y^e\right)\tilde{\partial}_b
\end{aligned}
$$

and

$$
\begin{aligned}
\left[y^a\dot{\tilde{\partial}}_a,\left(G^b-\tfrac{1}{4}F^b\right)\dot{\tilde{\partial}}_b\right]_{(\rho,\eta)TE} & = y^a\frac{\partial\left(G^b-\frac{1}{4}F^b\right)}{\partial y^a}\dot{\tilde{\partial}}_b-\left(G^b-\frac{1}{4}F^b\right)\delta_b^a\dot{\tilde{\partial}}_a\\
& = y^a\frac{\partial\left(G^b-\frac{1}{4}F^b\right)}{\partial y^a}\dot{\tilde{\partial}}_b-\left(G^b-\frac{1}{4}F^b\right)\dot{\tilde{\partial}}_b
\end{aligned}
$$

it results that

$$
[\mathbb{C},S]_{(\rho,\eta)TE}-S=2\left(-y^f\frac{\partial\left(G^a-\frac{1}{4}F^a\right)}{y^f}+2\left(G^a-\frac{1}{4}F^a\right)\right)\dot{\tilde{\partial}}_a
\tag{S_1}
$$

Using equality (5.11.1.4), it results that

$$
\begin{aligned}
\frac{\partial\left(G^a-\frac{1}{4}F^a\right)}{y^f} & = \left(\rho,\eta\right)\Gamma_c^a\cdot g_f^c\circ h\circ\pi\\
& +\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi\cdot g_f^c\circ h\circ\pi.
\end{aligned}
\tag{S_2}
$$

Using equalities (S_1) and (S_2), it results the conclusion of the theorem. *q.e.d.*

Remark 5.11.1.2. If $(\rho,\eta)=(id_{TM},Id_M)$ and $(g,h)=(Id_E,Id_M)$, then we get the canonical spray associated to connection Γ.

Theorem 5.11.1.8 *All (g,h)-lifts solutions of the following system of equations:*

$$
\begin{aligned}
& \frac{dy^a}{dt}+\left(\rho,\eta\right)\Gamma_c^a\left(g_f^c\circ h\circ\pi\cdot y^f\right)\\
& +\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi\left(g_f^c\circ h\circ\pi\cdot y^f\right)=0,
\end{aligned}
\tag{5.11.1.17}
$$

are the integral curves of canonical (ρ,η)-spray associated to mechanical (ρ,η)-system $\left(\left(E,\pi,M\right),F_e,\left(\rho,\eta\right)\Gamma\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).

5.11.2 The Lagrangian formalism for Lagrange mechanical (ρ,η)-systems

Let $((E,\pi,M),F_e,L)$ be an arbitrarily Lagrange mechanical (ρ,η)-system.

The *natural dual* (ρ,η)*-base* $(d\tilde{z}^\alpha, d\tilde{y}^a)$ of natural (ρ,η)-base $\left(\frac{\partial}{\partial\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{y}^a}\right)$ is determined by the equations

$$\begin{cases} \left\langle d\tilde{z}^\alpha, \frac{\partial}{\partial\tilde{z}^\beta}\right\rangle = \delta^\alpha_\beta, & \left\langle d\tilde{z}^\alpha, \frac{\partial}{\partial\tilde{y}^a}\right\rangle = 0,\\ \left\langle d\tilde{y}^a, \frac{\partial}{\partial\tilde{z}^\beta}\right\rangle = 0, & \left\langle d\tilde{y}^a, \frac{\partial}{\partial\tilde{y}^b}\right\rangle = \delta^a_b. \end{cases}$$

It is very important to remark that the 1-forms $d\tilde{z}^\alpha,\ \alpha\in\overline{1,p}$ and $d\tilde{y}^a,\ a\in\overline{1,n}$ are not the differentials of coordinates functions as in the classical case, but we will use the same notations. In this case

$$(d\tilde{z}^\alpha)\neq d^{(\rho,\eta)TE}(\tilde{z}^\alpha)=0,$$

where $d^{(\rho,\eta)TE}$ is the exterior differentiation operator associated to exterior differential $\mathcal{F}(E)$-algebra

$$(\Lambda((\rho,\eta)TE,(\rho,\eta)\tau_E,E),+,\cdot,\wedge).$$

Let L be a regular Lagrangian and let (g,h) be a locally invertible $\mathbf{B^v}$-morphism of (E,π,M) source and (E,π,M) target.

Definition 5.11.2.1 The 1-form

$$\theta_L=(\tilde{g}^e_a\circ h\circ\pi\cdot L_e)\,d\tilde{z}^a \tag{5.11.2.1}$$

will be called the 1*-form of Poincaré-Cartan type associated to the Lagrangian L and to the locally invertible* $\mathbf{B^v}$*-morphism* (g,h).

We obtain easily:

$$\theta_L\left(\frac{\partial}{\partial\tilde{z}^b}\right)=\tilde{g}^e_b\circ h\circ\pi\cdot L_e,\quad \theta_L\left(\frac{\partial}{\partial\tilde{y}^b}\right)=0. \tag{5.11.2.2}$$

Definition 5.11.2.2 The 2-form

$$\omega_L=d^{(\rho,\eta)TE}\theta_L$$

will be called the 2*-form of Poincaré-Cartan type associated to the Lagrangian L and to the locally invertible* $\mathbf{B^v}$*-morphism* (g,h).

By the definition of $d^{(\rho,\eta)TE}$, we obtain:

$$\begin{aligned}\omega_L(U,V) &= \Gamma(\tilde{\rho},Id_E)(U)(\theta_L(V))\\ &\quad -\Gamma(\tilde{\rho},Id_E)(V)(\theta_L(U))-\theta_L\left([U,V]_{(\rho,\eta)TE}\right),\end{aligned} \tag{5.11.2.3}$$

for any $U,V\in\Gamma((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$.

It follows:

$$
(5.11.2.4)\qquad \begin{cases}
\omega_L\left(\frac{\partial}{\partial\tilde{z}^a},\frac{\partial}{\partial\tilde{z}^b}\right) &= \left(\rho_a^i\circ h\circ\pi\right)\cdot L_{ib}\\
&\quad -\left(\rho_b^i\circ h\circ\pi\right)\cdot L_{ia}-L_{ab}^c\circ h\circ\pi\cdot\tilde{g}_c^e\circ h\circ\pi\cdot L_e;\\
\omega_L\left(\frac{\partial}{\partial\tilde{z}^a},\frac{\partial}{\partial\tilde{y}^b}\right) &= -\tilde{g}_a^e\circ h\circ\pi\cdot L_{eb};\\
\omega_L\left(\frac{\partial}{\partial\tilde{y}^a},\frac{\partial}{\partial\tilde{y}^b}\right) &= 0.
\end{cases}
$$

Definition 5.11.2.3 The real function

$$
(5.11.2.5)\qquad \mathcal{E}_L=y^a\cdot L_a-L
$$

will be called the *energy of regular Lagrangian L.*

Theorem 5.11.2.1 *The equation*

$$
(5.11.2.6)\qquad i_S\left(\omega_L\right)=-d^{(\rho,\eta)TE}\left(\mathcal{E}_L\right),\ S\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right),
$$

has an unique solution $S_L\left(g,h\right)$ of the type:

$$
(5.11.2.7)\qquad \left(g_e^a\circ h\circ\pi\right)y^e\frac{\partial}{\partial\tilde{z}^a}-2\left(G^a-\frac{1}{4}F^a\right)\frac{\partial}{\partial\tilde{y}^a},
$$

where

$$
(5.11.2.8)\qquad 2G^a=g_e^a\circ h\circ\pi\cdot\tilde{L}^{eb}\cdot E_b\left(L,g,h\right)+\frac{1}{2}F^a
$$

and

$$
(5.11.2.9)\qquad \begin{aligned}
E_b(L,g,h)\ &=\rho_b^i{\circ}h{\circ}\pi\cdot L_i-g_e^a\circ h\circ\pi\cdot y^e\cdot\rho_a^i{\circ}h{\circ}\pi\cdot\frac{\partial\left(\tilde{g}_b^e\circ h\circ\pi\cdot L_e\right)}{\partial x^i}\\
&+\ g_e^a\circ h\circ\pi\cdot y^e\cdot L_{ab}^d{\circ}h{\circ}\pi\cdot\left(\tilde{g}_d^e\circ h\circ\pi\cdot L_e\right).
\end{aligned}
$$

$S_L\left(g,h\right)$ will be called *the canonical (ρ,η)-semispray associated to Lagrange mechanical (ρ,η)-system $\left(\left(E,\pi,M\right),F_e,L\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).*

Proof. We obtain that

$$
i_S\left(\omega_L\right)=-d^{(\rho,\eta)TE}\left(\mathcal{E}_L\right)
$$

if and only if

$$
\omega_L\left(S,X\right)=-\Gamma\left(\tilde{\rho},Id_E\right)\left(X\right)\left(\mathcal{E}_L\right),
$$

for any $X\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_E,E\right).$

Particularly, we obtain:

$$
\omega_L\left(S,\frac{\partial}{\partial\tilde{z}^b}\right)=-\Gamma\left(\tilde{\rho},Id_E\right)\left(\frac{\partial}{\partial\tilde{z}^b}\right)\left(\mathcal{E}_L\right).
$$

If we expand this equality by using (5.11.2.2) and (5.11.2.4), we obtain

$$\begin{aligned}&g_e^a\circ h\circ\pi\cdot y^e\cdot\Big[\rho_a^i\circ h\circ\pi\cdot\frac{\partial\left(\tilde{g}_b^e\circ h\circ\pi\cdot L_e\right)}{\partial x^i}-\rho_b^i\circ h\circ\pi\cdot\frac{\partial\left(\tilde{g}_a^e\circ h\circ\pi\cdot L_e\right)}{\partial x^i}\\&\quad-L_{ab}^d\circ h\circ\pi\cdot\left(\tilde{g}_d^e\circ h\circ\pi\cdot L_e\right)\Big]+2\left(G^a-\frac{1}{4}F^a\right)\left(\tilde{g}_a^e\circ h\circ\pi\right)\cdot L_{eb}\\&\quad=-\rho_b^i\circ h\circ\pi\cdot\left(g_e^a\circ h\circ\pi\cdot y^e\right)\cdot\frac{\partial\left(\tilde{g}_a^e\circ h\circ\pi\cdot L_e\right)}{\partial x^i}+\rho_b^i\circ h\circ\pi\cdot L_i.\end{aligned}$$

After some calculations, we obtain

$$2\left(G^a-\frac{1}{4}F^a\right)=g_e^a\circ h\circ\pi\cdot\tilde{L}^{eb}\cdot E_b\left(L,g,h\right),$$

where

$$\begin{aligned}E_b\left(L,g,h\right)&=\rho_b^i\circ h\circ\pi\cdot L_i-g_e^a\circ h\circ\pi\cdot y^e\cdot\rho_a^i\circ h\circ\pi\cdot\frac{\partial\left(\tilde{g}_b^e\circ h\circ\pi\cdot L_e\right)}{\partial x^i}+\\&+g_e^a\circ h\circ\pi\cdot y^e\cdot L_{ab}^d\circ h\circ\pi\cdot\left(\tilde{g}_d^e\circ h\circ\pi\cdot L_e\right).\end{aligned}$$

q.e.d.

Remarks

1. If $F_e=0$ and $\eta=Id_M$, then $S_L\left(Id_E,Id_M\right)\overset{put}{=}S_L$ is the canonical ρ-semispray associated to regular Lagrangian L which is similar with the semispray presented in [27] by M. de Leon, J. Marrero and E. Martinez.

2. If $F_e\neq 0$ and $(\rho,\eta)=(Id_{TM},Id_M)$, then $S_L\left(Id_E,Id_M\right)\overset{put}{=}S_L$ will be called *the canonical semispray* which is not the same canonical semispray presented by I. Bucataru and R. Miron in [7].

3. If $F_e=0$ and $(\rho,\eta)=(Id_{TM},Id_M)$, then $S_L\left(Id_M,Id_E\right)\overset{put}{=}S_L$ will be called *the canonical semispray* which is not the same canonical semispray presented by R. Miron and M. Anastasiei in [41].

Theorem 5.11.2.2 *The real local functions*

$$\begin{aligned}(\rho,\eta)\,\Gamma_c^a&=\frac{1}{2}\tilde{g}_c^e\circ h\circ\pi\frac{\partial\left(g_e^a\circ h\circ\pi\cdot L^{eb}\cdot E_b\left(L,g,h\right)\right)}{\partial y^e}\\&-\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi,\ a,c\in\overline{1,r}.\end{aligned}\tag{5.11.2.10}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\,\Gamma$ for the vector bundle (E,π,M) which will be called the (ρ,η)-connection associated to Lagrange mechanical (ρ,η)-system $((E,\pi,M),F_e,L)$ and from $\mathbf{B^v}$-morphism (g,h).

Corollary 5.11.2.1 *The real local functions*

$$\begin{aligned}(\rho,\eta)\mathring{\Gamma}^a{}_c&=\left(\tilde{g}_c^b\circ h\circ\pi\right)\frac{\partial G^a}{\partial y^b}\\&-\frac{1}{2}\left(g_e^d\circ h\circ\pi\cdot y^e\right)L_{dc}^b\circ h\circ\pi\cdot\tilde{g}_b^a\circ h\circ\pi,\ a,c\in\overline{1,r}\end{aligned}\tag{5.11.2.11}$$

are the components of a (ρ,η)*-connection* $(\rho,\eta)\overset{\circ}{\Gamma}$ *for the vector bundle* (E,π,M).

In addition, we have

$$(\rho,\eta)\overset{\circ}{\Gamma}{}_{c}^{a}=(\rho,\eta)\Gamma_{c}^{a}+\frac{1}{4}(\tilde{g}_{c}^{b}\circ h\circ\pi)\cdot\frac{\partial F^{a}}{\partial y^{b}},\ \forall a,c\in\overline{1,r}. \tag{5.11.2.12}$$

Theorem 5.11.2.3 *The parallel* (g,h)*-lifts with respect to* (ρ,η)*-connection* $(\rho,\eta)\Gamma$ *are the integral curves of the canonical* (ρ,η)*-semispray associated to mechanical* (ρ,η)*-system* $((E,\pi,M),F_e,L)$ *and from locally invertible* $\mathbf{B^v}$*-morphism* (g,h).

Definition 5.11.2.4 The equations

$$\frac{dy^{a}(t)}{dt}+\left(g_{e}^{a}\circ h\circ\pi\cdot\tilde{L}^{eb}\cdot E_{b}(L,g,h)\right)\circ u(c,\dot{c})(x(t))=0, \tag{5.11.2.13}$$

where $x(t)=\eta\circ h\circ c(t)$, will be called the *equations of Euler-Lagrange type associated to Lagrange mechanical* (ρ,η)*-system* $((E,\pi,M),F_e,L)$ *and from locally invertible* $\mathbf{B^v}$*-morphism* (g,h).

The equations

$$\frac{dy^{a}(t)}{dt}+\left(\tilde{L}^{ab}\cdot E_{b}(L,Id_{E},Id_{M})\right)\circ u(c,\dot{c})(x(t))=0, \tag{5.11.2.13$'$}$$

where $x(t)=h\circ\eta\circ c(t)$, will be called the *equations of Euler-Lagrange type associated to Lagrange mechanical* (ρ,η)*-system* $((E,\pi,M),F_e,L)$.

Remark 5.11.2.1 The integral curves of the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,L)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h) are the (g,h)-lifts solutions for the equations of Euler-Lagrange type (5.11.2.13).

It is known that, in classical sense, a geodesic with respect to a Finsler metric

$$TM\ \overset{F}{\longrightarrow}\ \mathbb{R}_{+}$$

is a curve c on the manifold M such that the components of its tangent lift

$$\frac{dc^{i}}{dt}\cdot\frac{\partial}{\partial x^{i}}$$

are solutions for the Euler-Lagrange equations

$$\tfrac{d}{dt}\left(\tfrac{\partial F^{2}}{\partial y^{i}}\right)-\tfrac{\partial F^{2}}{\partial x^{i}}=0,\ i\in\overline{1,m}\ . \tag{5.11.2.14}$$

If

$$\left((TM,\tau_{M},M),\widetilde{[,]}_{TM,h},(\rho,\eta)\right)$$

is a generalized Lie algebroid different by the generalized Lie algebroid

$$\left((TM,\tau_{M},M),[,]_{TM,Id_{M}},(Id_{TM},Id_{M})\right),$$

then, using the classical method by work, we can not determine the geodesics on the manifold M such that the components of their lifts (different by the tangent lift) are solutions for the Euler-Lagrange equations (5.11.2.14).

Using our theory, we obtain the following

Theorem 5.11.2.4 *If F is a Finsler fundamental function, then the geodesics on the manifold M are the curves such that the components of their (g,h)-lifts are solutions for the equations of Euler-Lagrange type* (5.11.2.13) .

Therefore, it is natural to propose to extend the study of the Finsler geometry from the usual Lie algebroid

$$\left(\left(TM,\tau_M,M\right),[,]_{TM},\left(Id_{TM},Id_M\right)\right),$$

to an arbitrary (generalized) Lie algebroid

$$\left(\left(E,\pi,M\right),[,]_{E,h},\left(\rho,\eta\right)\right).$$

6 The geometry of total space of the generalized tangent bundle for dual vector bundle

6.1 Adapted (ρ,η)-basis and adapted dual (ρ,η)-basis

In the following we consider the following diagram:

$$\begin{array}{ccc} \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \overset{*}{\pi}\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E,\pi,M)\in|\mathbf{B}^{\mathbf{v}}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

If we put the problem of finding a base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module

$$\left(\Gamma\left(H\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

of the type

$$\frac{\overset{*}{\delta}}{\delta\tilde{z}^{\alpha}}=\tilde{Z}_{\alpha}^{\beta}\frac{\overset{*}{\partial}}{\partial\tilde{z}^{\beta}}+Y_{b\alpha}\frac{\partial}{\partial\tilde{p}_b},\alpha\in\overline{1,p}$$

which satisfies the following conditions:

$$\begin{array}{rcl}\Gamma\left(\left(\rho,\eta\right)\overset{*}{\pi}!,Id_{\overset{*}{E}}\right)\left(\dfrac{\overset{*}{\delta}}{\delta\tilde{z}^{\alpha}}\right) & = & \overset{*}{\tilde{T}}_{\alpha}\\ \Gamma\left(\left(\rho,\eta\right)\overset{*}{\Gamma},Id_{\overset{*}{E}}\right)\left(\dfrac{\overset{*}{\delta}}{\delta\tilde{z}^{\alpha}}\right) & = & 0,\end{array}\tag{6.1.1}$$

then we obtain the sections

$$\frac{\overset{*}{\delta}}{\delta\tilde{z}^{\alpha}}=\frac{\overset{*}{\partial}}{\partial\tilde{z}^{\alpha}}+\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\frac{\partial}{\partial\tilde{p}_b}\ .\tag{6.1.2}$$

We observe that their law of change is a tensorial law under a change of vector fiber charts.

Definition 6.1.1 The base

$$\left(\frac{\overset{*}{\delta}}{\delta\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{p}_a}\right)\overset{put}{=}\left(\overset{*}{\tilde{\delta}}_{\alpha},\overset{\cdot}{\tilde{\partial}}{}^{a}\right)$$

will be called the *adapted* (ρ,η)*-base.*

The following equality holds good

$$\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\alpha}\right)=\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\partial}_i+(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\dot{\partial}^{b},\tag{6.1.3}$$

where $\left(\overset{*}{\partial}_i,\dot{\partial}^a\right)$ is the natural base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module $\left(\Gamma\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$.

Moreover, if $(\rho,\eta)\,\overset{*}{\Gamma}$ is the (ρ,η)-connection associated to connection $\overset{*}{\Gamma}$, then we obtain

$$\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\alpha}\right)=\left(\rho_{\alpha}^{i}\circ h\circ\pi\right)\overset{*}{\delta}_i,\tag{6.1.4}$$

where $\left(\overset{*}{\delta}_i,\dot{\partial}_a\right)$ is the adapted base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module $\left(\Gamma\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$.

Theorem 6.1.1 *The following equality holds good*

$$\left[\overset{*}{\tilde{\delta}}_{\alpha},\overset{*}{\tilde{\delta}}_{\beta}\right]_{(\rho,\eta)T\overset{*}{E}}=L_{\alpha\beta}^{\gamma}\circ\left(h\circ\overset{*}{\pi}\right)\overset{*}{\tilde{\delta}}_{\gamma}+(\rho,\eta,h)\,\overset{*}{\mathbb{R}}_{b\ \ \alpha\beta}\overset{\cdot}{\tilde{\partial}}{}^{b},\tag{6.1.5}$$

where

$$\begin{aligned}(\rho,\eta,h)\,\overset{*}{\mathbb{R}}_{b\ \ \alpha\beta}&=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\beta}\right)\left((\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\right)\\&+\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\alpha}\right)\left((\rho,\eta)\,\overset{*}{\Gamma}_{b\beta}\right)-\left(L_{\alpha\beta}^{\gamma}\circ h\circ\overset{*}{\pi}\right)(\rho,\eta)\,\overset{*}{\Gamma}_{b\gamma},\end{aligned}\tag{6.1.6}$$

Moreover, we have:

$$\left[\overset{*}{\tilde{\delta}}_{\alpha},\overset{\cdot}{\tilde{\partial}}{}^{a}\right]_{(\rho,\eta)T\overset{*}{E}}=-\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{\cdot}{\tilde{\partial}}{}^{a}\right)\left((\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\right)\overset{\cdot}{\tilde{\partial}}{}^{b},\tag{6.1.7}$$

and

$$\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left[\overset{*}{\tilde{\delta}}_{\alpha},\overset{*}{\tilde{\delta}}_{\beta}\right]_{(\rho,\eta)T\overset{*}{E}}=\left[\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\alpha}\right),\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\beta}\right)\right]_{T\overset{*}{E}}.\tag{6.1.8}$$

Let $(d\tilde{z}^{\alpha},d\tilde{p}_a)$ be the natural dual (ρ,η)-base.

If we consider the problem of finding a base for the $\mathcal{F}\left(\overset{*}{E}\right)$-module

$$\left(\Gamma\left(\left(V\left(\rho,\eta\right)T\overset{*}{E}\right)^{*},\left(\left(\rho,\eta\right)\tau_{\overset{*}{E}}\right)^{*},\overset{*}{E}\right),+,\cdot\right)$$

of the type

$$\delta\tilde{p}_a = \theta_{a\alpha} d\tilde{z}^\alpha + \omega_a^b d\tilde{p}_b,\ \ a \in \overline{1,r}$$

which satisfies the following conditions:

$$\left\langle \delta\tilde{p}_a, \tilde{\dot{\partial}}^{b} \right\rangle = \delta_a^b \wedge \left\langle \delta\tilde{p}_a, \tilde{\overset{*}{\delta}}_\alpha \right\rangle = 0, \tag{6.1.9}$$

then we obtain the sections

$$\delta\tilde{p}_a = -\left(\rho,\eta\right)\overset{*}{\Gamma}_{a\alpha} d\tilde{z}^\alpha + d\tilde{p}_a, a \in \overline{1,r}. \tag{6.1.10}$$

We observe that their changing rule is tensorial under a change of vector fiber charts.

Definition 6.1.2 The base $\left(d\tilde{z}^\alpha, \delta\tilde{p}_a\right)$ will be called the *adapted dual* (ρ,η)*-base.*

6.2 Remarkable endomorphisms

Now, let us consider the following diagram:

$$\begin{tikzcd}
\overset{*}{E} \arrow[d, "\overset{*}{\pi}"'] & \left(F, \left[,\right]_{F,h}, \left(\rho,\eta\right)\right) \arrow[d, "\nu"] \\
M \arrow[r, "h"] & N
\end{tikzcd}$$

where $(E,\pi,M) \in |\mathbf{B^v}|$ and $\left(\left(F,\nu,N\right), \left[,\right]_{F,h}, \left(\rho,\eta\right)\right)$ is a generalized Lie algebroid.

Definition 6.2.1 For any **Mod**-endomorphism e of

$$\left(\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E}, \left(\rho,\eta\right)\tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot\right)$$

we define the application of Nijenhuis type

$$\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E}, \left(\rho,\eta\right)\tau_{\overset{*}{E}}, \overset{*}{E}\right)^2 \xrightarrow{N_e} \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E}, \left(\rho,\eta\right)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

defined by

$$N_e\left(X,Y\right) = \left[eX, eY\right]_{(\rho,\eta)T\overset{*}{E}} + e^2\left[X,Y\right]_{(\rho,\eta)T\overset{*}{E}} - e\left[eX,Y\right]_{(\rho,\eta)T\overset{*}{E}} - e\left[X,eY\right]_{(\rho,\eta)T\overset{*}{E}},$$

for any $X,Y \in \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E}, \left(\rho,\eta\right)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$.

Remark 6.2.1 The vertical and the horizontal vector subbundles are interior differential systems for the Lie algebroid generalized tangent bundle

$$\left(\left(\left(\rho,\eta\right)T\overset{*}{E}, \left(\rho,\eta\right)\tau_{\overset{*}{E}}, \overset{*}{E}\right), \left[,\right]_{(\rho,\eta)T\overset{*}{E}}, \left(\tilde{\overset{*}{\rho}}, Id_{\overset{*}{E}}\right)\right).$$

These interior differential systems will be called *vertical* and *horizontal interior differential systems.*

6.2.1 Projectors

Definition 6.2.1.1 Any **Mod**-endomorphism e of

$$\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

with the property

$$e^2=e \tag{6.2.1.1}$$

will be called a *projector.*

Example 6.2.1.1 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{V}}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \tilde{Z}^{\alpha}\overset{*}{\tilde{\delta}}_{\alpha}+Y_a\tilde{\partial}^{\cdot a} & \longmapsto & Y_a\tilde{\partial}^{\cdot a}\end{array}$$

is a projector which will be called the *the vertical projector.*

Remark 6.2.1.1 We have $\overset{*}{\mathcal{V}}\left(\overset{*}{\tilde{\delta}}_{\alpha}\right)=0$ and $\overset{*}{\mathcal{V}}\left(\tilde{\partial}^{\cdot a}\right)=\tilde{\partial}^{\cdot a}$. Therefore, it follows

$$\overset{*}{\mathcal{V}}\left(\overset{*}{\tilde{\partial}}_{\alpha}\right)=-\,(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\tilde{\partial}^{\cdot b}.$$

Theorem 6.2.1.1 *A (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is characterized by the existence of a* **Mod**-*endomorphism $\overset{*}{\mathcal{V}}$ of*

$$\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\overset{*}{\mathcal{V}}\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\right)\subset\Gamma\left(\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\right)\\ \overset{*}{\mathcal{V}}\,(X)=X\iff X\in\Gamma\left(\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\right)\end{array} \tag{6.2.1.2}$$

Example 6.2.1.2 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{H}}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \tilde{Z}^{\alpha}\overset{*}{\tilde{\delta}}_{\alpha}+Y_a\tilde{\partial}^{\cdot a} & \longmapsto & \tilde{Z}^{\alpha}\overset{*}{\tilde{\delta}}_{\alpha}\end{array}$$

is a projector which will be called the *horizontal projector.*

Remark 6.2.1.2 We have $\overset{*}{\mathcal{H}}\left(\overset{*}{\tilde{\delta}}_{\alpha}\right)=\overset{*}{\tilde{\delta}}_{\alpha}$ and $\overset{*}{\mathcal{H}}(\tilde{\partial}^{\cdot a})=0$. Therefore, we obtain $\overset{*}{\mathcal{H}}\left(\overset{*}{\tilde{\partial}}_{\alpha}\right)=\overset{*}{\tilde{\delta}}_{\alpha}$.

Theorem 6.2.1.2 *A (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is characterized by the existence of a* **Mod**-*endomorphism $\overset{*}{\mathcal{H}}$ of*

$$\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\subset\Gamma\left(H\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \overset{*}{\mathcal{H}}\,(X)=X\Longleftrightarrow X\in\Gamma\left(H\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right).\end{array}\tag{6.2.1.3}$$

Corollary 6.2.1.1 *A (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is characterized by the existence of a* **Mod**-*endomorphism $\overset{*}{\mathcal{H}}$ of*

$$\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

with the properties:

$$\begin{array}{c}\overset{*}{\mathcal{H}}^{2}=\overset{*}{\mathcal{H}}\\ Ker\left(\overset{*}{\mathcal{H}}\right)=\left(\Gamma\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right).\end{array}\tag{6.2.1.4}$$

Remark 6.2.1.3 For any

$$X\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

we obtain the following unique decomposition

$$X=\overset{*}{\mathcal{H}}X+\overset{*}{\mathcal{V}}X.$$

Proposition 6.2.1.1 *After some calculations we obtain*

$$N_{\overset{*}{\mathcal{V}}}\,(X,Y)=\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{H}}X,\ \overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}=N_{\overset{*}{\mathcal{H}}}\,(X,Y)\,,\tag{6.2.1.5}$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Corollary 6.2.1.2 *The horizontal interior differential system*

$$\left(H\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is involutive if and only if $N_{\overset{*}{\mathcal{V}}}=0$ or $N_{\overset{*}{\mathcal{H}}}=0$.

6.2.2 The almost product structure

Definition 6.2.2.1 Any **Mod**-endomorphism e of

$$\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

with the property

$$e^2=Id \tag{6.2.2.1}$$

will be called the *almost product structure.*

Example 6.2.2.1 The **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{P}}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \tilde{Z}^{\alpha}\overset{*}{\tilde{\delta}}_{\alpha}+Y_a\overset{\cdot\,a}{\tilde{\partial}} & \longmapsto & \tilde{Z}^{\alpha}\overset{*}{\tilde{\delta}}_{\alpha}-Y_a\overset{\cdot\,a}{\tilde{\partial}}\end{array}$$

is an almost product structure**.**

Remark 6.2.2.1 The previous almost product structure has the properties:

$$\begin{array}{l}\overset{*}{\mathcal{P}}=2\overset{*}{\mathcal{H}}-Id;\\ \overset{*}{\mathcal{P}}=Id-2\overset{*}{\mathcal{V}};\\ \overset{*}{\mathcal{P}}=\overset{*}{\mathcal{H}}-\overset{*}{\mathcal{V}}.\end{array} \tag{6.2.2.2}$$

Remark 6.2.2.2 We obtain that $\overset{*}{\mathcal{P}}\left(\overset{*}{\tilde{\delta}}_{\alpha}\right)=\overset{*}{\tilde{\delta}}_{\alpha}$ and $\overset{*}{\mathcal{P}}\left(\overset{\cdot\,a}{\tilde{\partial}}\right)=-\overset{\cdot\,a}{\tilde{\partial}}$. Therefore, it follows

$$\overset{*}{\mathcal{P}}\left(\overset{*}{\tilde{\partial}}_{\alpha}\right)=\overset{*}{\tilde{\delta}}_{\alpha}-\rho\overset{*}{\Gamma}_{b\alpha}\overset{\cdot\,b}{\tilde{\partial}}.$$

Theorem 6.2.2.1 *A (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is characterized by the existence of a* **Mod**-*endomorphism $\overset{*}{\mathcal{P}}$ of*

$$\left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

with the following property:

$$\overset{*}{\mathcal{P}}(X)=-X\Longleftrightarrow X\in\Gamma\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right). \tag{6.2.2.3}$$

Proposition 6.2.2.1 *After some calculations, we obtain*

$$N_{\overset{*}{\mathcal{P}}}(X,Y)=4\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right],$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Corollary 6.2.2.1 *The horizontal interior differential system*

$$\left(H\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is involutive if and only if $N_{\overset{*}{\mathcal{P}}}=0$.

6.2.3 The almost tangent structure

Definition 6.2.3.1 Any **Mod**-endomorphism e of

$$\left(\Gamma((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}),+,\cdot\right)$$

with the property

$$e^2 = 0 \tag{6.2.3.1}$$

will be called the *almost tangent structure.*

Example 6.2.3.1 If $(E,\pi,M)=(F,\nu,N)$ and $g\in\mathbf{Man}\left(\overset{*}{E},E\right)$ such that (g,h) is a $\mathbf{B^v}$-morphism locally invertible, then the **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{J}}_{(g,h)}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \tilde{Z}^a\overset{*}{\tilde{\delta}}_a+Y_b\overset{\cdot\,b}{\tilde{\partial}} & \longmapsto & \left(\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^a\overset{\cdot\,b}{\tilde{\partial}}\end{array}$$

is an almost tangent structure which will be called the *almost tangent structure associated to* $\mathbf{B^v}$*-morphism* (g,h). (See: **Definition 4.4.2.3**)

Remark 6.2.3.1 We obtain that

$$\overset{*}{\mathcal{J}}_{(g,h)}\left(\overset{*}{\tilde{\delta}}_a\right)=\overset{*}{\mathcal{J}}_{(g,h)}\left(\overset{*}{\tilde{\partial}}_a\right)=\left(\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\,b}{\tilde{\partial}}$$

and

$$\overset{*}{\mathcal{J}}_{(g,h)}\left(\overset{\cdot\,b}{\tilde{\partial}}\right)=0.$$

Remark 6.2.3.2 The previous almost tangent structure has the following properties:

$$\begin{array}{rcl}\overset{*}{\mathcal{J}}_{(g,h)}\circ\overset{*}{\mathcal{P}} & = & \overset{*}{\mathcal{J}}_{(g,h)};\\ \overset{*}{\mathcal{P}}\circ\overset{*}{\mathcal{J}}_{(g,h)} & = & -\overset{*}{\mathcal{J}}_{(g,h)};\\ \overset{*}{\mathcal{J}}_{(g,h)}\circ\overset{*}{\mathcal{H}} & = & \overset{*}{\mathcal{J}}_{(g,h)};\\ \overset{*}{\mathcal{H}}\circ\overset{*}{\mathcal{J}}_{(g,h)} & = & 0;\\ \overset{*}{\mathcal{J}}_{(g,h)}\circ\overset{*}{\mathcal{V}} & = & 0;\\ \overset{*}{\mathcal{V}}\circ\overset{*}{\mathcal{J}}_{(g,h)} & = & \overset{*}{\mathcal{J}}_{(g,h)};\\ N_{\overset{*}{\mathcal{J}}_{(g,h)}} & = & 0.\end{array} \tag{6.2.3.2}$$

6.2.4 The almost complex structure

Let us consider in the case $(E,\pi,M)=(F,\nu,N)$.

bk**Definition 6.2.4.1** Any **Mod**-endomorphism e of

$$\left(\Gamma((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}),+,\cdot\right)$$

with the property

$$e^2=-Id \tag{6.2.4.1}$$

will be called the *almost complex structure.*

Example 6.2.4.1 If (g,h) is a $\mathbf{B^v}$-morphism of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ source and (E,π,M) target locally invertible, then the **Mod**-endomorphism

$$\begin{array}{ccc}\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{F}}_{(g,h)}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \tilde{Z}^a\overset{*}{\tilde{\delta}}_a+Y_a\overset{\cdot\,a}{\tilde{\partial}} & \longmapsto & \left(g^{ab}\circ h\circ\overset{*}{\pi}\right)Y_b\overset{*}{\tilde{\delta}}_a-\left(\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^a\overset{\cdot\,b}{\tilde{\partial}}\end{array}$$

is an almost complex structure.

Remark 6.2.4.1 We have

$$\overset{*}{\mathcal{F}}_{(g,h)}\left(\overset{*}{\tilde{\delta}}_a\right)=-\left(\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\,b}{\tilde{\partial}}$$

and

$$\overset{*}{\mathcal{F}}_{(g,h)}\left(\overset{\cdot\,b}{\tilde{\partial}}\right)=\left(g^{ab}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\tilde{\delta}}_a.$$

Therefore, we obtain:

$$\overset{*}{\mathcal{F}}_{(g,h)}\left(\overset{*}{\tilde{\partial}}_c\right)=-\,(\rho,\eta)\,\overset{*}{\Gamma}_{bc}\left(g^{ab}\circ h\circ\overset{*}{\pi}\right)\overset{*}{\tilde{\delta}}_a-\left(\tilde{g}_{bc}\circ h\circ\overset{*}{\pi}\right)\overset{\cdot\,b}{\tilde{\partial}}\,.$$

Remark 6.2.4.2 The previous almost complex structure has the following properties:

$$\begin{array}{rcl}\overset{*}{\mathcal{F}}_{(g,h)}\circ\overset{*}{\mathcal{J}}_{(g,h)} & = & \overset{*}{\mathcal{H}};\\ \overset{*}{\mathcal{F}}_{(g,h)}\circ\overset{*}{\mathcal{H}} & = & -\overset{*}{\mathcal{J}}_{(g,h)};\\ \overset{*}{\mathcal{J}}_{(g,h)}\circ\overset{*}{\mathcal{F}}_{(g,h)} & = & \overset{*}{\mathcal{V}}.\end{array}\tag{6.2.4.2}$$

6.2.5 The (ρ,η)-tension endomorphism

Since

$$\frac{\partial (\rho,\eta)\overset{*}{\Gamma}_{b'\alpha'}}{\partial p_{a'}} = M_{b'}^{b}\circ\overset{*}{\pi}\left(-\rho_{\alpha}^{i}\circ h\frac{\partial M_{b}^{a'}\circ\overset{*}{\pi}}{\partial x^{i}}+\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{bc}}{\partial p_a}M_a^{a'}\circ\overset{*}{\pi}\right)\Lambda_{\alpha'}^{\alpha}\circ h,$$

it results that

$$(\rho,\eta)\overset{*}{\Gamma}_{b'\alpha'}-p_{a'}\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{b'\alpha'}}{\partial p_{a'}} = M_{b'}^{b}\circ\overset{*}{\pi}\left((\rho,\eta)\overset{*}{\Gamma}_{b\alpha}-p_a\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}}{\partial p_a}\right)\Lambda_{\alpha'}^{\alpha}\circ h\circ\overset{*}{\pi},$$

Therefore, we can introduce the following

Definition 6.2.5.1 The **Mod**-endomorphism

$$\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\ \xrightarrow{(\rho,\eta)\overset{*}{\mathbb{H}}}\ \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

defined by

$$(6.2.5.1)\qquad \begin{array}{rcl} (\rho,\eta)\,\overset{*}{\mathbb{H}}\left(\overset{*}{\tilde{\delta}}_{\alpha}\right) & = & \left((\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}-p_a\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{b\alpha}}{\partial p_a}\right)\tilde{\partial}^{\cdot b}, \\ (\rho,\eta)\,\overset{*}{\mathbb{H}}\left(\tilde{\partial}^{\cdot a}\right) & = & 0_{(\rho,\eta)T\overset{*}{E}} \end{array}$$

will be called the *(ρ,η)-tension of (ρ,η)-connection* $(\rho,\eta)\,\overset{*}{\Gamma}$.

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, then we obtain the *tension of connection* $\overset{*}{\Gamma}$.

Proposition 6.2.5.1 *We obtain the following equalities*

$$\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}\circ(\rho,\eta)\,\overset{*}{\mathbb{H}}=0=(\rho,\eta)\,\overset{*}{\mathbb{H}}\circ\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}.$$

6.3 The (ρ,η,h)-torsion and the (ρ,η,h)-curvature of a (ρ,η)-connection

We consider the following diagram:

$$\begin{tikzcd} \overset{*}{E} \arrow[d, "\overset{*}{\pi}"'] & \left(F,[,]_{F,h},(\rho,\eta)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Definition 6.3.1 If $(E,\pi,M)=(F,\nu,N)$, then the $\mathcal{F}\left(\overset{*}{E}\right)$-bilinear application

$$\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)^2\ \xrightarrow{(\rho,\eta,h)\overset{*}{\mathbb{T}}}\ \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

defined by

$$
\begin{aligned}
(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\tilde{\delta}}_b,\overset{*}{\tilde{\delta}}_c\right) &= \left(\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{bc}}{\partial p_a}-\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{cb}}{\partial p_a}-L_{bc}^a\circ h\circ\overset{*}{\pi}\right)\overset{*}{\tilde{\delta}}_a;\\
(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\tilde{\delta}}_b,\overset{\cdot\, c}{\tilde{\partial}}\right) &= 0=(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{\cdot\, b}{\tilde{\partial}},\overset{*}{\tilde{\delta}}_c\right);\\
(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{\cdot\, b}{\tilde{\partial}},\overset{\cdot\, c}{\tilde{\partial}}\right) &= 0;
\end{aligned}
\tag{6.3.1}
$$

will be called the (ρ,η,h)-*torsion associated to* (ρ,η)-*connection* $(\rho,\eta)\,\Gamma$.

In particular, if $h=Id_M$, then we obtain the (ρ,η)-*torsion associated to* (ρ,η)-*connection* $(\rho,\eta)\,\overset{*}{\Gamma}$.

Moreover, if $(\rho,\eta)=(Id_{TM},Id_M)$, then we obtain the *torsion associated to connection* $\overset{*}{\Gamma}$.

Remark 6.3.1 If $(\rho,\eta,h)\overset{*}{\mathbb{T}}$ is the (ρ,η,h)-torsion associated to (ρ,η)-connection $(\rho,\eta)\,\overset{*}{\Gamma}$, then

$$
(\rho,\eta,h)\overset{*}{\mathbb{T}}(X,Y)=-\,(\rho,\eta,h)\overset{*}{\mathbb{T}}(Y,X),
\tag{6.3.2}
$$

for any $X,Y\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Definition 6.3.2 If we consider the notation

$$
(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^a_{\ bc}\overset{put}{=}\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{bc}}{\partial p_a}-\frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{cb}}{\partial p_a}-L_{bc}^a\circ h\circ\overset{*}{\pi}
\tag{6.3.3}
$$

then the tensor field

$$
(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^a_{\ bc}\overset{*}{\tilde{\delta}}_a\otimes d\tilde{z}^b\otimes d\tilde{z}^c
\tag{6.3.4}
$$

will be called the (ρ,η,h)-*torsion tensor field associated to* (ρ,η)-*connection* $(\rho,\eta)\,\overset{*}{\Gamma}$.

Proposition 6.3.1 *We obtain*

$$
\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}\circ(\rho,\eta)\overset{*}{\mathbb{T}}=0
$$

and

$$
\begin{aligned}
(\rho,\eta)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}X,Y\right) &= (\rho,\eta)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}X,\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}Y\right)\\
&= (\rho,\eta)\overset{*}{\mathbb{T}}\left(X,\overset{*}{\mathcal{J}}_{\left(Id_{\overset{*}{E}},Id_M\right)}Y\right),
\end{aligned}
$$

for any $X, Y \in \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Theorem 6.3.1 *Using the* (ρ,η)*-tension tensor field*

$$(6.3.5)\qquad \left(\rho,\eta\right)\overset{*}{\mathbb{H}}_{ba}\overset{\cdot}{\tilde{\partial}}^{b}\otimes d\tilde{z}^{a}=\left(\left(\rho,\eta\right)\overset{*}{\Gamma}_{ba}-p_{c}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{ba}}{\partial p_{c}}\right)\overset{\cdot}{\tilde{\partial}}^{b}\otimes d\tilde{z}^{a},$$

and the (ρ,η,h)*-deflection of the* (ρ,η)*-connection* $\left(\rho,\eta\right)\overset{*}{\Gamma}$

$$(6.3.6)\qquad \left(\rho,\eta,h\right)\overset{*}{\mathbb{D}}_{bc}=-\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}+p_{a}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{cb}}{\partial p_{a}}+p_{a}\cdot L_{bc}^{a}\circ h\circ\overset{*}{\pi},$$

we obtain that $(\rho,\eta,h)\overset{*}{\mathbb{D}}_{bc}$=0 *if and only if* $(\rho,\eta)\overset{*}{\mathbb{H}}_{bc}$=0 *and* $(\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{bc}^{a}$=0.

Proof. If $\left(\rho,\eta,h\right)\overset{*}{\mathbb{D}}_{bc}$=0, then deriving with respect to p_a, we obtain:

$$-\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}}{\partial p_{a}}+\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{cb}}{\partial p_{a}}+L_{bc}^{a}\circ h\circ\overset{*}{\pi}=0\Longleftrightarrow\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}{}_{bc}^{a}=0.$$

The equality $\left(\rho,\eta,h\right)\overset{*}{\mathbb{D}}_{bc}$=0 implies:

$$(1)\qquad \left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}=p_{a}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{cb}}{\partial p_{a}}+p_{a}L_{bc}^{a}\circ h\circ\overset{*}{\pi}.$$

Since

$$\begin{aligned}\left(\rho,\eta\right)\overset{*}{\mathbb{H}}_{bc}&=\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}-p_{a}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}}{\partial p_{a}}\\&=p_{a}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{cb}}{\partial p_{a}}-p_{a}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}}{\partial p_{a}}+p_{a}L_{bc}^{a}\circ h\circ\overset{*}{\pi}=p_{a}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}{}_{bc}^{a}\end{aligned}$$

it results the equality $\left(\rho,\eta\right)\overset{*}{\mathbb{H}}_{bc}$=0.

Conservely, if $\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}{}_{bc}^{a}$=0, then, multiplying with p_a, we obtain:

$$(2)\qquad p_{a}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{cb}}{\partial p_{a}}-p_{a}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}}{\partial p_{a}}+p_{a}L_{bc}^{a}\circ h\circ\overset{*}{\pi}=0.$$

The equality $\left(\rho,\eta\right)\overset{*}{\mathbb{H}}_{bc}$=0 is equivalent with:

$$(3)\qquad \left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}=p_{a}\frac{\partial\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}}{\partial p_{a}}.$$

Using (2) and (3), it results the equality $\left(\rho,\eta,h\right)\overset{*}{\mathbb{D}}_{bc}=0$. *q.e.d.*

Definition 6.3.3 The $\mathcal{F}\left(\overset{*}{E}\right)$-bilinear application

$$\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)^{2}\ \xrightarrow{\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}}\ \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

defined by

$$\begin{aligned}
&\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(\overset{*}{\tilde{\delta}}_{\alpha},\overset{*}{\tilde{\delta}}_{\beta}\right)=\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}_{b\ \ \alpha\beta}\overset{\cdot\, b}{\tilde{\partial}}\ ;\\
&\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(\overset{*}{\tilde{\delta}}_{\alpha},\overset{\cdot\, b}{\tilde{\partial}}\right)=0=\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(\overset{\cdot\, b}{\tilde{\partial}},\overset{*}{\tilde{\delta}}_{\alpha}\right);\\
&\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(\overset{\cdot\, a}{\tilde{\partial}},\overset{\cdot\, b}{\tilde{\partial}}\right)=0;
\end{aligned}\tag{6.3.7}$$

will be called the (ρ,η,h)-*curvature associated to* (ρ,η)-*connection* $\left(\rho,\eta\right)\overset{*}{\Gamma}$.

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, then we obtain the *curvature associated to connection* $\overset{*}{\Gamma}$.

Remark 6.3.2 If $\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}$ is the (ρ,η,h)-curvature associated to (ρ,η)-connection $\left(\rho,\eta\right)\overset{*}{\Gamma}$, then

$$\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)=-\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(Y,X\right),\tag{6.3.8}$$

for any $X,Y\in\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Definition 4.3.4 The tensor field

$$\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}_{b\ \ \alpha\beta}\overset{\cdot\, b}{\tilde{\partial}}\otimes d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}\tag{4.3.9}$$

will be called the (ρ,η,h)-*curvature tensor field associated to the* (ρ,η)-*connection* $\left(\rho,\eta\right)\overset{*}{\Gamma}$.

Using equality (4.1.5) we obtain

Remark 6.3.3 The horizontal interior differential system $\left(H\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$ is involutive if and only if the (ρ,η,h)-curvature tensor field associated to the (ρ,η)-connection $\left(\rho,\eta\right)\overset{*}{\Gamma}$ is null.

6.4 Tensor d-fields. Distinguished linear (ρ,η)-connections

We consider the following diagram:

$$\begin{array}{ccc}
\overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\
\downarrow{\scriptstyle\overset{*}{\pi}} & & \downarrow{\scriptstyle\nu}\\
M & \xrightarrow{h} & N
\end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid.

Let

$$\left(\mathcal{T}\ {}^{p,r}_{q,s}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$$

be the $\mathcal{F}\left(\overset{*}{E}\right)$-module of tensor fields by $\binom{p,r}{q,s}$-type from the generalized tangent bundle

$$\left(H\,(\rho,\eta)\,T\overset{*}{E}\oplus V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right).$$

An arbitrarily tensor field T is written by the form:

$$T=T^{\alpha_1\ldots\alpha_p a_1\ldots a_r}_{\beta_1\ldots\beta_q b_1\ldots b_s}\overset{*}{\tilde{\delta}}_{\alpha_1}\otimes\ldots\otimes\overset{*}{\tilde{\delta}}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes\ldots\otimes d\tilde{z}^{\beta_q}\otimes\tilde{\partial}^{\cdot\,b_1}\otimes\ldots\otimes\tilde{\partial}^{\cdot\,b_s}\otimes\delta\tilde{p}_{a_1}\otimes\ldots\otimes\delta\tilde{p}_{a_r}.$$

Let

$$\left(\ \mathcal{T}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot,\otimes\right)$$

be the tensor fields algebra of generalized tangent bundle $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

If $T_1{\in}\mathcal{T}^{p_1,r_1}_{q_1,s_1}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ and $T_2{\in}\mathcal{T}^{p_2,r_2}_{q_2,s_2}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$, then the components of product tensor field $T_1\otimes T_2$ are the products of local components of T_1 and T_2.

Therefore, we obtain $T_1\otimes T_2\in\mathcal{T}^{p_1+p_2,r_1+r_2}_{q_1+q_2,s_1+s_2}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Let $\mathcal{DT}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ be the family of tensor fields

$$T\in\mathcal{T}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

for which there exists

$$T_1{\in}\mathcal{T}^{p,0}_{q,0}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and

$$T_2{\in}\mathcal{T}^{0,r}_{0,s}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

such that $T=T_1+T_2$.

The $\mathcal{F}\left(\overset{*}{E}\right)$-module $\left(\mathcal{DT}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot\right)$ will be called the *module of distinguished tensor fields* or the *module of tensor d-fields.*

Remark 6.4.1 The elements of

$$\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

respectively

$$\Gamma\left(\left((\rho,\eta)\,T\overset{*}{E}\right)^{*},((\rho,\eta)\tau_{\overset{*}{E}})^{*},\overset{*}{E}\right)$$

are tensor d-fields.

Definition 6.4.1 Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and let

$$(X,T)\xrightarrow{(\rho,\eta)\overset{*}{D}}(\rho,\eta)\overset{*}{D}_XT \tag{6.4.1}$$

be a covariant (ρ,η)-derivative for the tensor algebra of generalized tangent bundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

which preserves the horizontal and vertical distributions by parallelism.

If $\left(U,\overset{*}{s}_U\right)$ is a vector local $(m+r)$-chart for $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then the real local functions

$$\left((\rho,\eta)\,\overset{*}{H}{}^{\alpha}_{\beta\gamma},(\rho,\eta)\,\overset{*}{H}{}^{a}_{b\gamma},(\rho,\eta)\,\overset{*}{V}{}^{\alpha c}_{\beta},(\rho,\eta)\,\overset{*}{V}{}^{bc}_{a}\right)$$

defined on $\overset{*}{\pi}{}^{-1}(U)$ and determined by the following equalities:

$$\begin{array}{ll}(\rho,\eta)\,\overset{*}{D}_{\overset{*}{\tilde{\delta}}_\gamma}\overset{*}{\tilde{\delta}}_\beta=(\rho,\eta)\,\overset{*}{H}{}^{\alpha}_{\beta\gamma}\overset{*}{\tilde{\delta}}_\alpha, & (\rho,\eta)\,\overset{*}{D}_{\overset{*}{\tilde{\delta}}_\gamma}\tilde{\dot{\partial}}^{a}=(\rho,\eta)\,\overset{*}{H}{}^{a}_{b\gamma}\tilde{\dot{\partial}}^{b}\\ (\rho,\eta)\,\overset{*}{D}_{\tilde{\dot{\partial}}^{c}}\overset{*}{\tilde{\delta}}_\beta=(\rho,\eta)\,\overset{*}{V}{}^{\alpha c}_{\beta}\overset{*}{\tilde{\delta}}_\alpha, & (\rho,\eta)\,\overset{*}{D}_{\tilde{\dot{\partial}}^{c}}\tilde{\dot{\partial}}^{b}=(\rho,\eta)\,\overset{*}{V}{}^{bc}_{a}\tilde{\dot{\partial}}^{a}\end{array} \tag{6.4.2}$$

are the components of a linear (ρ,η)-connection

$$\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$$

for the generalized tangent bundle $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ which will be called the *distinguished linear (ρ,η)-connection.*

If $h=Id_M$, then the distinguished linear (Id_{TM},Id_M)-connection will be called the *distinguished linear connection.*

The components of a distinguished linear connection $\left(\overset{*}{H},\overset{*}{V}\right)$ will be denoted

$$\left(\overset{*}{H}{}^{i}_{jk},\overset{*}{H}{}^{a}_{bk},\overset{*}{V}{}^{ic}_{j},\overset{*}{V}{}^{bc}_{a}\right).$$

Theorem 6.4.1 *If $((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V})$ is a distinguished linear (ρ,η)-connection for the generalized tangent bundle $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$, then its components satisfy the*

change relations:

$$
(6.4.3)\quad
\begin{aligned}
(\rho,\eta)\overset{*}{H}{}^{\acute{\alpha}}_{\acute{\beta}\acute{\gamma}} &= \Lambda^{\acute{\alpha}}_{\alpha}\circ h\circ\overset{*}{\pi}\left[\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)\left(\Lambda^{\alpha}_{\acute{\beta}}\circ h\circ\overset{*}{\pi}\right)+\right.\\
&\left.+(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\gamma}\cdot\Lambda^{\beta}_{\acute{\beta}}\circ h\circ\overset{*}{\pi}\right]\cdot\Lambda^{\gamma}_{\acute{\gamma}}\circ h\circ\overset{*}{\pi},\\
(\rho,\eta)\overset{*}{H}{}^{\acute{a}}_{\acute{b}\acute{\gamma}} &= M^{\acute{a}}_{a}\circ\overset{*}{\pi}\left[\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)\left(M^{a}_{\acute{b}}\circ\overset{*}{\pi}\right)+\right.\\
&\left.+(\rho,\eta)\overset{*}{H}{}^{a}_{b\gamma}\cdot M^{b}_{\acute{b}}\circ\overset{*}{\pi}\right]\cdot\Lambda^{\gamma}_{\acute{\gamma}}\circ h\circ\overset{*}{\pi},\\
(\rho,\eta)\overset{*}{V}{}^{\acute{\alpha}\acute{c}}_{\acute{\beta}} &= \Lambda^{\acute{\alpha}}_{\alpha}\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\beta}\cdot\Lambda^{\beta}_{\acute{\beta}}\circ h\circ\overset{*}{\pi}\cdot M^{\acute{c}}_{c}\circ\overset{*}{\pi},\\
(\rho,\eta)\overset{*}{V}{}^{\acute{a}\acute{c}}_{\acute{b}} &= M^{\acute{a}}_{a}\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{V}{}^{ac}_{b}\cdot M^{b}_{\acute{b}}\circ\overset{*}{\pi}\cdot M^{\acute{c}}_{c}\circ\overset{*}{\pi}.
\end{aligned}
$$

The components of a distinguished linear connection $\left(\overset{*}{H},\overset{*}{V}\right)$ *verify the change relations:*

$$
(6.4.3)'\quad
\begin{aligned}
\overset{*}{H}{}^{\acute{i}}_{\acute{j}\acute{k}} &= \frac{\partial x^{\acute{i}}}{\partial x^{i}}\circ\overset{*}{\pi}\cdot\left[\frac{\delta}{\delta x^{k}}\left(\frac{\partial x^{i}}{\partial x^{\acute{j}}}\circ\overset{*}{\pi}\right)+\overset{*}{H}{}^{i}_{jk}\cdot\frac{\partial x^{j}}{\partial x^{\acute{j}}}\circ\overset{*}{\pi}\right]\cdot\frac{\partial x^{k}}{\partial x^{\acute{k}}}\circ\overset{*}{\pi},\\
\overset{*}{H}{}^{\acute{a}}_{\acute{b}\acute{k}} &= M^{\acute{a}}_{a}\circ\overset{*}{\pi}\cdot\left[\frac{\delta}{\delta x^{k}}\left(M^{a}_{\acute{b}}\circ\overset{*}{\pi}\right)+\overset{*}{H}{}^{a}_{bk}\cdot M^{b}_{\acute{b}}\circ\overset{*}{\pi}\right]\cdot\frac{\partial x^{k}}{\partial x^{\acute{k}}}\circ\overset{*}{\pi},\\
\overset{*}{V}{}^{\acute{i}\acute{c}}_{\acute{j}} &= \frac{\partial x^{\acute{i}}}{\partial x^{i}}\circ\overset{*}{\pi}\cdot\overset{*}{V}{}^{ic}_{j}\frac{\partial x^{j}}{\partial x^{\acute{j}}}\circ\overset{*}{\pi}\cdot M^{c}_{\acute{c}}\circ\overset{*}{\pi},\\
\overset{*}{V}{}^{\acute{a}\acute{c}}_{\acute{b}} &= M^{\acute{a}}_{a}\circ\overset{*}{\pi}\cdot\overset{*}{V}{}^{ac}_{b}\cdot M^{b}_{\acute{b}}\circ\overset{*}{\pi}M^{c}_{\acute{c}}\circ\overset{*}{\pi}.
\end{aligned}
$$

Example 6.4.1 If $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is endowed with the (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$, then the local real functions

$$\left(\frac{\partial\,(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}}{\partial p_a},\frac{\partial\,(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}}{\partial p_a},0,0\right)$$

are the components of a distinguished linear (ρ,η)-connection for the generalized tangent bundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

which will by called the *Berwald linear* (ρ,η)*-connection.*

The Berwald linear (Id_{TM}, Id_M)-connection will be called the *Berwald linear connection.*

Theorem 6.4.2 *If the generalized tangent bundle* $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ *is endowed with a distinguished linear* (ρ,η)*-connection* $((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V})$*, then, for any*

$$X=\tilde{Z}^{\gamma}\overset{*}{\tilde{\delta}}_{\gamma}+Y_a\overset{\cdot}{\tilde{\partial}}{}^{a}\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and for any

$$T\in\mathcal{T}\,^{pr}_{qs}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

we obtain the formula:

$$\begin{array}{l}(\rho,\eta)\,D_X\left(T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_s}\overset{*}{\tilde{\delta}}_{\alpha_1}\otimes...\otimes\overset{*}{\tilde{\delta}}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes...\otimes\right.\\ \qquad\left.\otimes d\tilde{z}^{\beta_q}\otimes\tilde{\partial}^{.\,b_1}\otimes...\otimes\tilde{\partial}^{.\,b_s}\otimes\delta\tilde{p}_{a_1}\otimes...\otimes\delta\tilde{p}_{a_r}\right)=\\ \qquad=\tilde{Z}^{\gamma}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_s|\gamma}\overset{*}{\tilde{\delta}}_{\alpha_1}\otimes...\otimes\overset{*}{\tilde{\delta}}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes...\otimes d\tilde{z}^{\beta_q}\otimes\tilde{\partial}^{.\,b_1}\otimes...\otimes\\ \qquad\otimes\tilde{\partial}^{.\,b_s}\otimes\delta\tilde{p}_{a_1}\otimes...\otimes\delta\tilde{p}_{a_r}+Y_cT^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_s}\mid^c\overset{*}{\tilde{\delta}}_{\alpha_1}\otimes...\otimes\\ \qquad\otimes\overset{*}{\tilde{\delta}}_{\alpha_p}\otimes d\tilde{z}^{\beta_1}\otimes...\otimes d\tilde{z}^{\beta_q}\otimes\tilde{\partial}^{.\,b_1}\otimes...\otimes\tilde{\partial}^{.\,b_s}\otimes\delta\tilde{p}_{a_1}\otimes...\otimes\delta\tilde{p}_{a_r},\end{array}$$

where

$$\begin{array}{l}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_s|\gamma}=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_s}\\ \qquad+(\rho,\eta)\,\overset{*}{H}{}^{\alpha_1}_{\alpha\gamma}T^{\alpha\alpha_2...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_s}+...+(\rho,\eta)\,\overset{*}{H}{}^{\alpha_p}_{\alpha\gamma}T^{\alpha_1...\alpha_{p-1}\alpha a_1...a_r}_{\beta_1...\beta_qb_1...b_s}\\ \qquad-(\rho,\eta)\,\overset{*}{H}{}^{\beta}_{\beta_1\gamma}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta\beta_2...\beta_qb_1...b_s}-...-(\rho,\eta)\,\overset{*}{H}{}^{\beta}_{\beta_q\gamma}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_{q-1}\beta b_1...b_s}\\ \qquad-(\rho,\eta)\,\overset{*}{H}{}^{a_1}_{a\gamma}T^{\alpha_1...\alpha_paa_2...a_r}_{\beta_1...\beta_qb_1...b_s}-...-(\rho,\eta)\,\overset{*}{H}{}^{a_r}_{a\gamma}T^{\alpha_1...\alpha_pa_1...a_{r-1}a}_{\beta_1...\beta_qb_1...b_s}\\ \qquad+(\rho,\eta)\,\overset{*}{H}{}^{b}_{b_1\gamma}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qbb_2...b_s}+...+(\rho,\eta)\,\overset{*}{H}{}^{b}_{b_s\gamma}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_{s-1}b}\end{array}$$

and

$$\begin{array}{l}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_s}\mid^c=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\partial}^{.\,c}\right)T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_s}+\\ \qquad+(\rho,\eta)\,\overset{*}{V}{}^{\alpha_1c}_{\alpha}\,T^{\alpha\alpha_2...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_s}+...+(\rho,\eta)\,\overset{*}{V}{}^{\alpha_pc}_{\alpha}\,T^{\alpha_1...\alpha_{p-1}\alpha a_1...a_r}_{\beta_1...\beta_qb_1...b_s}\\ \qquad-(\rho,\eta)\,\overset{*}{V}{}^{\beta c}_{\beta_1}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta\beta_2...\beta_qb_1...b_s}-...-(\rho,\eta)\,\overset{*}{V}{}^{\beta c}_{\beta_q}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_{q-1}\beta b_1...b_s}\\ \qquad-(\rho,\eta)\,\overset{*}{V}{}^{a_1c}_{a}\,T^{\alpha_1...\alpha_paa_2...a_r}_{\beta_1...\beta_qb_1...b_s}-...-(\rho,\eta)\,\overset{*}{V}{}^{a_rc}_{a}\,T^{\alpha_1...\alpha_pa_1...a_{r-1}a}_{\beta_1...\beta_qb_1...b_s}\\ \qquad+(\rho,\eta)\,\overset{*}{V}{}^{bc}_{b_1}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qbb_2...b_s}...+(\rho,\eta)\,\overset{*}{V}{}^{bc}_{b_s}T^{\alpha_1...\alpha_pa_1...a_r}_{\beta_1...\beta_qb_1...b_{s-1}b}.\end{array}$$

Definition 6.4.2 We assume that $(E,\pi,M)=(F,\nu,N)$.

If $(\rho,\eta)\,\overset{*}{\Gamma}$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and

$$\left((\rho,\eta)\,\overset{*}{H}{}^{a}_{bc},(\rho,\eta)\,\overset{*}{\tilde{H}}{}^{a}_{bc},(\rho,\eta)\,\overset{*}{V}{}^{ac}_{b},(\rho,\eta)\,\overset{*}{\tilde{V}}{}^{ac}_{b}\right)$$

are the components of a distinguished linear (ρ,η)-connection for the generalized tangent bundle $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ such that

$$(\rho,\eta)\,\overset{*}{H}{}^{a}_{bc}=(\rho,\eta)\,\overset{*}{\tilde{H}}{}^{a}_{bc}\text{ and }(\rho,\eta)\,\overset{*}{V}{}^{ac}_{b}=(\rho,\eta)\,\overset{*}{\tilde{V}}{}^{ac}_{b}\,,$$

then we will say that *the generalized tangent bundle* $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ *is endowed with a normal distinguished linear* (ρ,η)*-connection on components*

$$\left((\rho,\eta)\,\overset{*}{H}{}^{a}_{bc},(\rho,\eta)\,\overset{*}{V}{}^{ac}_{b}\right).$$

The components of a normal distinguished linear (Id_{TM},Id_M)-connection $\left(\overset{*}{H},\overset{*}{V}\right)$ will be denoted $\left(\overset{*}{H}{}^{i}_{jk},\overset{*}{V}{}^{i}_{jk}\right)$.

6.5 The lift of accelerations for a differentiable curve

We consider the following diagram:

$$\begin{array}{ccc}\overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\ \overset{*}{\pi}\downarrow & & \downarrow\nu\\ M & \xrightarrow{\;\;h\;\;} & N\end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,N),[,]_{F,h},(\rho,\eta)\right)\in|\mathbf{GLA}|$.

Let $(\rho,\eta)\,\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

We admit that $\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$ is a distinguished linear (ρ,η)-connection for the vector bundle $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Let $g\in\mathbf{Man}\left(\overset{*}{E},E\right)$ be such that (g,h) is a $\mathbf{B^v}$-morphism of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ source and (E,π,M) target.

Let

$$\begin{array}{ccc}I & \xrightarrow{\dot{c}} & \overset{*}{E}_{|\,\mathrm{Im}(\eta\circ h\circ c)}\\ t & \longmapsto & p_a(t)\,s^a(\eta\circ h\circ c(t))\end{array}\tag{6.5.1}$$

be the (g,h)-lift of differentiable curve $I\xrightarrow{c}M$.

Definition 6.5.1 The differentiable curve

$$\begin{array}{ccc}I & \xrightarrow{\ddot{c}} & (\rho,\eta)\,T\overset{*}{E}_{|\,\mathrm{Im}\,\dot{c}}\\ t & \longmapsto & g^{\alpha a}(h\circ c(t))\,p_a(t)\,\overset{*}{\tilde{\partial}}_{\alpha}(\dot{c}(t))+\dfrac{dp_a(t)}{dt}\overset{\cdot}{\tilde{\partial}}{}^{a}(\dot{c}(t))\end{array}\tag{6.5.2}$$

will be called the *differentiable* (g,h)*-lift of accelerations of the differentiable curve* c.

The section

$$\begin{array}{lccl}(6.5.3) & \mathrm{Im}\,(\dot{c}) & \xrightarrow{\overset{*}{u}(c,\dot{c},\ddot{c})} & (\rho,\eta)\,T\overset{*}{E}_{|\,\mathrm{Im}(\dot{c})} \\ & \dot{c}\,(t) & \longmapsto & g^{\alpha a}\,(\eta\circ h\circ c\,(t))\cdot p_a\,(t)\,\overset{*}{\tilde{\delta}}_{\alpha}\,(\dot{c}\,(t)) + \dfrac{dp_a\,(t)}{dt}\tilde{\overset{\cdot}{\partial}}{}^{a}\,(\dot{c}\,(t))\end{array}$$

will be called the *canonical section associated to the triple* $(c,\dot{c},\ddot{c})$.

Remark 6.5.1 For any $t\in I$, we obtain:

$$\begin{array}{ll}(6.5.4) & \begin{array}{l}\overset{*}{u}\,(c,\dot{c},\ddot{c})\,(\dot{c}\,(t)) = g^{\alpha a}\,(\eta\circ h\circ c\,(t))\cdot p_a\,(t)\,\overset{*}{\tilde{\delta}}_{\alpha}\,(\dot{c}\,(t)) + \dfrac{dp_b\,(t)}{dt}\tilde{\overset{\cdot}{\partial}}{}^{b}\,(\dot{c}\,(t)) \\ \qquad + (\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\circ\overset{*}{u}\,(c,\dot{c})\circ\eta\circ h\circ c\,(t)\cdot g^{\alpha a}\circ h\circ c\,(t)\cdot p_a\,(t)\,\tilde{\overset{\cdot}{\partial}}{}^{b}\,(\dot{c}\,(t)).\end{array}\end{array}$$

We observe easily that $\overset{*}{u}\,(c,\dot{c},\ddot{c})\,(\dot{c}\,(t))\in H\,(\rho,\eta)\,T\overset{*}{E}_{|\,\mathrm{Im}(\dot{c})}$if and only if the components functions $\left(p_a,\ a\in\overline{1,r}\right)$ are solutions for the differentiable equations

$$(6.5.5)\qquad \frac{du_b}{dt} + (\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\circ\overset{*}{u}\,(c,\dot{c})\circ\eta\circ h\circ c\cdot g^{\alpha a}\circ h\circ c\cdot u_a,\ a\in\overline{1,r}.$$

Remark 6.5.2 In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, then, using the differentiable (g,Id_M)-lift

$$\begin{array}{lccl}(6.5.6) & I & \xrightarrow{\dot{c}} & T\overset{*}{M} \\ & t & \longmapsto & \tilde{g}_{ji}\,(c\,(t))\,\dfrac{dc^j\,(t)}{dt}\cdot dx^i\,(c\,(t))\,,\end{array}$$

we obtain the (g,Id_M)-lift of accelerations of the differentiable curve c as being

$$\begin{array}{lccl}(6.5.7) & I & \xrightarrow{\ddot{c}} & (Id_{TM},Id_M)\,T\overset{*}{E}_{|\,\mathrm{Im}(\dot{c})} \\ & t & \longmapsto & \dfrac{dc^i\,(t)}{dt}\cdot\dfrac{\partial}{\partial\tilde{z}^i}\,(\dot{c}\,(t)) + \tilde{g}_{ji}\,(c\,(t))\,\dfrac{dc^j\,(t)}{dt}\cdot\dfrac{\partial}{\partial\tilde{p}_i}\,(\dot{c}\,(t))\end{array}$$

Definition 6.5.2 If the component functions

$$\left((g^{\alpha a}\circ h\circ c)\cdot p_a,\ a\in\overline{1,r}\right)$$

are solutions for the differentiable system of equations

$$(6.5.8)\qquad \frac{dz^{\alpha}}{dt} + (\rho,\eta)\,\overset{*}{H}{}^{\alpha}_{\beta\gamma}\circ\overset{*}{u}\,(c,\dot{c})\circ\eta\circ h\circ c\cdot z^{\beta}\cdot z^{\gamma}=0,\ \alpha\in\overline{1,p},$$

then the differentiable curve $\dot{c}$ will be called *horizontal parallel with respect to the distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$.

If the component functions $\left(p_a,\ a\in\overline{1,r}\right)$ are solutions for the differentiable system of equations

$$(6.5.9)\qquad \frac{du_b}{dt} - (\rho,\eta)\,\overset{*}{V}{}_{b}^{ac}\circ\overset{*}{u}\,(c,\dot{c})\circ\eta\circ h\circ c\cdot u_a\cdot u_c=0,\ b\in\overline{1,r}.$$

then the differentiable curve $\dot{c}$ will be called *vertical parallel with respect to the distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$.

Remark 6.5.3 In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, then the differentiable (g,Id_M)-lift (6.5.6) is horizontal parallel with respect to the distinguished linear connection $\left(\overset{*}{H},\overset{*}{V}\right)$ if the component functions $\left(\frac{dc^j(t)}{dt},\ i\in\overline{1,m}\right)$ are solutions for the differentiable system of equations

$$\frac{dz^i(t)}{dt}+\overset{*}{H}{}^i_{jk}\circ\overset{*}{u}(c,\dot{c})\circ c\cdot z^j\cdot z^k=0,\ i\in\overline{1,m}. \tag{6.5.11}$$

Moreover, the differentiable (g,Id_M)-lift (4.5.6) is vertical parallel with respect to the distinguished linear connection $\left(\overset{*}{H},\overset{*}{V}\right)$ if the component functions $\left(\tilde{g}_{ji}\circ c\cdot\frac{dc^j}{dt},\ i\in\overline{1,m}\right)$ are solutions for the differentiable system of equations

$$\frac{du_j}{dt}+\overset{*}{V}{}_j^{ik}\circ\overset{*}{u}(c,\dot{c})\circ c\cdot u_i\cdot u_k=0,\ j\in\overline{1,m}. \tag{6.5.12}$$

6.6 The (ρ,η,h)-torsion and the (ρ,η,h)-curvature of a distinguished linear (ρ,η)-connection

We consider the following diagram:

$$\begin{tikzcd} \overset{*}{E} \arrow[d, "\overset{*}{\pi}"'] & \left(F,[,]_{F,h},(\rho,\eta)\right) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $\left((F,\nu,M),[,]_{F,h},(\rho,\eta)\right)\in|\mathbf{GLA}|$.

Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and let

$$\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$$

be a distinguished linear (ρ,η)-connection for the generalized tangent bundle

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right).$$

Definition 6.6.1 The application

$$\begin{array}{ccc} \Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)^2 & \xrightarrow{(\rho,\eta,h)\mathbb{T}} & \Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right) \\ (X,Y) & \longmapsto & (\rho,\eta,h)\overset{*}{\mathbb{T}}(X,Y) \end{array}$$

defined by

$$(\rho,\eta,h)\overset{*}{\mathbb{T}}(X,Y)=(\rho,\eta)\overset{*}{D}_XY-(\rho,\eta)\overset{*}{D}_YX-[X,Y]_{(\rho,\eta)T\overset{*}{E}}, \tag{6.6.1}$$

for any $X,Y\in\Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$, will be called the (ρ,η,h)*-torsion associated to distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$.

The applications

$$\overset{*}{\mathcal{H}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{H}}(\cdot),\overset{*}{\mathcal{H}}(\cdot)\right),\overset{*}{\mathcal{H}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{H}}(\cdot),\overset{*}{\mathcal{H}}(\cdot)\right),...,\overset{*}{\mathcal{V}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{V}}(\cdot),\overset{*}{\mathcal{V}}(\cdot)\right)$$

are called $\overset{*}{\mathcal{H}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right),\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right),...,\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{V}}\overset{*}{\mathcal{V}}\right)$ (ρ,η,h)*-torsions associated to distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$.

Proposition 6.6.1 *The* (ρ,η,h)*-torsion* $(\rho,\eta,h)\overset{*}{\mathbb{T}}$ *associated to distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$*, is* $\mathbb{R}$*-bilinear and antisymmetric in the lower indices.*

Using the notations:

$$\begin{aligned}
\overset{*}{\mathcal{H}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\tilde{\delta}}_\gamma,\overset{*}{\tilde{\delta}}_\beta\right)&=(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{\alpha}{}_{\beta\gamma}\overset{*}{\tilde{\delta}}_\alpha,\\
\overset{*}{\mathcal{V}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{*}{\tilde{\delta}}_\gamma,\overset{*}{\tilde{\delta}}_\beta\right)&=(\rho,\eta,h)\overset{*}{\mathbb{T}}_{b\beta\gamma}\overset{\cdot\,b}{\tilde{\partial}},\\
\overset{*}{\mathcal{H}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{\cdot\,c}{\tilde{\partial}},\overset{*}{\tilde{\delta}}_\beta\right)&=(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\alpha}{}_{\beta}{}^{c}\overset{*}{\tilde{\delta}}_\alpha,\\
\overset{*}{\mathcal{V}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{\cdot\,c}{\tilde{\partial}},\overset{*}{\tilde{\delta}}_\beta\right)&=(\rho,\eta,h)\overset{*}{\mathbb{P}}_{b\beta}{}^{c}\overset{\cdot\,b}{\tilde{\partial}},\\
\overset{*}{\mathcal{V}}(\rho,\eta,h)\overset{*}{\mathbb{T}}\left(\overset{\cdot\,c}{\tilde{\partial}},\overset{\cdot\,b}{\tilde{\partial}}\right)&=(\rho,\eta,h)\overset{*}{\mathbb{S}}_a{}^{bc}\overset{\cdot\,a}{\tilde{\partial}},
\end{aligned} \tag{6.6.2}$$

we can easily prove the following

Theorem 6.6.1 *The* (ρ,η,h)*-torsion* $(\rho,\eta,h)\overset{*}{\mathbb{T}}$ *associated to the distinguished linear* (ρ,η)*-connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$*, is characterized by the tensor fields with local*

components:

$$(6.6.3)\quad \begin{aligned} (\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{\alpha}{}_{\beta\gamma} &= (\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\gamma} - (\rho,\eta)\overset{*}{H}{}^{\alpha}_{\gamma\beta} - L^{\alpha}_{\beta\gamma}\circ h\circ\overset{*}{\pi},\\ (\rho,\eta,h)\overset{*}{\mathbb{T}}_{b\beta\gamma} &= -(\rho,\eta,h)\overset{*}{\mathbb{R}}_{b\ \beta\gamma},\\ (\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\alpha\ c}_{\ \beta} &= (\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\beta},\\ (\rho,\eta,h)\overset{*}{\mathbb{P}}_{b\beta}{}^{c} &= \frac{\partial}{\partial p_c}\left((\rho,\eta)\overset{*}{\Gamma}_{b\beta}\right) - (\rho,\eta)\overset{*}{H}{}^{c}_{b\beta},\\ (\rho,\eta,h)\overset{*}{\mathbb{S}}_{a}{}^{bc} &= (\rho,\eta)\overset{*}{V}{}^{bc}_{a} - (\rho,\eta)\overset{*}{V}{}^{cb}_{a}. \end{aligned}$$

In particular, when $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$, *we regain the local components of torsion associated to distinguished linear connection* $\left(\overset{*}{H},\overset{*}{V}\right)$*:*

$$(6.6.3)'\quad \begin{aligned} \overset{*}{\mathbb{T}}{}^{i}{}_{jk} &= \overset{*}{H}{}^{i}_{jk} - \overset{*}{H}{}^{i}_{kj}, & \overset{*}{\mathbb{T}}_{bjk} &= -\overset{*}{\mathbb{R}}_{b\ jk},\\ \overset{*}{\mathbb{P}}{}^{i\ c}_{\ j} &= \overset{*}{V}{}^{ic}_{j}, & \overset{*}{\mathbb{P}}_{b\ k}^{\ c} &= \frac{\partial\overset{*}{\Gamma}_{bk}}{\partial p_c} - \overset{*}{H}{}^{c}_{bk},\\ \overset{*}{\mathbb{S}}_{a}{}^{bc} &= \overset{*}{V}{}^{bc}_{a} - \overset{*}{V}{}^{bc}_{a}. \end{aligned}$$

Definition 6.6.2 The application

$$\begin{array}{ccc} \left(\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\right)^{3} & \xrightarrow{(\rho,\eta,h)\overset{*}{\mathbb{R}}} & \Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ \left(\left(\tilde{Y},\tilde{Z}\right),\tilde{X}\right) & \longmapsto & (\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\tilde{Y},\tilde{Z}\right)\tilde{X} \end{array}$$

defined by:

$$(6.6.4)\quad \begin{aligned} (\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\tilde{Y},\tilde{Z}\right)\tilde{X} &= (\rho,\eta)\overset{*}{D}_{\tilde{Y}}\left((\rho,\eta)\overset{*}{D}_{\tilde{Z}}\tilde{X}\right) -\\ &- (\rho,\eta)\overset{*}{D}_{\tilde{Z}}\left((\rho,\eta)\overset{*}{D}_{\tilde{Y}}\tilde{X}\right) - (\rho,\eta)\overset{*}{D}_{\left[\tilde{Y},\tilde{Z}\right]_{(\rho,\eta)T\overset{*}{E}}}\tilde{X}, \end{aligned}$$

for any $\tilde{X},\tilde{Y},\tilde{Z}\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$, will be called the (ρ,η,h)-*curvature associated to distinguished linear* (ρ,η)-*connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$.

Proposition 6.6.2 *The* (ρ,η,h)-*curvature* $(\rho,\eta,h)\overset{*}{\mathbb{R}}$ *associated to distinguished linear* (ρ,η)-*connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$, *is* $\mathbb{R}$-*linear in each argument and antisymmetric in the first two arguments.*

Using the notations:

$$
(6.6.5)\qquad
\begin{aligned}
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\tilde{\delta}}_{\varepsilon},\overset{*}{\tilde{\delta}}_{\gamma}\right)\overset{*}{\tilde{\delta}}_{\beta} &= (\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{\beta\ \gamma\varepsilon}^{\ \alpha}\overset{*}{\tilde{\delta}}_{\alpha},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\tilde{\delta}}_{\varepsilon},\overset{*}{\tilde{\delta}}_{\gamma}\right)\overset{\cdot}{\tilde{\partial}}{}^{a} &= (\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{b\ \gamma\varepsilon}^{\ a}\overset{\cdot}{\tilde{\partial}}{}^{b},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\tilde{\delta}}_{\gamma},\overset{\cdot}{\tilde{\partial}}{}^{b}\right)\overset{*}{\tilde{\delta}}_{\varepsilon} &= (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{\varepsilon\ \gamma}^{\ \alpha\ b}\overset{*}{\tilde{\delta}}_{\alpha},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\tilde{\delta}}_{\gamma},\overset{\cdot}{\tilde{\partial}}{}^{b}\right)\overset{\cdot}{\tilde{\partial}}{}^{a} &= (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{c\ \ \gamma}^{\ ab}\overset{\cdot}{\tilde{\partial}}{}^{c},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{\cdot}{\tilde{\partial}}{}^{c},\overset{\cdot}{\tilde{\partial}}{}^{b}\right)\overset{*}{\tilde{\delta}}_{\beta} &= (\rho,\eta,h)\overset{*}{\mathbb{S}}{}_{\beta}^{\ \alpha\ bc}\overset{*}{\tilde{\delta}}_{\alpha},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{\cdot}{\tilde{\partial}}{}^{c},\overset{\cdot}{\tilde{\partial}}{}^{b}\right)\overset{\cdot}{\tilde{\partial}}{}^{a} &= (\rho,\eta,h)\overset{*}{\mathbb{S}}{}_{d}^{\ \ abc}\overset{\cdot}{\tilde{\partial}}{}^{d}.
\end{aligned}
$$

we can easily prove the following

Theorem 6.6.2 *The (ρ,η,h)-curvature $(\rho,\eta,h)\overset{*}{\mathbb{R}}$ associated to distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$, is characterized by the tensor fields with local components:*

$$
(6.6.6)\qquad
\left\{
\begin{aligned}
(\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{\beta\ \gamma\varepsilon}^{\ \alpha} ={}&\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\overset{*}{\tilde{\delta}}_{\varepsilon}\right)(\rho,\eta)\overset{*}{H}{}_{\beta\gamma}^{\alpha}-\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)(\rho,\eta)\overset{*}{H}{}_{\beta\varepsilon}^{\alpha}\\
&+(\rho,\eta)\overset{*}{H}{}_{\theta\varepsilon}^{\alpha}(\rho,\eta)\overset{*}{H}{}_{\beta\gamma}^{\theta}-(\rho,\eta)\overset{*}{H}{}_{\theta\gamma}^{\alpha}(\rho,\eta)\overset{*}{H}{}_{\beta\varepsilon}^{\theta}\\
&-(\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{b\ \varepsilon\gamma}(\rho,\eta)\overset{*}{V}{}_{\beta}^{\alpha b}-L_{\varepsilon\gamma}^{\theta}\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{H}{}_{\beta\theta}^{\alpha},\\
(\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{b\ \gamma\varepsilon}^{\ a} ={}&\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\overset{*}{\tilde{\delta}}_{\varepsilon}\right)(\rho,\eta)\overset{*}{H}{}_{b\gamma}^{a}-(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)(\rho,\eta)\overset{*}{H}{}_{b\varepsilon}^{a}\\
&+(\rho,\eta)\overset{*}{H}{}_{b\varepsilon}^{c}(\rho,\eta)\overset{*}{H}{}_{c\gamma}^{a}-(\rho,\eta)\overset{*}{H}{}_{b\gamma}^{c}(\rho,\eta)\overset{*}{H}{}_{c\varepsilon}^{a}\\
&-(\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{c\ \varepsilon\gamma}(\rho,\eta)\overset{*}{V}{}_{b}^{ac}-L_{\varepsilon\gamma}^{\theta}\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{H}{}_{b\theta}^{a},
\end{aligned}
\right.
$$

$$
(6.6.7)\qquad
\left\{
\begin{aligned}
(\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{\varepsilon\ \gamma}^{\ \alpha\ b} ={}&\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})(\overset{*}{\tilde{\delta}}_{\gamma})(\rho,\eta)\overset{*}{V}{}_{\varepsilon}^{\alpha b}-\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\overset{\cdot}{\tilde{\partial}}{}^{b}\right)(\rho,\eta)\overset{*}{H}{}_{\varepsilon\gamma}^{\alpha}\\
&+(\rho,\eta)\overset{*}{H}{}_{\theta\gamma}^{\alpha}\cdot(\rho,\eta)\overset{*}{V}{}_{\varepsilon}^{\theta b}-(\rho,\eta)\overset{*}{V}{}_{\theta}^{\alpha b}\cdot(\rho,\eta)\overset{*}{H}{}_{\varepsilon\gamma}^{\theta}\\
&+\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\overset{\cdot}{\tilde{\partial}}{}^{b}\right)\left((\rho,\eta)\overset{*}{\Gamma}_{c\gamma}\right)\cdot(\rho,\eta)\overset{*}{V}{}_{\varepsilon}^{\alpha c},\\
(\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{c\ \ \gamma}^{\ ab} ={}&\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)(\rho,\eta)\overset{*}{V}{}_{c}^{ab}-\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\overset{\cdot}{\tilde{\partial}}{}^{b}\right)(\rho,\eta)\overset{*}{H}{}_{c\gamma}^{a}\\
&+(\rho,\eta)\overset{*}{H}{}_{c\gamma}^{d}(\rho,\eta)\overset{*}{V}{}_{d}^{ab}-(\rho,\eta)\overset{*}{V}{}_{c}^{db}(\rho,\eta)\overset{*}{H}{}_{d\gamma}^{a}\\
&+\Gamma(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})\left(\overset{\cdot}{\tilde{\partial}}{}^{b}\right)\left((\rho,\eta)\overset{*}{\Gamma}_{d\gamma}\right)(\rho,\eta)\overset{*}{V}{}_{c}^{ad},
\end{aligned}
\right.
$$

$$
(6.6.8)\quad \begin{cases}
(\rho,\eta,h)\overset{*}{\mathbb{S}}{}_{\beta}^{\alpha\ bc} &= \Gamma(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}})\left(\overset{\cdot\, c}{\tilde{\partial}}\right)(\rho,\eta)\overset{*}{V}{}_{\beta}^{\alpha b} - \Gamma(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}})\left(\overset{\cdot\, b}{\tilde{\partial}}\right)(\rho,\eta)\overset{*}{V}{}_{\beta}^{\alpha c} \\
&\quad + (\rho,\eta)\overset{*}{V}{}_{\theta}^{\alpha c}(\rho,\eta)\overset{*}{V}{}_{\beta}^{\theta b} - (\rho,\eta)\overset{*}{V}{}_{\theta}^{\alpha b}(\rho,\eta)\overset{*}{V}{}_{\beta}^{\theta c}, \\
(\rho,\eta,h)\overset{*}{\mathbb{S}}{}_{d}^{abc} &= \Gamma(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}})\left(\overset{\cdot\, c}{\tilde{\partial}}\right)(\rho,\eta)\overset{*}{V}{}_{d}^{ab} - \Gamma(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}})\left(\overset{\cdot\, b}{\tilde{\partial}}\right)(\rho,\eta)\overset{*}{V}{}_{d}^{ac} \\
&\quad + (\rho,\eta)\overset{*}{V}{}_{d}^{ec}(\rho,\eta)\overset{*}{V}{}_{e}^{ab} - (\rho,\eta)\overset{*}{V}{}_{bd}^{e}(\rho,\eta)\overset{*}{V}{}_{e}^{ac}.
\end{cases}
$$

In particular, when $(\rho,\eta,h) = (Id_{TM}, Id_M, Id_M)$, we see the local components of the curvature associated to distinguished linear connection $\left(\overset{*}{H},\overset{*}{V}\right)$ in the followings:

$$
(6.6.6)'\quad \begin{aligned}
\overset{*}{\mathbb{R}}{}^{i}_{j\ kl} &= \overset{*}{\delta}_l\left(\overset{*}{H}{}^{i}_{jk}\right) - \overset{*}{\delta}_k\left(\overset{*}{H}{}^{i}_{jl}\right) \\
&\quad + \overset{*}{H}{}^{i}_{hl}\cdot\overset{*}{H}{}^{h}_{jk} - \overset{*}{H}{}^{i}_{hk}\cdot\overset{*}{H}{}^{h}_{jl} - \overset{*}{\mathbb{R}}_{b\ lk}\cdot\overset{*}{V}{}^{ib}_{j}, \\
\overset{*}{\mathbb{R}}{}_{b\ kl}^{\ a} &= \overset{*}{\delta}_l\left(\overset{*}{H}{}^{a}_{bk}\right) - \overset{*}{\delta}_k\left(\overset{*}{H}{}^{a}_{bl}\right) \\
&\quad + \overset{*}{H}{}^{c}_{bl}\cdot\overset{*}{H}{}^{a}_{ck} - \overset{*}{H}{}^{c}_{bk}\cdot\overset{*}{H}{}^{a}_{cl} - \overset{*}{\mathbb{R}}_{c\ lk}\cdot\overset{*}{V}{}^{ac}_{b},
\end{aligned}
$$

$$
(6.6.7)'\quad \begin{aligned}
\overset{*}{\mathbb{P}}{}^{i\ b}_{l\ k} &= \overset{*}{\delta}_k\left(\overset{*}{V}{}^{ib}_{l}\right) - \overset{\cdot\, b}{\tilde{\partial}}\left(\overset{*}{H}{}^{i}_{lk}\right) \\
&\quad + \overset{*}{H}{}^{i}_{hk}\cdot\overset{*}{V}{}^{hb}_{l} - \overset{*}{V}{}^{ib}_{h}\cdot\overset{*}{H}{}^{h}_{lk} + \overset{\cdot\, b}{\tilde{\partial}}\left(\overset{*}{\Gamma}_{ck}\right)\cdot\overset{*}{V}{}^{ic}_{l}, \\
\overset{*}{\mathbb{P}}{}_{c\ \ k}^{\ ab} &= \overset{*}{\delta}_k\left(\overset{*}{V}{}^{ab}_{c}\right) - \overset{\cdot\, b}{\tilde{\partial}}\left(\overset{*}{H}{}^{a}_{ck}\right) \\
&\quad + \overset{*}{H}{}^{d}_{ck}\cdot\overset{*}{V}{}^{ab}_{d} - \overset{*}{V}{}^{db}_{c}\cdot\overset{*}{H}{}^{a}_{dk} + \overset{\cdot\, b}{\tilde{\partial}}\left(\overset{*}{\Gamma}_{dk}\right)\cdot\overset{*}{V}{}^{ad}_{c},
\end{aligned}
$$

$$
(6.6.8)'\quad \begin{aligned}
\overset{*}{\mathbb{S}}{}^{i\ bc}_{j} &= \overset{\cdot\, c}{\tilde{\partial}}\left(\overset{*}{V}{}^{ib}_{j}\right) - \overset{\cdot\, b}{\tilde{\partial}}\left(\overset{*}{V}{}^{ic}_{j}\right) + \overset{*}{V}{}^{ic}_{h}\cdot\overset{*}{V}{}^{hb}_{j} - \overset{*}{V}{}^{ib}_{h}\cdot\overset{*}{V}{}^{hc}_{j}, \\
\overset{*}{\mathbb{S}}{}_{d}^{abc} &= \overset{\cdot\, c}{\tilde{\partial}}\left(\overset{*}{V}{}^{ab}_{d}\right) - \overset{\cdot\, b}{\tilde{\partial}}\left(\overset{*}{V}{}^{ac}_{d}\right) + \overset{*}{V}{}^{ae}_{d}\cdot\overset{*}{V}{}^{ab}_{e} - \overset{*}{V}{}^{eb}_{d}\cdot\overset{*}{V}{}^{ac}_{d}.
\end{aligned}
$$

Definition 6.6.3 The tensor field

$$
(6.6.9)\quad \begin{aligned}
&\mathbf{Ric}\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right) = \\
&= (\rho,\eta,h)\overset{*}{\mathbb{R}}{}_{\alpha\ \beta}\, d\overset{*}{\tilde{z}}{}^{\alpha}\otimes d\overset{*}{\tilde{z}}{}^{\beta} + (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{\alpha}^{\ \ b}\, d\overset{*}{\tilde{z}}{}^{\alpha}\otimes\delta\tilde{p}_b \\
&+ (\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{a}_{\ \ \beta}\,\delta\tilde{p}_a\otimes d\overset{*}{\tilde{z}}{}^{\beta} + (\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{a\ b}\,\delta\tilde{p}_a\otimes\delta\tilde{p}_b,
\end{aligned}
$$

$$
(6.6.10)\quad \begin{array}{ll}
(\rho,\eta,h)\overset{*}{\mathbb{R}}_{\alpha\beta} = (\rho,\eta,h)\overset{*}{\mathbb{R}}{}^{\ \gamma}_{\alpha\ \beta\gamma} & (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{\alpha}^{\ \ b} = (\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\beta\ b}_{\alpha\ \beta}, \\
(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{a}_{\ \ \beta} = (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{c\ \ \beta}^{\ ac} & (\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{ab} = (\rho,\eta,h)\overset{*}{\mathbb{S}}{}_{c}^{acb}
\end{array}
$$

will be called *the Ricci tensor field associated to distinguished linear (ρ,η)-connection* $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$.

This tensor field will be used for writing the Einstein equations in Subsection 6.10.

6.7 Formulas of Ricci type. Identities of Cartan and Bianchi type

We consider the following diagram:

$$\begin{array}{ccc} \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \overset{*}{\pi}\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $((F,\nu,M),[,]_{F.h},(\rho,\eta))\in|\mathbf{GLA}|$.

Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and let

$$\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$$

be a distinguished linear (ρ,η)-connection for the generalized tangent bundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right).$$

Theorem 6.7.1 Using the definition of (ρ,η,h)-curvature associated to the distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$, it results the following formulas:

$$(\mathcal{R}_1)\qquad\left\{\begin{array}{l}
(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}\overset{*}{\mathcal{H}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}X}\overset{*}{\mathcal{H}}Z\\
\qquad=(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right)\overset{*}{\mathcal{H}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}\left[\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{H}}Z\\
\qquad+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{H}}Z,\\
(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}\overset{*}{\mathcal{H}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}\overset{*}{\mathcal{H}}Z\\
\qquad=(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right)\overset{*}{\mathcal{H}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{H}}Z\\
\qquad+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{H}}Z,\\
(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}Y}\overset{*}{\mathcal{H}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}\overset{*}{\mathcal{H}}Z\\
\qquad=(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{V}}Y\right)\overset{*}{\mathcal{H}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{V}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{H}}Z,
\end{array}\right.$$

and

$$(\mathcal{R}_2)\quad \left\{\begin{array}{l}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}\overset{*}{\mathcal{V}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}X}\overset{*}{\mathcal{V}}Z\\ \qquad =(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right)\overset{*}{\mathcal{V}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}\left[\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{V}}Z\\ \qquad +(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{H}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{V}}Z,\\ (\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}\overset{*}{\mathcal{V}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}\overset{*}{\mathcal{V}}Z\\ \qquad =(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right)\overset{*}{\mathcal{V}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{H}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{V}}Z\\ \qquad +(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{H}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{V}}Z,\\ (\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}Y}\overset{*}{\mathcal{V}}Z-(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}Y}(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}X}\overset{*}{\mathcal{V}}Z\\ \qquad =(\rho,\eta,h)\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{V}}Y\right)\overset{*}{\mathcal{V}}Z+(\rho,\eta)\overset{*}{D}_{\overset{*}{\mathcal{V}}\left[\overset{*}{\mathcal{V}}X,\overset{*}{\mathcal{V}}Y\right]_{(\rho,\eta)T\overset{*}{E}}}\overset{*}{\mathcal{V}}Z.\end{array}\right.$$

Using the previous theorem, the horizontal and vertical sections of adapted base and an arbitrary section

$$\tilde{Z}^{\alpha}\frac{\partial}{\partial\tilde{z}^{\alpha}}+Y^{a}\frac{\partial}{\partial\tilde{y}^{a}}\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

we can propose the following

Theorem 6.7.2 *We obtain the following formulas of Ricci type:*

$$(\mathcal{R}_1)\quad \left\{\begin{array}{ll}\tilde{Z}^{\alpha}{}_{|\gamma|\beta}-\tilde{Z}^{\alpha}{}_{|\beta|\gamma} & =(\rho,\eta,h)\overset{*}{\mathbb{R}}{}^{\alpha}{}_{\theta\,\gamma\beta}\cdot\tilde{Z}^{\theta}-\left(L^{\theta}_{\gamma\beta}\circ h\circ\overset{*}{\pi}\right)\cdot\tilde{Z}^{\alpha}{}_{|\theta}\\ & \quad -(\rho,\eta,h)\overset{*}{\mathbb{T}}_{b\gamma\beta}\cdot\tilde{Z}^{\alpha}|^{b}-(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{\theta}{}_{\gamma\beta}\cdot\tilde{Z}^{\alpha}{}_{|\theta},\\ \tilde{Z}^{\alpha}{}_{|\gamma}|^{b}-\tilde{Z}^{\alpha}|^{b}{}_{|\gamma} & =(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\alpha}{}_{\theta}{}^{b}{}_{\gamma}\cdot\tilde{Z}^{\theta}-(\rho,\eta,h)\overset{*}{\mathbb{P}}_{c\gamma}{}^{b}\cdot\tilde{Z}^{\alpha}|^{c}\\ & \quad -(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\theta}{}_{\gamma}{}^{b}\cdot\tilde{Z}^{\alpha}{}_{|\theta},\\ \tilde{Z}^{\alpha}|^{c}|^{b}-\tilde{Z}^{\alpha}|^{b}|^{c} & =(\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{\alpha}{}_{\theta}{}^{cb}\cdot\tilde{Z}^{\theta}-(\rho,\eta,h)\overset{*}{\mathbb{S}}_{a}{}^{bc}\cdot\tilde{Z}^{\alpha}|^{a},\end{array}\right.$$

and

$$(\mathcal{R}_2)\quad \left\{\begin{array}{ll}Y_{a\ |\gamma|\beta}-Y_{a\ |\beta|\gamma} & =(\rho,\eta,h)\overset{*}{\mathbb{R}}_{a}{}^{c}{}_{\gamma\beta}\cdot Y_{c}-\left(L^{\theta}_{\gamma\beta}\circ h\circ\pi\right)\cdot Y_{a\ |\theta}\\ & \quad -(\rho,\eta)\overset{*}{\mathbb{T}}_{c\gamma\beta}\cdot Y_{a}|^{c}-(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{\theta}{}_{\gamma\beta}\cdot Y_{a\ |\theta},\\ Y_{a\ |\gamma}|^{b}-Y_{a}|^{b}{}_{|\gamma} & =(\rho,\eta,h)\overset{*}{\mathbb{P}}_{a}{}^{cb}{}_{\gamma}\cdot Y_{c}-(\rho,\eta,h)\overset{*}{\mathbb{P}}_{c\gamma}{}^{b}\cdot Y_{a}|^{c}\\ & \quad -(\rho,\eta)\overset{*}{\mathbb{P}}{}^{\theta}{}_{\gamma}{}^{b}\cdot Y_{a\ |\theta},\\ Y_{a}|^{c}|^{b}-Y_{a}|^{b}|^{c} & =(\rho,\eta,h)\overset{*}{\mathbb{S}}_{a}{}^{dbc}\cdot Y_{d}-(\rho,\eta,h)\overset{*}{\mathbb{S}}_{d}{}^{bc}\cdot Y_{a}|^{d}.\end{array}\right.$$

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,id_M)$ *and the Lie bracket* $[,]_{TM}$ *is the usual Lie bracket, then the formulas of Ricci type* $(\mathcal{R}_1)$ *and* $(\mathcal{R}_2)$ *become:*

$$(\mathcal{R}_1)'\quad\left\{\begin{array}{ll}\tilde{Z}^i{}_{|k|j}-\tilde{Z}^i{}_{|j|k} & =\overset{*}{\mathbb{R}}{}^i{}_{h\ kj}\cdot\tilde{Z}^h-\overset{*}{\mathbb{T}}_{bkj}\cdot\tilde{Z}^i|^b-\overset{*}{\mathbb{T}}{}^h{}_{kj}\cdot\tilde{Z}^i{}_{|h},\\ \tilde{Z}^i{}_{|k}|^b-\tilde{Z}^i|^b{}_{|k} & =\overset{*}{\mathbb{P}}{}^{i\ b}_{h\ k}\cdot\tilde{Z}^h-\overset{*}{\mathbb{P}}{}_{ck}{}^{b}\cdot\tilde{Z}^i|^c-\overset{*}{\mathbb{P}}{}^{h\ b}_{\ k}\cdot\tilde{Z}^i{}_{|h},\\ \tilde{Z}^i|^c|^b-\tilde{Z}^i|^b|^c & =\overset{*}{\mathbb{S}}{}^{i\ cb}_{h}\cdot\tilde{Z}^h-\overset{*}{\mathbb{S}}{}_a{}^{bc}\cdot\tilde{Z}^i|^a\end{array}\right.$$

and

$$(\mathcal{R}_2)'\quad\left\{\begin{array}{ll}Y_{a\ |k|j}-Y_{a\ |j|k} & =\overset{*}{\mathbb{R}}{}_{a\ kj}^{\ c}\cdot Y_c-\overset{*}{\mathbb{T}}_{ckj}\cdot Y_a|^c-\overset{*}{\mathbb{T}}{}^i_{kj}\cdot Y_{a\ |i},\\ Y_{a\ |k}|^b-Y_a|^b{}_{|k} & =\overset{*}{\mathbb{P}}{}_{a\ \ k}^{\ cb}\cdot Y_c-\overset{*}{\mathbb{P}}{}_{ck}{}^{b}\cdot Y_a|^c-\overset{*}{\mathbb{P}}{}^{h\ b}_{\ k}\cdot Y_{a\ |h},\\ Y_a|^c|^b-Y_a|^b|^c & =\overset{*}{\mathbb{S}}{}_a{}^{dbc}\cdot Y_d-\overset{*}{\mathbb{S}}{}_d{}^{bc}\cdot Y_a|^d.\end{array}\right.$$

Using the 1-forms associated to distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$

$$(6.7.1)\qquad\begin{array}{c}(\rho,\eta)\overset{*}{\omega}{}^\alpha_\beta=(\rho,\eta)\overset{*}{H}{}^\alpha_{\beta\gamma}d\tilde{z}^\gamma+(\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\beta}\,\delta\tilde{p}_c,\\ (\rho,\eta)\overset{*}{\omega}{}^a_b=(\rho,\eta)\overset{*}{H}{}^a_{b\gamma}d\tilde{z}^\gamma+(\rho,\eta)\overset{*}{V}{}^{ac}_{b}\,\delta\tilde{p}_c,\end{array}$$

the torsion 2-forms

$$(6.7.2)\qquad\left\{\begin{array}{ll}(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^\alpha & =\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^\alpha_{\ \beta\gamma}d\tilde{z}^\beta\wedge d\tilde{z}^\gamma+(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\alpha\ c}_{\ \beta}\,d\tilde{z}^\beta\wedge\delta\tilde{p}_c,\\ (\rho,\eta,h)\overset{*}{\mathbb{T}}_b & =\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{T}}_{b\beta\gamma}d\tilde{z}^\beta\wedge d\tilde{z}^\gamma+(\rho,\eta,h)\overset{*}{\mathbb{P}}_{b\beta}{}^{c}\,d\tilde{z}^\beta\wedge\delta\tilde{p}_c\\ & +\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{S}}_b{}^{ac}\,\delta\tilde{p}_a\wedge\delta\tilde{p}_c\end{array}\right.$$

and the curvature 2-forms

$$(6.7.3)\qquad\left\{\begin{array}{ll}(\rho,\eta,h)\overset{*}{\mathbb{R}}{}^\alpha_{\ \beta} & =\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{R}}{}^\alpha_{\ \beta\ \gamma\theta}d\tilde{z}^\gamma\wedge d\tilde{z}^\theta+(\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{\alpha\ c}_{\ \beta\ \gamma}d\tilde{z}^\gamma\wedge\delta\tilde{p}_c\\ & +\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{\alpha\ bc}_{\ \beta}\,\delta\tilde{p}_b\wedge\delta\tilde{p}_c,\\ (\rho,\eta,h)\overset{*}{\mathbb{R}}{}_b^{\ a} & =\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{R}}{}_b^{\ a}{}_{\gamma\theta}d\tilde{z}^\gamma\wedge d\tilde{z}^\theta+(\rho,\eta,h)\overset{*}{\mathbb{P}}{}_b^{\ ac}{}_{\gamma}d\tilde{z}^\gamma\wedge\delta\tilde{p}_c\\ & +\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{a\ cd}_{\ b}\,\delta\tilde{p}_c\wedge\delta\tilde{p}_d,\end{array}\right.$$

we obtain the following

Theorem 4.7.3 *Identities of Cartan type hold good:*

$$(\mathcal{C}_1)\qquad\begin{array}{c}(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^\alpha=d^{(\rho,\eta)T\overset{*}{E}}(d\tilde{z}^\alpha)+(\rho,\eta)\overset{*}{\omega}{}^\alpha_\beta\wedge d\tilde{z}^\beta\\ (\rho,\eta,h)\overset{*}{\mathbb{T}}_b=d^{(\rho,\eta)T\overset{*}{E}}(\delta\tilde{p}_b)+(\rho,\eta)\overset{*}{\omega}{}^a_b\wedge\delta\tilde{p}_a\end{array},$$

$$(\mathcal{C}_2)\qquad\begin{array}{c}(\rho,\eta,h)\overset{*}{\mathbb{R}}{}^\alpha_{\ \beta}=d^{(\rho,\eta)T\overset{*}{E}}\left((\rho,\eta)\overset{*}{\omega}{}^\alpha_\beta\right)+(\rho,\eta)\overset{*}{\omega}{}^\alpha_\gamma\wedge(\rho,\eta)\overset{*}{\omega}{}^\gamma_\beta\\ (\rho,\eta,h)\overset{*}{\mathbb{R}}{}_b^{\ a}=d^{(\rho,\eta)T\overset{*}{E}}\left((\rho,\eta)\overset{*}{\omega}{}^a_b\right)+(\rho,\eta)\overset{*}{\omega}{}^a_c\wedge(\rho,\eta)\overset{*}{\omega}{}^c_b.\end{array}$$

In particular, if $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$ and the Lie bracket $[,]_{TM}$ is the usual Lie bracket, then the identities of Cartan type $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ become:

$$(\mathcal{C}_1)' \qquad \begin{aligned} \overset{*}{\mathbb{T}}{}^i &= d^{\left(Id_{T\overset{*}{E}},Id_{\overset{*}{E}}\right)T\overset{*}{E}}\left(d\tilde{z}^i\right)+\overset{*}{\omega}{}^i_j\wedge d\tilde{z}^j\\ \overset{*}{\mathbb{T}}_b &= d^{\left(Id_{T\overset{*}{E}},Id_{\overset{*}{E}}\right)T\overset{*}{E}}\left(\delta\tilde{p}_b\right)+\omega^a_b\wedge\delta\tilde{p}_a\end{aligned}$$

and

$$(\mathcal{C}_2)' \qquad \begin{aligned} \overset{*}{\mathbb{R}}{}^i_{\ j} &= d^{\left(Id_{T\overset{*}{E}},Id_{\overset{*}{E}}\right)T\overset{*}{E}}\left(\overset{*}{\omega}{}^i_j\right)+\overset{*}{\omega}{}^i_k\wedge\overset{*}{\omega}{}^k_j\\ \overset{*}{\mathbb{R}}{}_b^{\ a} &= d^{\left(Id_{T\overset{*}{E}},Id_{\overset{*}{E}}\right)T\overset{*}{E}}\left(\overset{*}{\omega}{}^a_b\right)+\overset{*}{\omega}{}^a_c\wedge\overset{*}{\omega}{}^c_b.\end{aligned}$$

Remark 6.7.1 For any

$$X,Y,Z\in\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

the following identities hold good:

$$(6.7.4) \qquad \begin{aligned} \overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)\overset{*}{\mathcal{H}}Z &= 0\\ \overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)\overset{*}{\mathcal{V}}Z &= 0,\end{aligned}$$

$$(6.7.5) \qquad \begin{aligned} \overset{*}{\mathcal{V}}D_X\left(\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(Y,Z\right)\overset{*}{\mathcal{H}}U\right) &= 0\\ \overset{*}{\mathcal{H}}D_X\left(\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(Y,Z\right)\overset{*}{\mathcal{V}}U\right) &= 0,\end{aligned}$$

and

$$(6.7.6) \qquad \left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)Z = \overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)\overset{*}{\mathcal{H}}Z+\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)\overset{*}{\mathcal{V}}Z.$$

Using the formulas of Bianchi type presented in Subsection 4.2 of our paper and the Remark 6.7.1 we obtain the following

Theorem 6.7.4 *The identities of Bianchi type:*

$$(\mathcal{B}_1) \qquad \left\{\begin{aligned} &\sum_{cyclic(X,Y,Z)}\left\{\overset{*}{\mathcal{H}}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(Y,Z\right)\right)-\overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)Z\right.\\ &\qquad+\overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(X,Y\right),Z\right)\\ &\qquad\left.+\overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(X,Y\right),Z\right)\right\}=0,\\ &\sum_{cyclic(X,Y,Z)}\left\{\overset{*}{\mathcal{V}}\left(\rho,\eta\right)D_X\left(\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(Y,Z\right)\right)-\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{R}}\left(X,Y\right)Z\right.\\ &\qquad+\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{H}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(X,Y\right),Z\right)\\ &\qquad\left.+\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(\overset{*}{\mathcal{V}}\left(\rho,\eta,h\right)\overset{*}{\mathbb{T}}\left(X,Y\right),Z\right)\right\}=0.\end{aligned}\right.$$

and

$$(\mathcal{B}_2)\quad\left\{\begin{array}{l}\displaystyle\sum_{cyclic(X,Y,Z,U)}\Big\{\overset{*}{\mathcal{H}}(\rho,\eta)\,D_X\left((\rho,\eta,h)\,\overset{*}{\mathbb{R}}(Y,Z)\,U\right)\\ \qquad -\overset{*}{\mathcal{H}}(\rho,\eta,h)\,\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{H}}(\rho,\eta,h)\,\overset{*}{\mathbb{T}}(X,Y)\,,Z\right)U\\ \qquad -\overset{*}{\mathcal{H}}(\rho,\eta,h)\,\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{V}}(\rho,\eta,h)\,\overset{*}{\mathbb{T}}(X,Y)\,,Z\right)U\Big\}=0,\\ \displaystyle\sum_{cyclic(X,Y,Z,U)}\Big\{\overset{*}{\mathcal{V}}(\rho,\eta)\,D_X\left((\rho,\eta,h)\,\overset{*}{\mathbb{R}}(Y,Z)\,U\right)\\ \qquad -\overset{*}{\mathcal{V}}(\rho,\eta,h)\,\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{H}}(\rho,\eta,h)\,\overset{*}{\mathbb{T}}(X,Y)\,,Z\right)U\\ \qquad -\overset{*}{\mathcal{V}}(\rho,\eta,h)\,\overset{*}{\mathbb{R}}\left(\overset{*}{\mathcal{V}}(\rho,\eta,h)\,\overset{*}{\mathbb{T}}(X,Y)\,,Z\right)U\Big\}=0,\end{array}\right.$$

hold good for any $X,Y,Z\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Corollary 6.7.1 *Using the following sections* $(\delta_\theta,\delta_\gamma,\delta_\beta)$*, the identities* $(\mathcal{B}_1)$ *become:*

$$(\mathcal{B}_1)'\quad\left\{\begin{array}{l}\displaystyle\sum_{cyclic(\beta,\gamma,\theta)}\Big\{(\rho,\eta,h)\,\overset{*}{\mathbb{T}}{}^{\alpha}_{\ \beta\gamma_{|\theta}}-(\rho,\eta,h)\,\overset{*}{\mathbb{R}}{}^{\alpha}_{\ \beta\ \gamma\theta}\\ \qquad +(\rho,\eta,h)\,\overset{*}{\mathbb{T}}{}^{\lambda}_{\ \gamma\theta}\,(\rho,\eta,h)\,\overset{*}{\mathbb{T}}{}^{\alpha}_{\ \beta\gamma}+(\rho,\eta,h)\,\overset{*}{\mathbb{T}}_{b\gamma\theta}\,(\rho,\eta,h)\,\overset{*}{\mathbb{P}}{}^{\alpha\ b}_{\ \beta}\Big\}=0,\\ \displaystyle\sum_{cyclic(\beta,\gamma,\theta)}\Big\{(\rho,\eta,h)\,\overset{*}{\mathbb{T}}_{b\beta\gamma\ |\theta}+(\rho,\eta,h)\,\overset{*}{\mathbb{T}}{}^{\alpha}_{\ \gamma\theta}\,(\rho,\eta,h)\,\overset{*}{\mathbb{T}}_{b\beta\alpha}\\ \qquad +(\rho,\eta,h)\,\overset{*}{\mathbb{T}}_{c\gamma\theta}\,(\rho,\eta,h)\,\overset{*}{\mathbb{P}}_{b\beta}{}^{c}\Big\}=0,\end{array}\right.$$

and using the following sections $(\delta_\lambda,\delta_\theta,\delta_\gamma,\delta_\beta)$*, the identities* $(\mathcal{B}_2)$ *become:*

$$(\mathcal{B}_2)'\quad\left\{\begin{array}{l}\displaystyle\sum_{cyclic(\beta,\gamma,\theta,\lambda)}\Big\{(\rho,\eta,h)\,\overset{*}{\mathbb{R}}{}^{\alpha}_{\ \beta\ \gamma\theta_{|\lambda}}-(\rho,\eta,h)\,\overset{*}{\mathbb{T}}{}^{\mu}_{\ \theta\lambda}\,(\rho,\eta,h)\,\overset{*}{\mathbb{T}}{}^{\alpha}_{\ \beta\gamma\mu}\\ \qquad -(\rho,\eta,h)\,\overset{*}{\mathbb{T}}_{b\theta\lambda}\,(\rho,\eta,h)\,\overset{*}{\mathbb{P}}{}^{\alpha\ b}_{\ \beta\ \gamma}\Big\}=0,\\ \displaystyle\sum_{cyclic(\gamma,\theta,\lambda)}\Big\{(\rho,\eta,h)\,\overset{*}{\mathbb{R}}_{b\ \gamma\theta_{|\lambda}}^{\ a}-(\rho,\eta,h)\,\overset{*}{\mathbb{T}}{}^{\mu}_{\ \theta\lambda}\,(\rho,\eta,h)\,\overset{*}{\mathbb{R}}_{b\ \gamma\mu}^{\ a}\\ \qquad -(\rho,\eta,h)\,\overset{*}{\mathbb{T}}_{c\theta\lambda}\,(\rho,\eta,h)\,\overset{*}{\mathbb{P}}_{b\ \ \gamma}^{\ ac}\Big\}=0.\end{array}\right.$$

Using another base of sections, we shall obtain new identities of Bianchi type necessary in the applications.

6.8 The (ρ,η)-(pseudo)metrizability

We consider the following diagram:

$$\begin{array}{ccc}\overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\ \downarrow^{\overset{*}{\pi}} & & \downarrow^{\nu}\\ M & \xrightarrow{h} & N\end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $((F,\nu,M),[,]_{F,h},(\rho,\eta))\in|\mathbf{GLA}|$. Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and let $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$ be a distinguished linear (ρ,η)-connection for the generalized tangent bundle

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right).$$

Definition 6.8.1 A tensor d-field

$$\overset{*}{G}=g_{\alpha\beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+g^{ab}\delta\tilde{p}_a\otimes\delta\tilde{p}_b\in\mathcal{DT}_{20}^{02}\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

will be called a *pseudometrical structure* if its components are symmetric and the matrices $\left\|g_{\alpha\beta}\left(\overset{*}{u}_x\right)\right\|$and $\left\|g^{ab}\left(\overset{*}{u}_x\right)\right\|$ are nondegenerate, for any point $\overset{*}{u}_x\in\overset{*}{E}$.

Moreover, if the matrices $\left\|g_{\alpha\beta}\left(\overset{*}{u}_x\right)\right\|$ and $\left\|g^{ab}\left(\overset{*}{u}_x\right)\right\|$ has constant signature, then the tensor d-field $\overset{*}{G}$ will be called *metrical structure.*

Let

$$\overset{*}{G}=g_{\alpha\beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+g^{ab}\delta\tilde{p}_a\otimes\delta\tilde{p}_b$$

be a (pseudo)metrical structure. If $\alpha,\beta\in\overline{1,p}$ and $a,b\in\overline{1,r}$, then for any vector local $(m+r)$-chart $\left(U,\overset{*}{s}_U\right)$ of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, we consider the real functions

$$\overset{*}{\pi}^{-1}(U)\xrightarrow{\tilde{g}^{\beta\alpha}}\mathbb{R}$$

and

$$\overset{*}{\pi}^{-1}(U)\xrightarrow{\tilde{g}_{ba}}\mathbb{R}$$

such that

$$\tilde{g}^{\beta\alpha}\left(\overset{*}{u}_x\right)\cdot g_{\alpha\gamma}\left(\overset{*}{u}_x\right)=\delta_{\gamma}^{\beta}$$

and

$$\tilde{g}_{ba}\left(\overset{*}{u}_x\right)\cdot g^{ac}\left(\overset{*}{u}_x\right)=\delta_b^c,$$

for any $\overset{*}{u}_x\in\overset{*}{\pi}^{-1}(U)\setminus\{0_x\}$.

Definition 6.8.2 We will say that the *(pseudo)metrical structure*

$$\overset{*}{G}=g_{\alpha\beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+g^{ab}\delta\tilde{p}_a\otimes\delta\tilde{p}_b$$

is Riemannian (pseudo)metrical structure if around each point $x \in M$ it exists a local vector $m+r$-chart $\left(U, \overset{*}{s}_U\right)$ and a local m-chart (U, ξ_U) such that $g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,p)$ and $g^{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,p)$ depends only on x, for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$.

If only the condition is verified: *"$g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,p)$ depends only on x, for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$"* (respectively *"$g^{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,p)$ depend only on x, for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$"*), then we will say that the *(pseudo)metrical structure G is a Riemannian $\overset{*}{\mathcal{H}}$-(pseudo)metrical structure* respectively a *Riemannian $\overset{*}{\mathcal{V}}$-(pseudo)metrical structure.*

Definition 6.8.3 We will say that the *(pseudo)metrical structure*

$$\overset{*}{G} = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

is locally Minkowski structure if around each point $x \in M$ it exists a local vector $m+r$-chart $\left(U, \overset{*}{s}_U\right)$ and a local m-chart (U, ξ_U) such that $g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,p)$ and $g^{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,p)$ depends only on p, for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$.

If only the condition is verified: "$g_{\alpha\beta} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,p)$ depend only on p, for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$" respectively "$g^{ab} \circ s_U^{-1} \circ \left(\xi_U^{-1} \times Id_{\mathbb{R}^m}\right)(x,p)$ depends only on p, for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$", then we will say that *the (pseudo)metrical structure $\overset{*}{G}$ is a (pseudo)metrical structure $\overset{*}{\mathcal{H}}$-locally Minkowski* and *$\overset{*}{\mathcal{V}}$-locally Minkowski, respectively.*

Definition 6.8.4 The generalized tangent bundle $\left((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$ will be called (ρ,η)-*(pseudo)metrizable* if it exists a (pseudo)metrical structure

$$\overset{*}{G} = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

and a distinguished linear (ρ,η)-connection

$$\left((\rho,\eta)\overset{*}{H}, (\rho,\eta)\overset{*}{V}\right)$$

such that

$$(\rho,\eta) D_X G = 0,\ \forall X \in \Gamma\left((\rho,\eta) T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right). \tag{6.8.1}$$

The condition (6.8.1) is equivalent with the following equalities:

$$g_{\alpha\beta|\gamma} = 0,\ g^{ab}{}_{|\gamma} = 0,\ g_{\alpha\beta}|^{c} = 0\,,\ g^{ab}|^{c} = 0. \tag{6.8.2}$$

If $g_{\alpha\beta|\gamma}$=0 and $g^{ab}{}_{|\gamma}$=0, then we will say that *the vector bundle*

$$\left((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

is $\overset{*}{\mathcal{H}}$-(ρ,η)-(pseudo)metrizable.

If $g_{\alpha\beta}|^{c}=0$ and $g^{ab}|^{c}=0$, then we will say that *the vector bundle*

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is $\overset{*}{\mathcal{V}}$*-*(ρ,η)*-(pseudo)metrizable.*

Theorem 6.8.1 *If*

$$\left((\rho,\eta)\overset{*}{\mathring{H}},(\rho,\eta)\overset{*}{\mathring{V}}\right)$$

is a distinguished linear (ρ,η)*-connection for the generalized tangent bundle*

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and

$$\overset{*}{G}=g_{\alpha\beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+g^{ab}\delta\tilde{p}_{a}\otimes\delta\tilde{p}_{b}$$

is a (pseudo)metrical structure, then the real local functions:

$$\begin{array}{ll}(\rho,\eta)\overset{*}{H}{}_{\beta\gamma}^{\alpha} & =\dfrac{1}{2}\tilde{g}^{\alpha\varepsilon}\left(\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)g_{\varepsilon\beta}\right.\\ & +\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\beta}\right)g_{\varepsilon\gamma}-\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\varepsilon}\right)g_{\beta\gamma}+\\ & \left.+g_{\theta\varepsilon}L_{\gamma\beta}^{\theta}\circ h\circ\overset{*}{\pi}-g_{\beta\theta}L_{\gamma\varepsilon}^{\theta}\circ h\circ\overset{*}{\pi}-g_{\theta\gamma}L_{\beta\varepsilon}^{\theta}\circ h\circ\overset{*}{\pi}\right),\\ (\rho,\eta)\overset{*}{H}{}_{b\gamma}^{a} & =(\rho,\eta)\overset{*}{\mathring{H}}{}_{b\gamma}^{a}+\dfrac{1}{2}\tilde{g}_{bc}g^{ac}{}_{\overset{0}{|\gamma}},\\ (\rho,\eta)\overset{*}{V}{}_{\beta}^{\alpha c} & =(\rho,\eta)\overset{*}{\mathring{V}}{}_{\beta}^{\alpha c}+\dfrac{1}{2}g_{\beta\varepsilon}\tilde{g}^{\alpha\varepsilon}\overset{0^{c}}{|},\\ (\rho,\eta)\overset{*}{V}{}_{b}^{ac} & =\dfrac{1}{2}\tilde{g}_{be}\left(\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\partial}^{\cdot c}\right)g^{ea}\right.\\ & \left.+\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\partial}^{\cdot a}\right)g^{ec}-\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\tilde{\partial}^{\cdot e}\right)g^{ac}\right)\end{array}\tag{6.8.3}$$

are components of a distinguished linear (ρ,η)*-connection such that the generalized tangent bundle*

$$\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

becomes (ρ,η)*-(pseudo)metrizable.*

Corollary 6.8.1 *If the distinguished linear* (ρ,η)*-connection*

$$\left((\rho,\eta)\overset{*}{\mathring{H}},(\rho,\eta)\overset{*}{\mathring{V}}\right)$$

coincide with the Berwald linear (ρ,η)-connection, then the local real functions:

$$
\begin{array}{rl}
(\rho,\eta)\overset{c}{\overset{*}{H}}{}^{\alpha}_{\beta\gamma} & = \frac{1}{2}\tilde{g}^{\alpha\varepsilon}\left(\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\gamma}\right)g_{\varepsilon\beta}\right. \\
& +\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\beta}\right)g_{\varepsilon\gamma} - \Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_{\varepsilon}\right)g_{\beta\gamma} \\
& \left.+g_{\theta\varepsilon}L^{\theta}_{\gamma\beta}\circ h\circ\overset{*}{\pi} - g_{\beta\theta}L^{\theta}_{\gamma\varepsilon}\circ h\circ\overset{*}{\pi} - g_{\theta\gamma}L^{\theta}_{\beta\varepsilon}\circ h\circ\overset{*}{\pi}\right), \\
(\rho,\eta)\overset{c}{\overset{*}{H}}{}^{a}_{b\gamma} & = \frac{\partial(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}}{\partial p_a} + \frac{1}{2}\tilde{g}_{bc}g^{ac}{}_{\underset{|\gamma}{0}}, \\
(\rho,\eta)\overset{c}{\overset{*}{V}}{}^{\alpha c}_{\beta} & = \frac{1}{2}g_{\beta\varepsilon}\frac{\partial\tilde{g}^{\alpha\varepsilon}}{\partial p_c}, \\
(\rho,\eta)\overset{c}{\overset{*}{V}}{}^{bc}_{a} & = \frac{1}{2}\tilde{g}_{ae}\left(\frac{\partial g^{eb}}{\partial p_c} + \frac{\partial g^{ec}}{\partial p_b} - \frac{\partial g^{bc}}{\partial p_e}\right)
\end{array}
\tag{6.8.4}
$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle

$$\left((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

becomes (ρ,η)-(pseudo)metrizable.

Moreover, if the (pseudo)metrical structure $\overset{*}{G}$ is $\mathcal{H}$- and $\mathcal{V}$-Riemannian, then the local real functions:

$$
\begin{array}{l}
(\rho,\eta)\overset{c}{\overset{*}{H}}{}^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\varepsilon}\left(\rho^{k}_{\gamma}\circ h\circ\pi\frac{\partial g_{\varepsilon\beta}}{\partial x^k} + \rho^{j}_{\beta}\circ h\circ\pi\frac{\partial g_{\varepsilon\gamma}}{\partial x^j} - \rho^{e}_{\varepsilon}\circ h\circ\pi\frac{\partial g_{\beta\gamma}}{\partial x^e} + \right. \\
\qquad\qquad \left. + g_{\theta\varepsilon}L^{\theta}_{\gamma\beta}\circ h\circ\pi - g_{\beta\theta}L^{\theta}_{\gamma\varepsilon}\circ h\circ\pi - g_{\theta\gamma}L^{\theta}_{\beta\varepsilon}\circ h\circ\pi\right), \\
(\rho,\eta)\overset{c}{\overset{*}{H}}{}^{a}_{b\gamma} = \frac{\partial(\rho,\eta)\Gamma^{a}_{\gamma}}{\partial y^b} + \frac{1}{2}g^{ac}\left(\rho^{i}_{\gamma}\circ h\circ\pi\frac{\partial g_{bc}}{\partial x^i} - \frac{\partial\rho\Gamma^{e}_{\gamma}}{\partial y^b}g_{ec} - \frac{\partial\rho\Gamma^{e}_{\gamma}}{\partial y^c}g_{eb}\right), \\
(\rho,\eta)\overset{c}{\overset{*}{V}}{}^{\alpha}_{\beta c} = 0,\ (\rho,\eta)\overset{c}{\overset{*}{V}}{}^{a}_{bc} = 0
\end{array}
\tag{4.8.5}
$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle

$$\left((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

becomes (ρ,η)-(pseudo)metrizable.

Theorem 6.8.2 *Let $(\rho,\eta)\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$. Let*

$$\left((\rho,\eta)\overset{*}{\mathring{H}}, (\rho,\eta)\overset{*}{\mathring{V}}\right)$$

be a distinguished linear (ρ,η)-connection for

$$\left((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

and let

$$\overset{*}{G} = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

be a (pseudo)metrical structure.

Let

$$(6.8.6)\qquad \begin{array}{ll} O_{\beta\gamma}^{\alpha\varepsilon} = \frac{1}{2}\left(\delta_{\beta}^{\alpha}\delta_{\gamma}^{\varepsilon} - g_{\beta\gamma}\tilde{g}^{\alpha\varepsilon}\right), & O_{\beta\gamma}^{*\alpha\varepsilon} = \dfrac{1}{2}\left(\delta_{\beta}^{\alpha}\delta_{\gamma}^{\varepsilon} + g_{\beta\gamma}\tilde{g}^{\alpha\varepsilon}\right), \\ O_{bc}^{ae} = \dfrac{1}{2}\left(\delta_{b}^{a}\delta_{c}^{e} - \tilde{g}_{bc}g^{ae}\right), & O_{bc}^{*ae} = \frac{1}{2}\left(\delta_{b}^{a}\delta_{c}^{e} + \tilde{g}_{bc}g^{ae}\right), \end{array}$$

be the Obata operators

If the real local functions $X_{\beta\gamma}^{\alpha}, X_{\beta}^{\alpha c}, Y_{b\gamma}^{a}, Y_{b}^{ac}$ *are components of tensor fields, then the local real functions are given in the following:*

$$(6.8.7)\qquad \begin{array}{l} (\rho,\eta)\overset{*}{H}{}_{\beta\gamma}^{\alpha} = (\rho,\eta)\overset{c}{\overset{*}{H}}{}_{\beta\gamma}^{\alpha} + O_{\gamma\eta}^{\alpha\varepsilon}X_{\varepsilon\beta}^{\eta}, \\ (\rho,\eta)\overset{*}{H}{}_{b\gamma}^{a} = (\rho,\eta)\overset{c}{\overset{*}{H}}{}_{b\gamma}^{a} + O_{bd}^{ae}Y_{e\gamma}^{d}, \\ (\rho,\eta)\overset{*}{V}{}_{\beta}^{\alpha c} = (\rho,\eta)\overset{c}{\overset{*}{V}}{}_{\beta}^{\alpha c} + O_{\beta\eta}^{*\alpha\varepsilon}X_{\varepsilon}^{\eta c}, \\ (\rho,\eta)\overset{*}{V}{}_{b}^{ac} = (\rho,\eta)\overset{c}{\overset{*}{V}}{}_{b}^{ac} + O_{bd}^{*ae}Y_{e}^{dc}, \end{array}$$

are the components of a distinguished linear (ρ,η)*-connection such that the generalized tangent bundle*

$$\left((\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

becomes (ρ,η)*-(pseudo)metrizable.*

Theorem 6.8.3 *Let* $(\rho,\eta)\overset{*}{\Gamma}$ *be a* (ρ,η)*-connection for the vector bundle* $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$*. If*

$$\left((\rho,\eta)\,\overset{*}{\mathring{H}}, (\rho,\eta)\,\overset{*}{\mathring{V}}\right)$$

is a distinguished linear (ρ,η)*-connection for the generalized tangent bundle*

$$\left((\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

and

$$\overset{*}{G} = g_{\alpha\beta} d\tilde{z}^{\alpha} \otimes d\tilde{z}^{\beta} + g^{ab} \delta\tilde{p}_a \otimes \delta\tilde{p}_b$$

is a (pseudo)metrical structure, then the real local functions:

$$(6.8.8)\qquad \begin{array}{l} (\rho,\eta)\overset{*}{H}{}_{\beta\gamma}^{\alpha} = (\rho,\eta)\overset{*}{\mathring{H}}{}_{\beta\gamma}^{\alpha} + \frac{1}{2}g_{\beta\varepsilon}\tilde{g}^{\varepsilon\alpha}{}_{\stackrel{0}{|\gamma}}, \\ (\rho,\eta)\overset{*}{H}{}_{b\gamma}^{a} = (\rho,\eta)\overset{*}{\mathring{H}}{}_{b\gamma}^{a} + \frac{1}{2}\tilde{g}_{be}g^{ea}{}_{\stackrel{0}{|\gamma}}, \\ (\rho,\eta)\overset{*}{V}{}_{\beta}^{\alpha c} = (\rho,\eta)\overset{*}{\mathring{V}}{}_{\beta}^{\alpha c} + \frac{1}{2}g_{\beta\varepsilon}\tilde{g}^{\varepsilon\alpha}\overset{0^{c}}{|}, \\ (\rho,\eta)\overset{*}{V}{}_{b}^{ac} = (\rho,\eta)\overset{*}{\mathring{V}}{}_{b}^{ac} + \frac{1}{2}\tilde{g}_{be}g^{ea}\overset{0^{c}}{|} \end{array}$$

are the components of a distinguished linear (ρ,η)-connection such that the generalized tangent bundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

becomes (ρ,η)-(pseudo)metrizable.

6.9 Generalized Hamilton (ρ,η)-spaces, Hamilton (ρ,η)-spaces and Cartan (ρ,η)-spaces

We consider the following diagram:

$$\begin{array}{ccc}\overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\ \downarrow{\scriptstyle\overset{*}{\pi}} & & \downarrow{\scriptstyle\nu}\\ M & \xrightarrow{h} & N\end{array}$$

such that $(E,\pi,M)=(F,\nu,N)$ and the generalized tangent bundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is (ρ,η)-(pseudo)metrizable.

Definition 6.9.1 A smooth *Hamilton fundamental function* on the dual vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ is a mapping

$$\overset{*}{E}\ \xrightarrow{H}\ \mathbb{R}$$

which satisfies the following conditions:

1. $H\circ\overset{*}{u}\in C^{\infty}(M)$, for any $\overset{*}{u}\in\Gamma\left(\overset{*}{E},\overset{*}{\pi},M\right)\setminus\{0\}$;
2. $H\circ 0\in C^{0}(M)$, where 0 means the null section of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

Let H be a differentiable Hamiltonian defined on the total space of the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

If $\left(U,\overset{*}{s}_U\right)$ is a local vector $(m+r)$-chart for $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then we obtain the following real functions defined on $\overset{*}{\pi}^{-1}(U)$:

$$\begin{array}{ll}H_i\overset{put}{=}\dfrac{\partial H}{\partial x^i}\overset{put}{=}\dfrac{\partial}{\partial x^i}(H) & H_i^b\overset{put}{=}\dfrac{\partial^2 H}{\partial x^i\partial p_b}\overset{put}{=}\dfrac{\partial}{\partial x^i}\left(\dfrac{\partial}{\partial p_b}(H)\right)\\ H^a\overset{put}{=}\dfrac{\partial H}{\partial p_a}\overset{put}{=}\dfrac{\partial}{\partial p_a}(H) & H^{ab}\overset{put}{=}\dfrac{\partial^2 H}{\partial p_a\partial p_b}\overset{put}{=}\dfrac{\partial}{\partial p_a}\left(\dfrac{\partial}{\partial p_b}(H)\right)\end{array}.\tag{6.9.1}$$

Definition 6.9.2 If for any local vector $m+r$-chart $\left(U,\overset{*}{s}_U\right)$ of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, we have:

$$rank\left\|H^{ab}\left(\overset{*}{u}_x\right)\right\|=r,\tag{6.9.2}$$

for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U) \setminus \{0_x\}$, then we say that *the Hamiltonian H is regular.*

Proposition 6.9.1 If the Hamiltonian H is regular, then for any local vector $m+r$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, we obtain the real functions $\tilde{H}_{ba}$ locally defined by

$$\begin{array}{ccc} \overset{*}{\pi}^{-1}(U) & \xrightarrow{\tilde{H}_{ba}} & \mathbb{R} \\ \overset{*}{u}_x & \longmapsto & H_{ba}\left(\overset{*}{u}_x\right) \end{array} \tag{6.9.3}$$

where $\left\|\tilde{H}_{ba}\left(\overset{*}{u}_x\right)\right\| = \left\|H^{ab}\left(\overset{*}{u}_x\right)\right\|^{-1}$, for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U)$.

Definition 6.9.3 A smooth *Cartan fundamental function* on the vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ is a mapping

$$\overset{*}{E} \xrightarrow{K} \mathbb{R}_+$$

which satisfies the following conditions:

1. $K \circ \overset{*}{u} \in C^\infty(M)$, for any $\overset{*}{u} \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right) \setminus \{0\}$;
2. $K \circ 0 \in C^0(M)$, where 0 means the null section of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$;
3. K is positively 1-homogenous on the fibres of vector bundle $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$;
4. For any local vector $m+r$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, the hessian:

$$\left\|K^{2\ ab}\left(\overset{*}{u}_x\right)\right\| \tag{6.9.4}$$

is positively define for any $\overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U) \setminus \{0_x\}$.

Definition 6.9.4 If the (pseudo)metrical structure $\overset{*}{G}$ is determined by a (pseudo)metrical structure

$$g \in \mathcal{T}^{\,2}_{\,0}\left(V(\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right),$$

then the (ρ,η)-(pseudo)metrizable vector bundle

$$\left((\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

will be called the *generalized Hamilton (ρ,η)-space.*

In particular, if the (pseudo)metrical structure g is determined with the help of a Hamilton fundamental function or Cartan fundamental function, then the (ρ,η)-(pseudo)metrizable vector bundle

$$\left((\rho,\eta)\,T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

will be called the *Hamilton (ρ,η)-space* or the *Cartan (ρ,η)-space, respectively.*

The generalized Hamilton (Id_{TM}, Id_M)-space, the Hamilton (Id_{TM}, Id_M)-space, and the Cartan (Id_{TM}, Id_M)-space will be called the *generalized Hamilton space, Hamilton space, Cartan space.*

Definition 6.9.5 The normal distinguished linear (ρ, η)-connections of a Hamilton or Cartan (ρ, η)-space will be called *Hamilton* and *Cartan linear (ρ, η)-connections.*

The Hamilton and Cartan linear (Id_{TM}, Id_M)-connections will be called *Hamilton* and *Cartan linear connections*, respectively.

Theorem 6.9.1 *If the (pseudo)metrical structure $\overset{*}{G}$ is determined by a (pseudo)metrical structure*

$$g \in \mathcal{T}\,{}^2_0\left(V\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

then the real local functions:

$$\begin{array}{rl}
\left(\rho,\eta\right)\overset{*}{H}{}^{a}_{bc} & = \dfrac{1}{2}g^{ae}\left(\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_b\right)\tilde{g}_{ec} + \Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_c\right)\tilde{g}_{be}\right.\\
& -\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_e\right)\tilde{g}_{bc} - \tilde{g}_{cd}\cdot L^d_{be}\circ h\circ\overset{*}{\pi}\\
& \left.+\tilde{g}_{bd}\cdot L^d_{ec}\circ h\circ\overset{*}{\pi} - \tilde{g}_{ed}\cdot L^d_{bc}\circ h\circ\overset{*}{\pi}\right),\\
\left(\rho,\eta\right)\overset{*}{V}{}^{bc}_{a} & = \dfrac{1}{2}\tilde{g}_{ae}\left(\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{\cdot}{\tilde{\partial}}{}^{c}\right)g^{eb}\right.\\
& \left.+\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{\cdot}{\tilde{\partial}}{}^{b}\right)g^{ec} - \Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right)\left(\overset{\cdot}{\tilde{\partial}}{}^{e}\right)g^{bc}\right)
\end{array}\tag{6.9.5}$$

are the components of a normal distinguished linear (ρ, η)-connection with (ρ,η)-$\overset{*}{\mathcal{H}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right)$ and (ρ,η)-$\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{V}}\overset{*}{\mathcal{V}}\right)$ torsions free such that the generalized tangent bundle

$$\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

derives generalized Hamilton (ρ, η)-space.

This normal distinguished linear (ρ, η)-connection will be called *generalized linear (ρ, η)-connection of Levi-Civita type.*

If the *(pseudo)metrical structure* g is determined with the help of a Hamilton or Cartan fundamental function, then the Hamilton and the Cartan linear (ρ, η)-connections will be called *canonical Hamilton* and *Cartan linear (ρ, η)-connection, respectively.*

The canonical Hamilton and Cartan linear (Id_{TM}, Id_M)-connection will be called the *canonical Hamilton* and *Cartan linear connection* respectively.

Theorem 6.9.2 *Let $\left(\left(\rho,\eta\right)\overset{*}{H},\left(\rho,\eta\right)\overset{*}{V}\right)$ be the normal distinguished linear (ρ, η)-connection presented in the previous theorem.*

If

$$\overset{*}{\mathbb{T}}{}^{a}_{bc}\overset{*}{\tilde{\delta}}_a\otimes d\tilde{z}^b\otimes d\tilde{z}^c\in\mathcal{T}^{10}_{20}\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and

$$\overset{*}{\mathbb{S}}_{a}^{bc}\,\dot{\tilde{\partial}}^{a}\otimes\delta\tilde{p}_{b}\otimes\delta\tilde{p}_{c}\in\mathcal{T}_{01}^{02}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

such that they satisfy the following conditions:

$$\overset{*}{\mathbb{T}}{}_{bc}^{a}=-\overset{*}{\mathbb{T}}{}_{cb}^{a},\ \overset{*}{\mathbb{S}}{}_{a}^{bc}=-\overset{*}{\mathbb{S}}{}_{a}^{bc},\ \forall b,c\in\overline{1,r},$$

then the real local functions:

$$\text{(6.9.6)}\qquad \begin{array}{l}(\rho,\eta)\,\tilde{\overset{*}{H}}{}_{bc}^{a}=(\rho,\eta)\,\overset{*}{H}{}_{bc}^{a}+\dfrac{1}{2}g^{ae}\cdot\left(\tilde{g}_{ed}\overset{*}{\mathbb{T}}{}_{bc}^{d}-\tilde{g}_{bd}\overset{*}{\mathbb{T}}{}_{ec}^{d}+\tilde{g}_{cd}\overset{*}{\mathbb{T}}{}_{be}^{d}\right),\\ (\rho,\eta)\,\tilde{\overset{*}{V}}{}_{a}^{bc}=(\rho,\eta)\,\overset{*}{V}{}_{a}^{bc}+\dfrac{1}{2}\tilde{g}_{ae}\cdot\left(g^{ed}\overset{*}{\mathbb{S}}{}_{d}^{bc}-g^{bd}\overset{*}{\mathbb{S}}{}_{d}^{ec}+g^{cd}\overset{*}{\mathbb{S}}{}_{d}^{be}\right)\end{array}$$

are the components of a normal distinguished linear (ρ,η)-connection with (ρ,η)-$\overset{*}{\mathcal{H}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right)$ and (ρ,η)-$\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{V}}\overset{*}{\mathcal{V}}\right)$ torsions a priori given such that the generalized tangent bundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

derives generalized Hamilton (ρ,η)-space.

Moreover, we obtain:

$$\text{(6.9.7)}\qquad \begin{array}{l}\overset{*}{\mathbb{T}}{}_{bc}^{a}=(\rho,\eta)\,\overset{*}{H}{}_{bc}^{a}-(\rho,\eta)\,\overset{*}{H}{}_{cb}^{a}-L_{bc}^{a}\circ h\circ\overset{*}{\pi},\\ \overset{*}{\mathbb{S}}{}_{a}^{bc}=(\rho,\eta)\,\overset{*}{V}{}_{a}^{bc}-(\rho,\eta)\,\overset{*}{V}{}_{a}^{cb}.\end{array}$$

6.10 Einstein equations

We shall consider a metric structure

$$\overset{*}{G}=g_{\alpha\beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+g^{ab}\delta\tilde{p}_{a}\otimes\delta\tilde{p}_{b}$$

and a distinguished linear (ρ,η)-connection $\left((\rho,\eta)\,\overset{*}{H},(\rho,\eta)\,\overset{*}{V}\right)$ compatible with the structure metric $\overset{*}{G}$ having $\overset{*}{\mathcal{H}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right)$ and $\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{V}}\overset{*}{\mathcal{V}}\right)$-torsions prescribed.

Definition 6.10.1 If $(\rho,\eta,h)\,\overset{*}{\mathbb{R}}{}_{\alpha\ \beta}$ and $(\rho,\eta,h)\,\overset{*}{\mathbb{S}}{}^{a\ b}$ are the components of tensor Ricci associated to distinguished linear (ρ,η)-connection

$$((\rho,\eta)\,H,(\rho,\eta)\,V)\,,$$

then the scalar

$$\text{(6.10.1)}\qquad (\rho,\eta,h)\,\overset{*}{\mathbb{R}}=(\rho,\eta,h)\,\overset{*}{\mathbb{R}}{}_{\alpha\ \beta}g^{\alpha\beta}+(\rho,\eta,h)\,\overset{*}{\mathbb{S}}{}^{a\ b}\tilde{g}_{ab}$$

will be called the *scalar of curvature of distinguished linear (ρ,η)-connection* $((\rho,\eta)\,H,(\rho,\eta)\,V)$.

Definition 6.10.2 The tensor field

$$(6.10.2)\qquad \begin{aligned}(\rho,\eta,h)\overset{*}{\mathbb{T}} &= (\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{\alpha\ \beta}d\tilde{z}^{\alpha}\otimes d\tilde{z}^{\beta}+(\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{\alpha}^{\ \ b}d\tilde{z}^{\alpha}\otimes\delta\tilde{p}_{b}\\ &+(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{a}_{\ \ \beta}\delta\tilde{p}_{a}\otimes d\tilde{z}^{\beta}+(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{a\ b}\delta\tilde{p}_{a}\otimes\delta\tilde{p}_{b}\end{aligned}$$

such that its components verify the following conditions:

$$(6.10.3)\qquad \begin{aligned}\varkappa(\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{\alpha\ \beta} &= (\rho,\eta,h)\overset{*}{\mathbb{R}}_{\alpha\beta}-\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{R}}\cdot g_{\alpha\beta},\\ -\varkappa(\rho,\eta,h)\overset{*}{\mathbb{T}}{}_{\alpha}^{\ \ b} &= (\rho,\eta,h)\overset{*}{\mathbb{P}}{}_{\alpha}^{\ \ b},\\ \varkappa(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{a}_{\ \ \beta} &= (\rho,\eta,h)\overset{*}{\mathbb{P}}{}^{a}_{\ \ \beta},\\ \varkappa(\rho,\eta,h)\overset{*}{\mathbb{T}}{}^{a\ b} &= (\rho,\eta,h)\overset{*}{\mathbb{S}}{}^{a\ b}-\frac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{R}}\cdot g^{ab},\end{aligned}$$

where $\varkappa$ is a constant, will be called *the energy-momentum tensor field associated to distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$ and metrical structure $\overset{*}{G}$.*

The equations (4.10.3) will be called *the Einstein equations associated to distinguished linear (ρ,η)-connection $\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)$ and metrical structure $\overset{*}{G}$.*

Formally, the Einstein equations will be written

$$(6.10.3)'\qquad \mathbf{Ric}\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right)-\tfrac{1}{2}(\rho,\eta,h)\overset{*}{\mathbb{R}}\cdot\overset{*}{G}=\varkappa\cdot(\rho,\eta,h)\overset{*}{\mathbb{T}}.$$

6.11 Dual mechanical systems

Using the diagram:

$$(6.11.1)\qquad \begin{array}{ccc}\overset{*}{E} & & \left(E,[,]_{E,h},(\rho,\eta)\right)\\ \ \ \downarrow\overset{*}{\pi} & & \downarrow\pi\\ M & \xrightarrow{\ h\ } & M\end{array}$$

where $\left((E,\pi,M),[,]_{E,h},(\rho,\eta)\right)$ is a generalized Lie algebroid, we build the generalized tangent bundle

$$(((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}),[,]_{(\rho,\eta)T\overset{*}{E}},(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}})).$$

Definition 6.11.1 A triple

$$(6.11.2)\qquad \left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_{e},(\rho,\eta)\overset{*}{\Gamma}\right),$$

where

$$(6.11.3)\qquad \overset{*}{F}_{e}=F_{a}\overset{\cdot}{\tilde{\partial}}{}^{a}\in\Gamma\left(V(\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is an external force and $(\rho,\eta)\overset{*}{\Gamma}$ is a (ρ,η)-connection, will be called *dual mechanical (ρ,η)-system.*

A dual mechanical (ρ,η)-system

$$\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_{e},(\rho,\eta)\overset{*}{\Gamma}\right)$$

endowed with a (pseudo)metrical structure $\overset{*}{G}$ determined with the help of a (pseudo)metrical structure

$$g=g^{ab}\delta\tilde{p}_{a}\otimes\delta\tilde{p}_{a}\in\mathcal{T}\,_{00}^{02}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

will be denoted

$$\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_{e},(\rho,\eta)\overset{*}{\Gamma},\overset{*}{G}\right) \tag{6.11.4}$$

and will be called *generalized Hamilton mechanical (ρ,η)-system.*

Any dual mechanical (Id_{TM},Id_{M})-system and any generalized Hamilton mechanical (Id_{TM},Id_{M})-system will be called *mechanical system* and *generalized Hamilton mechanical system,* respectively.

Definition 6.11.2 If H respectively K is a smooth Hamilton respectively Cartan function, then we put the triple

$$\left(\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{\pi},M\right),\overset{*}{F}_{e},H\right),$$

respectively

$$\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_{e},K\right),$$

where

$$\overset{*}{F}_{e}=F_{a}\overset{\cdot\,a}{\tilde{\partial}}\in\Gamma\left(V\,(\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

is an external force. These are called *Hamilton mechanical (ρ,η)-system* and *Cartan mechanical (ρ,η)-system* respectively.

Any Hamilton mechanical (Id_{TM},Id_{M})-system and any Cartan mechanical (Id_{TM},Id_{M})-system will be called *Hamilton mechanical system* and *Cartan mechanical system, respectively.*

6.11.1 (ρ,η)-semisprays and (ρ,η)-sprays for dual mechanical (ρ,η)-systems

Let $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_{e},(\rho,\eta)\overset{*}{\Gamma}\right)$ be an arbitrary dual mechanical (ρ,η)-system.

Definition 6.11.1.1 The vertical section

$$\overset{*}{\mathbb{C}}=p_{a}\overset{\cdot\,a}{\tilde{\partial}}, \tag{6.11.1.1}$$

will be called *the Liouville section.*

Definition 6.11.1.2 The section $\overset{*}{S}\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ will be called (ρ,η)-*semispray* if there exists an almost tangent structure e such that

$$e\left(\overset{*}{S}\right)=\overset{*}{\mathbb{C}}. \tag{6.11.1.2}$$

Let (g,h) be a locally invertible $\mathbf{B^v}$-morphism of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ source and (E,π,M) target.

Theorem 6.11.1.1 *The section*

$$S=\left(g^{ab}\circ h\circ\overset{*}{\pi}\right)p_b\tfrac{\partial}{\partial\tilde{z}^a}-2\left(G_a-\tfrac{1}{4}F_a\right)\tfrac{\partial}{\partial\tilde{p}_a}\tag{6.11.1.3}$$

is a (ρ,η)-semispray such that the real local functions $G_a,\ a\in\overline{1,n}$, satisfy the following conditions

$$\begin{array}{ll}(\rho,\eta)\overset{*}{\Gamma}_{bc} & =-\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\right)\frac{\partial\left(G_c-\frac{1}{4}F_c\right)}{\partial p_e}\\ & +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)L^a_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ac}\circ h\circ\overset{*}{\pi},\ \ b,c\in\overline{1,r}.\end{array}\tag{6.11.1.4}$$

In addition, we remark that the local real functions

$$\begin{array}{ll}(\rho,\eta)\overset{\overset{*}{\circ}}{\Gamma}_{bc} & \overset{put}{=}-\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\right)\frac{\partial G_c}{\partial p_e}\\ & +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)L^a_{db}\circ h\circ\pi\cdot\tilde{g}_{ac}\circ h\circ\overset{*}{\pi},\ \ a,b\in\overline{1,r}\end{array}\tag{6.11.1.5}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\overset{\overset{*}{\circ}}{\Gamma}$ for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

The (ρ,η)-semispray $\overset{*}{S}$ will be called *the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).*

Proof. We consider the **Mod**-endomorphism

$$\begin{array}{rcl}\Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right) & \overset{\overset{*}{\mathbb{P}}}{\longrightarrow} & \Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)\\ X & \longmapsto & \overset{*}{\mathcal{J}}_{(g,h)}\left[\overset{*}{S},X\right]_{(\rho,\eta)T\overset{*}{E}}-\left[\overset{*}{S},\overset{*}{\mathcal{J}}_{(g,h)}X\right]_{(\rho,\eta)T\overset{*}{E}}.\end{array}$$

Let $X=\tilde{Z}^a\overset{*}{\tilde{\partial}}_a+Y_a\overset{\cdot}{\tilde{\partial}}{}^a$ be an arbitrary section.
Since

$$\begin{array}{ll}\left[\overset{*}{S},X\right]_{(\rho,\eta)T\overset{*}{E}} & =\left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,\tilde{Z}^b\overset{*}{\tilde{\partial}}_b\right]_{(\rho,\eta)T\overset{*}{E}}\\ & +\left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,Y_b\overset{\cdot}{\tilde{\partial}}{}^b\right]_{(\rho,\eta)T\overset{*}{E}}\\ & -\left[2\left(G_a-\frac{1}{4}F_a\right)\overset{\cdot}{\tilde{\partial}}{}^a,\tilde{Z}^b\overset{*}{\tilde{\partial}}_b\right]_{(\rho,\eta)T\overset{*}{E}}\\ & -\left[2\left(G_a-\frac{1}{4}F_a\right)\overset{\cdot}{\tilde{\partial}}{}^a,Y_b\overset{\cdot}{\tilde{\partial}}{}^b\right]_{(\rho,\eta)T\overset{*}{E}}\end{array}$$

and

$$\begin{aligned}\left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,\tilde{Z}^b\overset{*}{\tilde{\partial}}_b\right]_{(\rho,\eta)T\overset{*}{E}} &= \left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial\tilde{Z}^c}{\partial x^i}\overset{*}{\tilde{\partial}}_c\\ &-\tilde{Z}^b\rho_b^j\circ h\circ\overset{*}{\pi}\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial x^j}\overset{*}{\tilde{\partial}}_c\\ &+\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\tilde{Z}^b\left(L_{ab}^c\circ h\circ\overset{*}{\pi}\right)\overset{*}{\tilde{\partial}}_c,\end{aligned}$$

$$\begin{aligned}\left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,Y_b\dot{\tilde{\partial}}^b\right]_{(\rho,\eta)T\overset{*}{E}} &= \left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial Y_c}{\partial x^i}\dot{\tilde{\partial}}^c\\ &-Y_b\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_b}\overset{*}{\tilde{\partial}}_c,\end{aligned}$$

$$\begin{aligned}\left[2\left(G_a-\tfrac{1}{4}F_a\right)\dot{\tilde{\partial}}^a,\tilde{Z}^b\overset{*}{\tilde{\partial}}_b\right]_{(\rho,\eta)T\overset{*}{E}} &= 2\left(G_a-\tfrac{1}{4}F_a\right)\frac{\partial\tilde{Z}^c}{\partial p_a}\overset{*}{\tilde{\partial}}_c\\ &-2\tilde{Z}^b\rho_b^j\circ h\circ\overset{*}{\pi}\frac{\partial\left(G_c-\frac{1}{4}F_c\right)}{\partial x^j}\dot{\tilde{\partial}}^c,\end{aligned}$$

$$\begin{aligned}\left[2\left(G_a-\tfrac{1}{4}F_a\right)\dot{\tilde{\partial}}^a,Y_b\dot{\tilde{\partial}}^b\right]_{(\rho,\eta)T\overset{*}{E}} &= 2\left(G_a-\tfrac{1}{4}F_a\right)\frac{\partial Y_c}{\partial p_a}\dot{\tilde{\partial}}^c\\ &-2Y_b\frac{\partial\left(G_c-\frac{1}{4}F_c\right)}{\partial p_b}\dot{\tilde{\partial}}^c,\end{aligned}$$

it results that

$$(P_1)\qquad\begin{aligned}\overset{*}{\mathcal{J}}_{(g,h)}\left[\overset{*}{S},X\right]_{(\rho,\eta)T\overset{*}{E}} &= \left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial\tilde{Z}^c}{\partial x^i}\cdot\left(\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\right)\dot{\tilde{\partial}}^d\\ &-\tilde{Z}^b\rho_b^j\circ h\circ\overset{*}{\pi}\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial x^j}\cdot\left(\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\right)\dot{\tilde{\partial}}^d\\ &+\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\tilde{Z}^bL_{ab}^c\cdot\left(\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\right)\dot{\tilde{\partial}}^d\\ &-Y^b\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial y^b}\cdot\left(\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\right)\dot{\tilde{\partial}}^d\\ &-2\left(G_a-\tfrac{1}{4}F_a\right)\frac{\partial\tilde{Z}^c}{\partial p_a}\cdot\left(\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\right)\dot{\tilde{\partial}}^d.\end{aligned}$$

Since

$$\begin{aligned}\left[\overset{*}{S},\overset{*}{\mathcal{J}}_{(g,h)}X\right]_{(\rho,\eta)T\overset{*}{E}} &= \left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,\left(\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^b\dot{\tilde{\partial}}^c\right]_{(\rho,\eta)T\overset{*}{E}}\\ &-\left[2\left(G_a-\tfrac{1}{4}F_a\right)\dot{\tilde{\partial}}^a,\left(\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^b\dot{\tilde{\partial}}^c\right]_{(\rho,\eta)T\overset{*}{E}}\end{aligned}$$

and

$$\begin{aligned}\left[\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\overset{*}{\tilde{\partial}}_a,\left(\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^b\dot{\tilde{\partial}}_c\right]_{(\rho,\eta)T\overset{*}{E}} &= \left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial\left(\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\right)}{\partial x^i}\dot{\tilde{\partial}}^d\\ &-\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\frac{\partial\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_c}\overset{*}{\tilde{\partial}}_d,\end{aligned}$$

$$\left[2\left(G_a-\frac{1}{4}F_a\right)\dot{\tilde{\partial}}^{a},\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\tilde{Z}^b\dot{\tilde{\partial}}^{c}\right]_{(\rho,\eta)T\overset{*}{E}}=2\left(G_a-\frac{1}{4}F_a\right)\frac{\partial\left(\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\right)}{\partial p_a}\dot{\tilde{\partial}}^{d}$$
$$-2\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\frac{\partial\left(G_d-\frac{1}{4}F_d\right)}{\partial p_c}\dot{\tilde{\partial}}^{d}$$

it results that

$$(P_2)\qquad\begin{aligned}\left[\overset{*}{S},\overset{*}{\mathcal{J}}_{(g,h)}X\right]_{(\rho,\eta)T\overset{*}{E}}&=\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\pi\frac{\partial\left(\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\right)}{\partial x^i}\dot{\tilde{\partial}}^{d}\\&-\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\frac{\partial\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_c}\overset{*}{\tilde{\partial}}_d\\&-2\left(G_a-\frac{1}{4}F_a\right)\frac{\partial\left(\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\right)}{\partial p_a}\dot{\tilde{\partial}}^{d}\\&+2\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\frac{\partial\left(G_d-\frac{1}{4}F_d\right)}{\partial p_c}\dot{\tilde{\partial}}^{d}.\end{aligned}$$

We remark that

$$\begin{aligned}\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial\left(\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\right)}{\partial x^i}&=g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\left(\rho_a^i\circ h\circ\overset{*}{\pi}\right)\frac{\partial\tilde{Z}^c}{\partial x^i}\cdot\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}\\&-\tilde{Z}^b\rho_b^j\circ h\circ\overset{*}{\pi}\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial x^j}\cdot\tilde{g}_{dc}\circ h\circ\overset{*}{\pi},\end{aligned}$$

$$Y_d=Y_b\frac{\partial\left(g^{ce}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_b}\cdot\tilde{g}_{dc}\circ h\circ\overset{*}{\pi}$$

and

$$\tilde{Z}^d=\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\cdot\tilde{Z}^b\frac{\partial\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)}{\partial p_c}.$$

Using the equalities (P_1) and (P_2) we obtain:

$$\mathbb{P}\left(\tilde{Z}^a\overset{*}{\tilde{\partial}}_a+Y_a\dot{\tilde{\partial}}^{a}\right)=\tilde{Z}^a\overset{*}{\tilde{\partial}}_a+$$
$$+\left(-Y_a-2\tilde{g}_{cb}\circ h\circ\overset{*}{\pi}\frac{\partial\left(G_a-\frac{1}{4}F_a\right)}{\partial p_c}\tilde{Z}^b+\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\tilde{Z}^bL_{db}^c\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ac}\circ h\circ\overset{*}{\pi}\right)\dot{\tilde{\partial}}^{a}.$$

After some calculations, it results that $\mathbb{P}$ is an almost product structure.

Using the equality

$$\mathbb{P}=Id-2\left(\rho,\eta\right)\overset{*}{\Gamma},$$

we obtain that

$$\left(\rho,\eta\right)\Gamma\left(\tilde{Z}^a\overset{*}{\tilde{\partial}}_a+Y_a\dot{\tilde{\partial}}^{a}\right)=$$
$$=\left(Y_a+\tilde{g}_{ac}\circ h\circ\overset{*}{\pi}\frac{\partial\left(G_b-\frac{1}{4}F_b\right)}{\partial p_a}\tilde{Z}^c-\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\tilde{Z}^cL_{dc}^f\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{af}\circ h\circ\overset{*}{\pi}\right)\dot{\tilde{\partial}}^{a}$$

Since

$$\left(\rho,\eta\right)\Gamma\left(\tilde{Z}^a\overset{*}{\tilde{\partial}}_a+Y_a\dot{\tilde{\partial}}^{a}\right)=\left(Y_c-\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}\tilde{Z}^b\right)\dot{\tilde{\partial}}^{c}$$

it results that relations (4.11.1.4) are satisfied. In addition, since

$$(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}=(\rho,\eta)\overset{*}{\Gamma}_{bc}-\tfrac{1}{4}\tilde{g}_{db}\circ h\circ\overset{*}{\pi}\tfrac{\partial F_c}{\partial p_d}$$

and

$$\begin{array}{l}(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{b'c'}=(\rho,\eta)\overset{*}{\Gamma}_{b'c'}-\tfrac{1}{4}\tilde{g}_{e'c'}\circ h\circ\overset{*}{\pi}\tfrac{\partial F_{b'}}{\partial p_{e'}}\\
=M_{b'}^{b}\circ\overset{*}{\pi}\left(-\rho_c^i\circ h\circ\overset{*}{\pi}\cdot\tfrac{\partial M_b^{a'}}{\partial x^i}p_{a'}+(\rho,\eta)\overset{*}{\Gamma}_{bc}\right)M_{c'}^{c}\circ h\circ\overset{*}{\pi}\\
+M_{b'}^{b}\circ\overset{*}{\pi}\left(\tfrac{1}{4}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\cdot\tfrac{\partial F_b}{\partial p_e}\right)M_{c'}^{c}\circ h\circ\overset{*}{\pi}\\
=M_{b'}^{b}\circ\overset{*}{\pi}\left(-\rho_c^i\circ h\circ\overset{*}{\pi}\cdot\tfrac{\partial M_b^{a'}}{\partial x^i}p_{a'}+\left((\rho,\eta)\overset{*}{\Gamma}_{bc}-\tfrac{1}{4}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\cdot\tfrac{\partial F_b}{\partial p_e}\right)\right)M_{c'}^{c}\circ h\circ\overset{*}{\pi}\\
=M_{b'}^{b}\circ\overset{*}{\pi}\left(-\rho_c^i\circ h\circ\overset{*}{\pi}\cdot\tfrac{\partial M_b^{a'}}{\partial x^i}p_{a'}+(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}\right)M_{c'}^{c}\circ h\circ\overset{*}{\pi}\end{array}$$

it results the conclusion of the theorem. *q.e.d.*

Theorem 6.11.1.2 *The following properties hold:*

1° *Since*

$$\overset{*}{\mathring{\tilde{\delta}}}_c=\overset{*}{\tilde{\partial}}_c+(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}\overset{\cdot\, b}{\tilde{\partial}},\ c\in\overline{1,r},$$

it results that

(6.11.1.6) $$\overset{*}{\mathring{\tilde{\delta}}}_c=\overset{*}{\tilde{\delta}}_c-\tfrac{1}{4}\tilde{g}_{ec}\circ h\circ\pi\cdot\tfrac{\partial F_b}{\partial p_e}\overset{\cdot\, b}{\tilde{\partial}},\ c\in\overline{1,r}.$$

2° *Since*

$$\mathring{\delta}\tilde{p}_b=-(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}d\tilde{z}^c+d\tilde{p}_b,$$

it results that

(6.11.1.7) $$\mathring{\delta}\tilde{p}_b=\delta\tilde{p}_b+\tfrac{1}{4}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\tfrac{\partial F_b}{\partial\tilde{p}_e}d\tilde{z}^c,\ b\in\overline{1,r}.$$

Theorem 6.11.1.3 *The real local functions*

(6.11.1.8) $$\left(\tfrac{\partial(\rho,\eta)\overset{*}{\Gamma}_{bc}}{\partial p_a},\tfrac{\partial(\rho,\eta)\overset{*}{\Gamma}_{bc}}{\partial p_a},0,\ 0\right),\ a,b,c\in\overline{1,r}$$

and

(6.11.1.8)′ $$\left(\tfrac{\partial(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}}{\partial p_a},\tfrac{\partial(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}}{\partial p_a},0,\ 0\right),\ a,b,c\in\overline{1,r}$$

respectively are the coefficients to a normal Berwald linear (ρ,η)-connection for the generalized tangent bundle $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Theorem 6.11.1.4 *The tensor of integrability of the (ρ,η)-connection $(\rho,\eta)\overset{*}{\mathring{\Gamma}}$ is as follows:*

$$
\begin{array}{ll}
 & (\rho,\eta,h)\overset{*}{\mathring{\mathbb{R}}}_{b\ cd}=(\rho,\eta,h)\overset{*}{\mathbb{R}}_{b\ cd}+\frac{1}{4}\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}_{|c}-\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}_{|d}\right)+\\
(6.11.1.9) & +\frac{1}{16}\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_l}{\partial p_e}\tilde{g}_{fc}\circ h\circ\overset{*}{\pi}\frac{\partial^2 F_b}{\partial p_l\partial p_f}-\tilde{g}_{fc}\circ h\circ\overset{*}{\pi}\frac{\partial F_l}{\partial p_f}\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial^2 F_b}{\partial p_l\partial p_e}\right)+\\
 & +\frac{1}{4}\left(L_{cd}^{f}\circ h\circ\overset{*}{\pi}\right)\left(\tilde{g}_{ef}\circ h\circ\overset{*}{\pi}\right)\frac{\partial F_b}{\partial p_e},
\end{array}
$$

where $_{|c}$ is the h-covariant derivation with respect to the normal Berwald linear ρ-connection (6.11.1.8).

Proof. Since

$$
\begin{array}{rl}
(\rho,\eta,h)\overset{*}{\mathring{\mathbb{R}}}_{b\ cd}= & \Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\mathring{\tilde{\delta}}}_c\right)\left((\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bd}\right)-\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\mathring{\tilde{\delta}}}_d\right)\left((\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}\right)\\
 & -L_{cd}^{e}\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{be},
\end{array}
$$

and

$$
\begin{array}{rl}
\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\mathring{\tilde{\delta}}}_c\right)\left((\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bd}\right) & =\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_c\right)\left((\rho,\eta)\overset{*}{\Gamma}_{bd}\right)\\
 & +\frac{1}{4}\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_c\right)\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}\right)\\
 & -\frac{1}{4}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_f}{\partial p_e}\frac{\partial}{\partial p_f}\left((\rho,\eta)\overset{*}{\Gamma}_{bd}\right)\\
 & -\frac{1}{16}\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_f}{\partial p_e}\frac{\partial}{\partial p_f}\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}\right),
\end{array}
$$

$$
\begin{array}{rl}
\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\mathring{\tilde{\delta}}}_d\right)\left((\rho,\eta)\overset{*}{\mathring{\Gamma}}_{bc}\right) & =\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_d\right)\left((\rho,\eta)\overset{*}{\Gamma}_{bc}\right)\\
 & +\frac{1}{4}\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\overset{*}{\tilde{\delta}}_d\right)\left(\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}\right)\\
 & -\frac{1}{4}\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_f}{\partial p_e}\frac{\partial}{\partial p_f}\left((\rho,\eta)\overset{*}{\Gamma}_{bc}\right)\\
 & -\frac{1}{16}\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\frac{\partial F_f}{\partial p_e}\frac{\partial}{\partial p_f}\left(\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}\right),
\end{array}
$$

$$
\begin{array}{rl}
L_{cd}^{e}\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{\mathring{\Gamma}}_{be} & =L_{cd}^{e}\circ h\circ\overset{*}{\pi}\cdot(\rho,\eta)\overset{*}{\Gamma}_{be}\\
 & +L_{cd}^{e}\circ h\circ\overset{*}{\pi}\cdot\left(\tilde{g}_{fe}\circ h\circ\overset{*}{\pi}\frac{\partial F_b}{\partial p_e}\right)
\end{array}
$$

it results the conclusion of the theorem. *q.e.d.*

Theorem 6.11.1.5 *Let*

$$
\overset{*}{\mathbb{T}}{}_{bc}^{a}\delta_a\otimes d\tilde{z}^b\otimes d\tilde{z}^c\in\mathcal{T}_{20}^{10}\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)
$$

and

$$\overset{*}{\mathbb{S}}{}_{a}^{bc}\,\tilde{\partial}_a \otimes \delta\tilde{y}^b \otimes \delta\tilde{y}^c \in \mathcal{T}_{01}^{02}\left((\rho,\eta)\, T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

such that they verify the following conditions:

$$\overset{*}{\mathbb{T}}{}_{bc}^{a} = -\overset{*}{\mathbb{T}}{}_{cb}^{a},\ \overset{*}{\mathbb{S}}{}_{a}^{bc} = -\overset{*}{\mathbb{S}}{}_{a}^{bc},\ \forall b,c \in \overline{1,r}.$$

If $\left((\rho,\eta)\,\overset{*}{\tilde{H}}, (\rho,\eta)\,\overset{*}{\tilde{V}}\right)$ *is the distinguished linear* (ρ,η)*-connection presented in the Theorem 6.9.2, then the local real functions:*

$$\begin{aligned}(\rho,\eta)\,\overset{\overset{*}{\circ}}{\tilde{H}}{}_{bc}^{a} &= (\rho,\eta)\,\overset{*}{\tilde{H}}{}_{bc}^{a} + \tfrac{1}{8}g^{ae}\left(-\tilde{g}_{fc}\circ h\circ\overset{*}{\pi}\tfrac{\partial F_d}{\partial p_f}\tfrac{\partial\tilde{g}_{be}}{\partial p_d}\right.\\ &\qquad\left. + \tilde{g}_{fe}\circ h\circ\overset{*}{\pi}\tfrac{\partial F_d}{\partial p_f}\tfrac{\partial\tilde{g}_{bc}}{\partial p_d} - \tilde{g}_{fb}\circ h\circ\overset{*}{\pi}\tfrac{\partial F_d}{\partial p_f}\tfrac{\partial\tilde{g}_{ec}}{\partial p_d}\right),\\ (\rho,\eta)\,\overset{\overset{*}{\circ}}{\tilde{V}}{}_{bc}^{a} &= (\rho,\eta)\,\overset{*}{\tilde{V}}{}_{bc}^{a}\end{aligned} \tag{6.11.1.10}$$

are the components of a normal distinguished linear (ρ,η)*-connection with* (ρ,η)*-*$\overset{*}{\mathcal{H}}\left(\overset{*}{\mathcal{H}}\overset{*}{\mathcal{H}}\right)$ *and* (ρ,η)*-*$\overset{*}{\mathcal{V}}\left(\overset{*}{\mathcal{V}}\overset{*}{\mathcal{V}}\right)$ *torsions a priori given such that the generalized tangent bundle* $\left((\rho,\eta)\, T\overset{*}{E}, (\rho,\eta)\,\tau_{\overset{*}{E}}, \overset{*}{E}\right)$ *derives generalized Hamilton* (ρ,η)*-space.*

In addition, we have:

$$\begin{aligned}(\rho,\eta,h)\,\overset{\overset{*}{\circ}}{\mathbb{T}}{}_{bc}^{a} &= \overset{*}{\mathbb{T}}{}_{bc}^{a}\\ (\rho,\eta,h)\,\overset{\overset{*}{\circ}}{\mathbb{S}}{}_{a}^{bc} &= \overset{*}{\mathbb{S}}{}_{a}^{bc}.\end{aligned} \tag{6.11.1.11}$$

The local functions $\tilde{g}_{fc},\ \tilde{g}_{fe},\ \tilde{g}_{fb}$ *are the local functions associated to the locally invertible* $\mathbf{B^v}$*-morphism* (g,h).

Proposition 6.11.1.1 *If* $\overset{*}{S}$ *is the canonical* (ρ,η)*-semispray associated to the mechanical* (ρ,η)*-system* $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\,\overset{*}{\Gamma}\right)$ *and from locally invertible* $\mathbf{B^v}$*-morphism* (g,h)*, then*

$$2G_{\acute{b}} = 2G_b\cdot M_{\acute{b}}^{b}\circ h\circ\overset{*}{\pi} - \left(g^{ae}\circ h\circ\overset{*}{\pi}\right)p_e\left(\rho_a^i\circ h\circ\overset{*}{\pi}\right)\tfrac{\partial p_{\acute{b}}}{\partial x^i}. \tag{6.11.1.12}$$

Proof. Since the Jacobian matrix of coordinates transformation is

$$\left\Vert\begin{array}{cc} M_a^{\acute{a}}\circ h\circ\overset{*}{\pi} & 0\\ \rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial M_{\acute{b}}^{a}\circ\overset{*}{\pi}}{\partial x^i}p_a & M_{\acute{b}}^{b}\circ\overset{*}{\pi}\end{array}\right\Vert = \left\Vert\begin{array}{cc} M_a^{\acute{a}}\circ h\circ\overset{*}{\pi} & 0\\ \rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial p_{\acute{b}}}{\partial x^i} & M_{\acute{b}}^{b}\circ\overset{*}{\pi}\end{array}\right\Vert$$

and

$$\left\Vert\begin{array}{cc} M_a^{\acute{a}}\circ h\circ\overset{*}{\pi} & 0\\ \rho_a^i\circ h\circ\overset{*}{\pi}\frac{\partial p_{\acute{b}}}{\partial x^i} & M_{\acute{b}}^{b}\circ\overset{*}{\pi}\end{array}\right\Vert\cdot\left(\begin{array}{c}\left(g^{ae}\circ h\circ\overset{*}{\pi}\right)p_e\\ -2\left(G_b-\frac{1}{4}F_b\right)\end{array}\right) =$$

$$= \left(\begin{array}{c}\left(g^{\acute{a}\acute{e}}\circ h\circ\overset{*}{\pi}\right)p_{\acute{e}}\\ -2\left(G_{\acute{b}}-\frac{1}{4}F_{\acute{b}}\right)\end{array}\right),$$

the conclusion results immediately. *q.e.d.*

In the following we consider a differentiable curve $I \overset{c}{\to} M$ and its (g,h)-lift $\dot{c}$.

Definition 6.11.1.3 The curve $\dot{c}$ is an integral curve of the (ρ,η)-semispray $\overset{*}{S}$ of the dual mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$, if it is verify the following equality:

$$\frac{d\dot{c}(t)}{dt}=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\overset{*}{S}\left(\dot{c}\left(t\right)\right). \tag{6.11.1.13}$$

Theorem 6.11.1.6 *The integral curves of the canonical (ρ,η)-semispray associated to the mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible* $\mathbf{B}^{\mathbf{v}}$*-morphism (g,h), are the (g,h)-lifts solutions of the equations:*

$$\frac{dp_b(t)}{dt}+2G_b\circ\overset{*}{u}\left(c,\dot{c}\right)\left(x\left(t\right)\right)=\frac{1}{2}F_b\circ\overset{*}{u}\left(c,\dot{c}\right)\left(x\left(t\right)\right),\ b\in\overline{1,r}, \tag{6.11.1.14}$$

where $x\left(t\right)=\left(\eta\circ h\circ c\right)\left(t\right)$.

Proof. Since the equality

$$\frac{d\dot{c}(t)}{dt}=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\overset{*}{S}\left(\dot{c}\left(t\right)\right)$$

is equivalent with

$$\begin{array}{c}\frac{d}{dt}((\eta\circ h\circ c)^i(t),p_b(t))=\\ =\left(\rho_a^i\circ\eta\circ h\circ c(t)g^{ae}\circ h\circ c(t)p_e(t),-2\left(G_b-\frac{1}{4}F_b\right)((\eta\circ h\circ c)(t),p\left(t\right))\right),\end{array}$$

it results

$$\begin{array}{c}\frac{dp_b(t)}{dt}+2G_b(x\left(t\right),p\left(t\right))=\frac{1}{2}F_b(x\left(t\right),p\left(t\right)),\ b\in\overline{1,r},\\ \frac{dx^i(t)}{dt}=\rho_a^i\circ\eta\circ h\circ c\left(t\right)g^{ae}\circ h\circ c\left(t\right)p_e\left(t\right),\end{array}$$

where $x^i\left(t\right)=\left(\eta\circ h\circ c\right)^i\left(t\right)$. *q.e.d.*

Definition 6.11.1.4 If $\overset{*}{S}$ is a (ρ,η)-semispray, then the vector field

$$\left[\overset{*}{\mathbb{C}},\overset{*}{S}\right]_{(\rho,\eta)T\overset{*}{E}}-\overset{*}{S} \tag{6.11.1.15}$$

will be called the *derivation of (ρ,η)-semispray* $\overset{*}{S}$.

The (ρ,η)-semispray $\overset{*}{S}$ will be called *(ρ,η)-spray* if there are verified the following conditions:

1. $\overset{*}{S}\circ 0\in C^1$, where 0 is the null section;
2. Its derivation is the null vector field.

The (ρ,η)-semispray $\overset{*}{S}$ will be called *quadratic (ρ,η)-spray* if there are verified the following conditions:

1. $\overset{*}{S}\circ 0\in C^{2}$, where 0 is the null section;
2. Its derivation is the null vector field.

In particular, if $(\rho,\eta)=(id_{TM},Id_{M})$ and $(g,h)=(Id_{E},Id_{M})$, then we obtain the *spray* and the *quadratic spray* which is similar with the classical spray and quadratic spray.

Theorem 6.11.1.7 *If $\overset{*}{S}$ is the canonical (ρ,η)-semispray associated to mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_{e},(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible $\mathbf{B^{v}}$-morphism (g,h), then*

$$\begin{array}{rl}2\left(G_{b}-\frac{1}{4}F_{b}\right) & =(\rho,\eta)\overset{*}{\Gamma}_{bc}\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_{f}\right)\\ & +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_{e}\right)L_{dc}^{a}\circ h\circ\overset{*}{\pi}\\ & \cdot\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_{f}\right),\ b\in\overline{1,r}.\end{array}\tag{6.11.1.16}$$

Then we obtain the spray

$$\begin{array}{l}\overset{*}{S}=\left(g^{ae}\circ h\circ\overset{*}{\pi}\right)p_{e}\frac{\partial}{\partial\tilde{z}^{a}}+(\rho,\eta)\overset{*}{\Gamma}_{bc}\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_{f}\right)\frac{\partial}{\partial\tilde{p}_{b}}\\ +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_{e}\right)L_{dc}^{a}\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_{f}\right)\frac{\partial}{\partial\tilde{p}_{b}}.\end{array}\tag{6.11.1.17}$$

This (ρ,η)-spray will be called the canonical (ρ,η)-spray associated to mechanical system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_{e},(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible $\mathbf{B^{v}}$-morphism (g,h).

In particular, if $(\rho,\eta,g,h)=(Id_{TM},Id_{M},Id_{M},Id_{M})$, then we get the canonical spray associated to connection $\overset{*}{\Gamma}$ which is similar with the classical canonical spray associated to connection $\overset{*}{\Gamma}$.

Proof. Since

$$\begin{array}{rl}\left[\overset{*}{\mathbb{C}},\overset{*}{S}\right]_{(\rho,\eta)T\overset{*}{E}} & =\left[p_{a}\overset{\cdot\,a}{\tilde{\partial}},\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot p_{e}\right)\overset{*}{\tilde{\partial}}_{b}\right]_{(\rho,\eta)T\overset{*}{E}}\\ & -2\left[p_{a}\overset{\cdot\,a}{\tilde{\partial}},\left(G_{b}-\frac{1}{4}F_{b}\right)\overset{\cdot\,b}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}},\end{array}$$

$$\begin{array}{rl}\left[p_{a}\overset{\cdot\,a}{\tilde{\partial}},\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot p_{e}\right)\overset{*}{\tilde{\partial}}_{b}\right]_{(\rho,\eta)T\overset{*}{E}} & =p_{a}\frac{\partial\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot p_{e}\right)}{\partial p_{a}}\overset{*}{\tilde{\partial}}_{b}\\ & -\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot p_{e}\right)\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\frac{\partial p_{a}}{\partial x^{i}}\overset{\cdot\,a}{\tilde{\partial}}\\ & =\left(p_{a}\cdot g^{be}\circ h\circ\overset{*}{\pi}\cdot\delta_{a}^{e}\right)\overset{*}{\tilde{\partial}}_{b}-0\\ & =\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot p_{e}\right)\overset{*}{\tilde{\partial}}_{b}\end{array}$$

and
$$\left[p_a\overset{\cdot\, a}{\tilde{\partial}}, \left(G_b-\tfrac{1}{4}F_b\right)\overset{\cdot\, b}{\tilde{\partial}}\right]_{(\rho,\eta)T\overset{*}{E}} \begin{array}{l} = p_a\frac{\partial\left(G_b-\frac{1}{4}F_b\right)}{\partial p_a}\overset{\cdot\, b}{\tilde{\partial}} \\ -\left(G_b-\frac{1}{4}F_b\right)\delta_a^b\overset{\cdot\, a}{\tilde{\partial}} \\ = p_a\frac{\partial\left(G_b-\frac{1}{4}F_b\right)}{\partial p_a}\overset{\cdot\, b}{\tilde{\partial}}-\left(G_b-\frac{1}{4}F_b\right)\overset{\cdot\, b}{\tilde{\partial}} \end{array}$$

it results that

$$(S_1)\qquad \left[\overset{*}{\mathbb{C}},\overset{*}{S}\right]_{(\rho,\eta)T\overset{*}{E}}-\overset{*}{S}\ =2\left(-p_f\frac{\partial\left(G_b-\frac{1}{4}F_b\right)}{p_f}+2\left(G_b-\tfrac{1}{4}F_b\right)\right)\overset{\cdot\, b}{\tilde{\partial}}$$

Using the equality (6.11.1.4) it results that

$$(S_2)\qquad \begin{array}{ll} \frac{\partial\left(G_b-\frac{1}{4}F_b\right)}{p_f} & =-\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}\circ\overset{*}{u}\left(c,\dot{c}\right)\circ\left(\eta\circ h\circ c\right)\cdot g^{cf}\circ h\circ\overset{*}{\pi} \\ & +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\cdot L_{dc}^a\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\cdot g^{cf}\circ h\circ\overset{*}{\pi}. \end{array}$$

Using the equalities (S_1) and (S_2) it results the conclusion of the theorem. *q.e.d.*

Remark 6.11.1.2. If $(\rho,\eta,h)=(id_{TM},Id_M,Id_M)$, then we get the canonical spray associated to connection $\overset{*}{\Gamma}$.

Theorem 6.11.1.8 *The integral curves of canonical (ρ,η)-spray associated to mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h) are the (g,h)-lifts solutions of the following system of equations:*

$$(6.11.1.17)\qquad \begin{array}{l} \frac{dp_b}{dt}-\left(\rho,\eta\right)\overset{*}{\Gamma}_{bc}\circ\overset{*}{u}\left(c,\dot{c}\right)\circ\left(\eta\circ h\circ c\right)\cdot\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_f\right) \\ +\frac{1}{2}\left(g^{de}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\cdot L_{dc}^a\circ h\circ\overset{*}{\pi}\cdot\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\cdot\left(g^{cf}\circ h\circ\overset{*}{\pi}\cdot p_f\right)=0, \end{array}$$

where $x(t)=\eta\circ h\circ c(t)$.

6.11.2 The Hamiltonian formalism for Hamilton mechanical (ρ,η)-systems

Let $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right)$ be an arbitrary Hamilton mechanical (ρ,η)-system.

The *natural dual (ρ,η)-base* $(d\tilde{z}^\alpha,d\tilde{p}_a)$ of natural (ρ,η)-base $\left(\frac{\partial}{\partial\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{p}_a}\right)$ is determined by the equations

$$\begin{cases} \left\langle d\tilde{z}^\alpha,\frac{\partial}{\partial\tilde{z}^\beta}\right\rangle=\delta_\beta^\alpha, & \left\langle d\tilde{z}^\alpha,\frac{\partial}{\partial\tilde{p}_b}\right\rangle=0, \\ \left\langle d\tilde{p}_a,\frac{\partial}{\partial\tilde{z}^\beta}\right\rangle=0, & \left\langle d\tilde{p}_a,\frac{\partial}{\partial\tilde{p}_b}\right\rangle=\delta_a^b. \end{cases}$$

It is very important to remark that the 1-forms $d\tilde{z}^\alpha,\ \alpha\in\overline{1,p}$ and $d\tilde{p}_a,\ a\in\overline{1,r}$ are not the differentials of coordinates functions as in the classical case, but we will use the same notations.

In this case

$$\left(d\tilde{z}^\alpha\right)\neq d^{(\rho,\eta)T\overset{*}{E}}\left(\tilde{z}^\alpha\right)=0,$$

where $d^{(\rho,\eta)T\overset{*}{E}}$ is the exterior differentiation operator associated to exterior differential $\mathcal{F}\left(\overset{*}{E}\right)$-algebra

$$\left(\Lambda\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot,\wedge\right).$$

Let H be a regular Hamiltonian and let (g,h) be a $\mathbf{B}^{\mathbf{v}}$-morphism locally invertible of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ source and (E,π,M) target.

Definition 6.11.2.1 The 1-form

$$\theta_H=\left(\tilde{g}_{ea}\circ h\circ\overset{*}{\pi}\cdot H^e\right)d\tilde{z}^a \tag{6.11.2.1}$$

will be called the 1-*form of Poincaré-Cartan type associated to the regular Hamiltonian* H *and to the locally invertible* $\mathbf{B}^{\mathbf{v}}$-*morphism* (g,h).

We obtain easily:

$$\theta_H\left(\overset{*}{\frac{\partial}{\partial\tilde{z}^b}}\right)=\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\cdot H^e,\quad\theta_H\left(\frac{\partial}{\partial\tilde{p}_b}\right)=0. \tag{6.11.2.2}$$

Definition 6.11.2.2 The 2-form

$$\omega_H=d^{(\rho,\eta)T\overset{*}{E}}\theta_L$$

will be called the 2-*form of Poincaré-Cartan type associated to the regular Hamiltonian* H *and to the locally invertible* $\mathbf{B}^{\mathbf{v}}$-*morphism* (g,h).

By the definition of $d^{(\rho,\eta)T\overset{*}{E}}$, we obtain:

$$\begin{array}{c}\omega_H(U,V)=\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)(U)\left(\theta_H(V)\right)-\\-\Gamma\left(\tilde{\rho},Id_E\right)(V)\left(\theta_H(U)\right)-\theta_H\left([U,V]_{(\rho,\eta)T\overset{*}{E}}\right),\end{array} \tag{6.11.2.3}$$

for any $U,V\in\Gamma\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

It follows:

(6.11.2.4)

$$\begin{cases}\omega_L\left(\overset{*}{\frac{\partial}{\partial\tilde{z}^a}},\overset{*}{\frac{\partial}{\partial\tilde{z}^b}}\right)&=\left(\rho_a^i\circ h\circ\overset{*}{\pi}\right)\cdot\dfrac{\partial\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i}\\&\quad-\left(\rho_b^i\circ h\circ\overset{*}{\pi}\right)\cdot\dfrac{\partial\left(\tilde{g}_{ea}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i}-L_{ab}^c\circ h\circ\overset{*}{\pi}\cdot\left(\tilde{g}_{ec}\circ h\circ\overset{*}{\pi}\cdot H^e\right);\\\omega_L\left(\overset{*}{\frac{\partial}{\partial\tilde{z}^a}},\frac{\partial}{\partial\tilde{p}_b}\right)&=\tilde{g}_{ea}\circ h\circ\overset{*}{\pi}\cdot H^{eb};\\\omega_L\left(\frac{\partial}{\partial\tilde{p}_a},\frac{\partial}{\partial\tilde{p}_b}\right)&=0.\end{cases}$$

Definition 6.11.2.3 The real function

$$\mathcal{E}_H=p_a\cdot H^a-H \tag{6.11.2.5}$$

will be called the *energy of regular Hamiltonian H.*

Theorem 6.11.2.1 *The equation*

$$(6.11.2.6)\quad i_{\overset{*}{S}}\left(\omega_H\right)=-d^{(\rho,\eta)T\overset{*}{E}}\left(\mathcal{E}_H\right),\ \overset{*}{S}\in\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

has an unique solution $\overset{*}{S}_H\left(g,h\right)$ of the type:

$$(6.11.2.7)\qquad \left(g^{ae}\circ h\circ\overset{*}{\pi}\right)p_e\frac{\overset{*}{\partial}}{\partial\tilde{z}^a}-2\left(G_a-\tfrac{1}{4}F_a\right)\frac{\partial}{\partial\tilde{p}_a},$$

where

$$(6.11.2.8)\qquad 2G_a=\left(g^{be}\circ h\circ\overset{*}{\pi}\cdot\tilde{H}_{ea}\right)\cdot E_b\left(H,g,h\right)+\tfrac{1}{2}F_a$$

and

$$(6.11.2.9)\qquad \begin{array}{c}E_b(H,g,h)=\rho_b^i\circ h\circ\overset{*}{\pi}\cdot H_i-g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\cdot\rho_a^i\circ h\circ\overset{*}{\pi}\cdot\dfrac{\partial\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i}\\ +g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\cdot L_{ab}^d\circ h\circ\overset{*}{\pi}\cdot\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\cdot H^e\right).\end{array}$$

$\overset{*}{S}_H\left(g,h\right)$ will be called *the canonical (ρ,η)-semispray associated to Hamilton mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).*

Proof. We obtain that

$$i_{\overset{*}{S}}\left(\omega_H\right)=-d^{(\rho,\eta)T\overset{*}{E}}\left(\mathcal{E}_H\right)$$

if and only if

$$\omega_H\left(\overset{*}{S},X\right)=-\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(X\right)\left(\mathcal{E}_H\right),$$

for any $X\in\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right)$.

Particularly, we obtain:

$$\omega_H\left(\overset{*}{S},\frac{\overset{*}{\partial}}{\partial\tilde{z}^b}\right)=-\Gamma\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\left(\frac{\overset{*}{\partial}}{\partial\tilde{z}^b}\right)\left(\mathcal{E}_H\right).$$

If we expand this equality using (6.11.2.2) and (6.11.2.4), we obtain

$$\begin{array}{c}g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\cdot\left[\rho_a^i\circ h\circ\overset{*}{\pi}\cdot\dfrac{\partial\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i}-\rho_b^i\circ h\circ\overset{*}{\pi}\cdot\dfrac{\partial\left(\tilde{g}_{ea}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i}\right.\\ \left.-L_{ab}^d\circ h\circ\overset{*}{\pi}\cdot\left(\tilde{g}_{ed}\circ h\circ\overset{*}{\pi}\cdot H^e\right)\right]+2\left(G_a-\tfrac{1}{4}F_a\right)\left(\tilde{g}_{eb}\circ h\circ\overset{*}{\pi}\right)\cdot H^{ea}\\ =-\rho_b^i\circ h\circ\overset{*}{\pi}\cdot\left(g^{ae}\circ h\circ\overset{*}{\pi}\cdot p_e\right)\cdot\dfrac{\partial\left(\tilde{g}_{ea}\circ h\circ\overset{*}{\pi}\cdot H^e\right)}{\partial x^i}+\rho_b^i\circ h\circ\overset{*}{\pi}\cdot H_i.\end{array}$$

After some calculations, we obtain

$$2\left(G_a-\frac{1}{4}F_a\right)=\left(g^{be}\circ h\circ\pi\cdot\tilde{H}_{ea}\right)\cdot E_b\left(H,g,h\right),$$

where

$$E_b(H,g,h) = \rho_b^i \circ h \circ \overset{*}{\pi} \cdot H_i - g^{ae} \circ h \circ \overset{*}{\pi} \cdot p_e \cdot \rho_a^i \circ h \circ \overset{*}{\pi} \cdot \frac{\partial\left(\tilde{g}_{eb} \circ h \circ \overset{*}{\pi} \cdot H^e\right)}{\partial x^i}$$
$$+ g^{ae} \circ h \circ \overset{*}{\pi} \cdot p_e \cdot L_{ab}^d \circ h \circ \overset{*}{\pi} \cdot \left(\tilde{g}_{ed} \circ h \circ \overset{*}{\pi} \cdot H^e\right).$$

q.e.d.

Theorem 6.11.2.2 *The real local functions*

(6.11.2.10)
$$\begin{aligned}(\rho,\eta)\overset{*}{\Gamma}_{bc} &= \tfrac{1}{2}\tilde{g}_{ec} \circ h \circ \overset{*}{\pi} \frac{\partial\left(\left(g^{ae} \circ h \circ \overset{*}{\pi} \cdot H_{eb}\right) E_a(H,g,h)\right)}{\partial p_e} \\ &-\tfrac{1}{2}\left(g^{de} \circ h \circ \overset{*}{\pi} \cdot p_e\right) L_{dc}^a \circ h \circ \overset{*}{\pi} \cdot \tilde{g}_{ab} \circ h \circ \overset{*}{\pi},\ b,c \in \overline{1,r}.\end{aligned}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$ for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ which will be called the (ρ,η)-connection associated to Hamilton mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).

Corollary 6.11.2.1 *The real local functions*

(6.11.2.11)
$$\begin{aligned}(\rho,\eta)\overset{\overset{*}{\circ}}{\Gamma}_{bc} &= (\tilde{g}_{ec} \circ h \circ \pi)\frac{\partial G_b}{\partial p_e} \\ &-\tfrac{1}{2}\left(g^{de} \circ h \circ \overset{*}{\pi} \cdot p_e\right) L_{dc}^a \circ h \circ \overset{*}{\pi} \cdot \tilde{g}_{ab} \circ h \circ \overset{*}{\pi},\ b,c \in \overline{1,r}\end{aligned}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\overset{\overset{*}{\circ}}{\Gamma}$ for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$.

In addition, we have

(6.11.2.12)
$$(\rho,\eta)\overset{\overset{*}{\circ}}{\Gamma}_{bc} = (\rho,\eta)\overset{*}{\Gamma}_{bc} + \tfrac{1}{4}(\tilde{g}_{ec} \circ h \circ \pi) \cdot \tfrac{\partial F_b}{\partial p_e},\ \forall a,c \in \overline{1,r}.$$

Theorem 6.11.2.3 *The integral curves of the canonical (ρ,η)-semispray associated to $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right)$ mechanical (ρ,η)-system and from locally invertible $\mathbf{B^v}$-morphism (g,h) are the autoparallel lifts with respect to (ρ,η)-connection $(\rho,\eta)\overset{*}{\Gamma}$.*

Definition 6.11.2.4 The equations

(6.11.2.13)
$$\frac{dp_b(t)}{dt} + \left(g^{ae} \circ h \circ \overset{*}{\pi} \cdot \tilde{H}_{eb} \cdot E_a(H,g,h)\right) \circ \overset{*}{u}(c,\dot{c}) \circ (\eta \circ h \circ c(t)) = 0,$$

will be called the *equations of Hamilton-Jacobi type associated to Hamilton mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h).*

Remark 6.11.2.1 The integral curves of the canonical (ρ,η)-semispray associated to dual mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_e,H\right)$ and from locally invertible $\mathbf{B^v}$-morphism (g,h) are the (g,h)-lifts solutions for the equations of Hamilton-Jacobi type (6.11.2.13).

7 The (horizontal) Legendre (ρ, η, h)-equivalence

Let (E, π, M) be a vector bundle.

We take $\left(x^i, y^a\right)$ as canonical local coordinates on (E, π, M), where $i \in \overline{1, m}$ and $a \in \overline{1, r}$.

Consider

$$\left(x^i, y^a\right) \longrightarrow \left(x^{\acute{i}}\left(x^i\right), y^{\acute{a}}\left(x^i, y^a\right)\right)$$

a change of coordinates on (E, π, M). Then the coordinates y^a change to $y^{\acute{a}}$ by the rule:

$$y^{\acute{a}} = M_a^{\acute{a}} y^a. \tag{7.1}$$

Let $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ be the dual vector bundle of (E, π, M).

We take $\left(x^i, p_a\right)$ as canonical local coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, where $i \in \overline{1, m}$ and $a \in \overline{1, r}$.

Consider

$$\left(x^i, p_a\right) \longrightarrow \left(x^{\acute{i}}\left(x^i\right), p_{\acute{a}}\left(x^i, p_a\right)\right)$$

a change of coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$. Then the coordinates p_a change to $p_{\acute{a}}$ by the rule:

$$p_{\acute{a}} = M_{\acute{a}}^a p_a. \tag{7.1'}$$

If (U, s_U) and $\left(U, \overset{*}{s}_U\right)$ are vector local $(m+r)$-charts then

$$M_{\acute{a}}^a(x) \cdot M_b^{\acute{a}}(x) = \delta_b^a, \ \forall x \in U.$$

Let L be a differentiable Lagrangian defined on the total space of the vector bundle (E, π, M).

If (U, s_U) is a vector local $(m+r)$-chart for (E, π, M), then we obtain the following real functions defined on $\pi^{-1}(U)$:

$$\begin{array}{ll} L_i \overset{put}{=} \frac{\partial L}{\partial x^i} & L_{ib} \overset{put}{=} \frac{\partial^2 L}{\partial x^i \partial y^b} \\ L_a \overset{put}{=} \frac{\partial L}{\partial y^a} & L_{ab} \overset{put}{=} \frac{\partial^2 L}{\partial y^a \partial y^b} \end{array}. \tag{7.3}$$

We build the fiber bundle morphism

$$\begin{array}{ccc} E & \xrightarrow{\varphi_L} & \overset{*}{E} \\ \pi \downarrow & & \downarrow \overset{*}{\pi} \\ M & \xrightarrow{Id_M} & M \end{array},$$

where φ_L is locally defined

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_L} & \overset{*}{\pi}^{-1}(U) \\ u_x & \longmapsto & L_b(u_x) s^a(x) \end{array}, \tag{7.4}$$

for any vector local $(m+r)$-chart (U, s_U) of (E, π, M) and for any vector local $(m+r)$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$.

Using the differentiable Lagrangian L, we build the differentiable Hamiltonian H, locally defined by

$$(7.2') \qquad \begin{array}{ccc} \overset{*}{\pi}^{-1}(U) & \xrightarrow{H} & \mathbb{R} \\ \overset{*}{u}_x = p_a s^a & \longmapsto & p_a y^a - L(u_x) \end{array},$$

for any vector local $(m+r)$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, where $\left(y^a,\ a \in \overline{1,r}\right)$ are the components solutions of the differentiable equations

$$p_b = L_b(u_x),\ u_x \in \pi^{-1}(U).$$

If $\left(U, \overset{*}{s}_U\right)$ is a vector local $(m+r)$-chart for $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, then we obtain the following real functions defined on $\overset{*}{\pi}^{-1}(U)$:

$$(7.3') \qquad \begin{array}{cc} H_i = \frac{\partial H}{\partial x^i} & H_i^b = \frac{\partial^2 H}{\partial x^i \partial p_b} \\ H^a = \frac{\partial H}{\partial p_a} & H^{ab} = \frac{\partial^2 H}{\partial p_a \partial p_b} \end{array}.$$

Using this Hamiltonian, we build the fiber bundle morphism

$$\begin{array}{ccc} \overset{*}{E} & \xrightarrow{\varphi_H} & E \\ \overset{*}{\pi} \downarrow & & \downarrow \pi \\ M & \xrightarrow{Id_M} & M \end{array},$$

where φ_H is locally defined

$$(7.4') \qquad \begin{array}{ccc} \overset{*}{\pi}^{-1}(U) & \xrightarrow{\varphi_H} & \pi^{-1}(U) \\ \overset{*}{u}_x & \longmapsto & H^a\left(\overset{*}{u}_x\right) s_a(x) \end{array},$$

for any vector local $(m+r)$-chart (U, s_U) of (E, π, M) and for any vector local $(m+r)$-chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$.

Using the **B**-morphism (φ_L, Id_M), we build the $\mathbf{B^v}$-morphism $((\rho,\eta)T\varphi_L, \varphi_L)$ given by the diagram

$$(7.5) \qquad \begin{array}{ccc} (\rho,\eta)TE & \xrightarrow{(\rho,\eta)T\varphi_L} & (\rho,\eta)T\overset{*}{E} \\ (\rho,\eta)\tau_E \downarrow & & \downarrow (\rho,\eta)\tau_{\overset{*}{E}} \\ E & \xrightarrow{\varphi_L} & \overset{*}{E} \end{array},$$

such that

$$(7.6) \qquad \begin{array}{ll} \Gamma((\rho,\eta)T\varphi_L, \varphi_L)\left(\tilde{Z}^\alpha \tilde{\partial}_\alpha\right) & = \left(\tilde{Z}^\alpha \circ \varphi_H\right) \overset{*}{\tilde{\partial}}_\alpha + \left[\left(\rho_\alpha^i \circ h \circ \pi\right) \tilde{Z}^\alpha L_{ib}\right] \circ \varphi_H \overset{\cdot\, b}{\tilde{\partial}}, \\ \Gamma((\rho,\eta)T\varphi_L, \varphi_L)\left(Y^a \overset{\cdot}{\tilde{\partial}}_a\right) & = (Y^a L_{ab}) \circ \varphi_H \overset{\cdot\, b}{\tilde{\partial}}, \end{array}$$

for any $\tilde{Z}^{\alpha}\tilde{\partial}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a}\in\Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right).$

The $\mathbf{B^{v}}$-morphism $\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)$ will be called the *(ρ,η)-tangent application of the Legendre bundle morphism associated to the Lagrangian L.*

Using the $\mathbf{B}$-morphism (φ_{H},Id_{M}), we build the $\mathbf{B^{v}}$-morphism $\left(\left(\rho,\eta\right)T\varphi_{H},\varphi_{H}\right)$ given by the diagram

$$\begin{array}{ccc}\left(\rho,\eta\right)T\overset{*}{E} & \xrightarrow{(\rho,\eta)T\varphi_{H}} & \left(\rho,\eta\right)TE\\ \left(\rho,\eta\right)\tau_{\overset{*}{E}}\downarrow & & \downarrow\left(\rho,\eta\right)\tau_{E}\\ E^{*} & \xrightarrow{\varphi_{H}} & E,\end{array}\tag{7.5'}$$

such that

$$\begin{array}{rl}\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\left(\tilde{Z}^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}\right) & =\left(\tilde{Z}^{\alpha}\circ\varphi_{L}\right)\tilde{\partial}_{\alpha}+\left[\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^{\alpha}H_{i}^{b}\right]\circ\varphi_{L}\dot{\tilde{\partial}}_{b},\\ \Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\left(Y_{a}\dot{\tilde{\partial}}^{a}\right) & =\left(Y_{a}H^{ab}\right)\circ\varphi_{L}\dot{\tilde{\partial}}_{b},\end{array}\tag{7.6'}$$

for any $\tilde{Z}^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}+Y_{a}\dot{\tilde{\partial}}^{a}\in\Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right).$

The $\mathbf{B^{v}}$-morphism $\left(\left(\rho,\eta\right)T\varphi_{H},\varphi_{H}\right)$ will be called the *(ρ,η)-tangent application of the Legendre bundle morphism associated to the Hamiltonian H.*

Let

$$\left(\frac{\partial}{\partial x^{i}},\frac{\partial}{\partial y^{a}}\right)\overset{put}{=}\left(\partial_{i},\dot{\partial}_{a}\right)\tag{7.7}$$

be the natural base for sections Lie algebra $\left(\Gamma\left(TE,\tau_{E},E\right),+,\cdot,[,]_{TE}\right).$

Let

$$\left(\frac{\partial}{\partial x^{i}},\frac{\partial}{\partial p_{a}}\right)\overset{put}{=}\left(\frac{\overset{*}{\partial}}{\partial x^{i}},\frac{\partial}{\partial p_{a}}\right)\overset{put}{=}\left(\overset{*}{\partial}_{i},\dot{\partial}^{a}\right)\tag{7.7'}$$

be the natural base for sections Lie algebra $\left(\Gamma\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),+,\cdot,[,]_{T\overset{*}{E}}\right).$

Using the diagram:

$$\begin{array}{ccc}E & & \left(F,[,]_{F,h},(\rho,\eta)\right)\\ \pi\downarrow & & \downarrow\nu\\ M & \xrightarrow{h} & N\end{array},\tag{7.8}$$

where $\left(\left(F,\nu,N\right),[,]_{F,h},\left(\rho,\eta\right)\right)$ is a generalized Lie algebroid, we build the generalized tangent bundle

$$\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right).\tag{7.9}$$

The natural (ρ,η)-base of sections is denoted

$$\left(\frac{\partial}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{y}^{a}}\right)\overset{put}{=}\left(\tilde{\partial}_{\alpha},\dot{\tilde{\partial}}_{a}\right).\tag{7.10}$$

Using the diagram:

$$(7.8')\qquad \begin{array}{ccc} \overset{*}{E} & & \left(F,[,]_{F,h},(\rho,\eta)\right) \\ \overset{*}{\pi}\downarrow & & \downarrow\nu \\ M & \xrightarrow{\ \ h\ \ } & N \end{array},$$

where $\left(\left(F,\nu,N\right),[,]_{F,h},(\rho,\eta)\right)$ is a generalized Lie algebroid, we build the generalized tangent bundle

$$(7.9')\qquad \left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right).$$

The natural (ρ,η)-base of sections is denoted

$$(7.10')\qquad \left(\frac{\overset{*}{\partial}}{\partial\tilde{z}^{\alpha}},\frac{\partial}{\partial\tilde{p}_{a}}\right)\overset{put}{=}\left(\overset{*}{\tilde{\partial}}_{\alpha},\overset{\cdot}{\tilde{\partial}}{}^{a}\right).$$

7.1 The duality between mechanical systems

Let $\left(\left(E,\pi,M\right),F_{e},(\rho,\eta)\Gamma\right)$ be a mechanical (ρ,η)-system.

Let $g\in\mathbf{Man}\left(E,E\right)$ such that (g,h) is a $\mathbf{B^{v}}$-morphism locally invertible of (E,π,M) source and (E,π,M) target, on components g_{b}^{a}.

The $\mathbf{Mod}$-endomorphism

$$(7.1.1)\qquad \begin{array}{ccc} \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right) & \xrightarrow{\mathcal{J}_{(g,h)}} & \Gamma\left(\left(\rho,\eta\right)TE,\left(\rho,\eta\right)\tau_{E},E\right) \\ \tilde{Z}^{a}\tilde{\partial}_{a}+Y^{b}\dot{\tilde{\partial}}_{b} & \longmapsto & \left(\tilde{g}_{a}^{b}\circ h\circ\pi\right)\tilde{Z}^{a}\dot{\tilde{\partial}}_{b} \end{array}$$

is the almost tangent structure associated to $\mathbf{B^{v}}$-morphism (g,h).

The vertical section

$$(7.1.2)\qquad \mathbb{C}=y^{a}\dot{\tilde{\partial}}_{a}$$

is the Liouville section.

Let $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),\overset{*}{F}_{e},(\rho,\eta)\overset{*}{\Gamma}\right)$ be a dual mechanical (ρ,η)-system.

Let $g\in\mathbf{Man}\left(\overset{*}{E},E\right)$ be such that (g,h) is a $\mathbf{B^{v}}$-morphism locally invertible of $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ source and (E,π,M) target, on components g^{ab}.

The $\mathbf{Mod}$-endomorphism

$$(7.1.1)'\qquad \begin{array}{ccc} \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right) & \xrightarrow{\overset{*}{\mathcal{J}}_{(g,h)}} & \Gamma\left(\left(\rho,\eta\right)T\overset{*}{E},\left(\rho,\eta\right)\tau_{\overset{*}{E}},\overset{*}{E}\right) \\ \tilde{Z}^{a}\overset{*}{\tilde{\partial}}_{a}+Y_{b}\overset{\cdot}{\tilde{\partial}}{}^{b} & \longmapsto & \left(\tilde{g}_{ba}\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^{a}\overset{\cdot}{\tilde{\partial}}{}^{b} \end{array}$$

is the almost tangent structure associated to $\mathbf{B^{v}}$-morphism (g,h).

The vertical section

$$(7.1.2)'\qquad \overset{*}{\mathbb{C}}=p_{b}\overset{\cdot}{\tilde{\partial}}{}^{b}$$

is the Liouville section.

Let

$$S=y^{b}\left(g_{b}^{a}\circ h\circ \pi\right)\frac{\partial}{\partial \tilde{z}^{a}}-2\left(G^{a}-\frac{1}{4}F^{a}\right)\frac{\partial}{\partial \tilde{y}^{a}} \tag{7.1.3}$$

be the (ρ,η)-semispray associated to mechanical (ρ,η)-system $((E,\pi,M),F_e,(\rho,\eta)\Gamma)$ and from locally invertible $\mathbf{B}^{\mathbf{v}}$-morphism (g,h) and let

$$\overset{*}{S}=p_{b}\left(g^{ab}\circ h\circ \overset{*}{\pi}\right)\frac{\overset{*}{\partial}}{\partial \tilde{z}^{a}}-2\left(G_{a}-\frac{1}{4}F_{a}\right)\frac{\partial}{\partial \tilde{p}_{a}} \tag{7.1.3$'$}$$

be the (ρ,η)-semispray associated to the mechanical (ρ,η)-system $\left(\left(\overset{*}{E},\overset{*}{\pi},M\right),F_e,(\rho,\eta)\overset{*}{\Gamma}\right)$ and from locally invertible $\mathbf{B}^{\mathbf{v}}$-morphism (g,h).

Theorem 7.1.1 *If*

$$\Gamma\left((\rho,\eta)T\varphi_L,\varphi_L\right)(S)=\overset{*}{S}, \tag{7.1.4}$$

then we obtain:

$$y^{b}\left(g_{b}^{a}\circ h\circ \pi\right)\circ\varphi_H \quad =p_{b}\left(g^{ab}\circ h\circ \overset{*}{\pi}\right) \tag{7.1.5}$$

and

$$\begin{array}{ll} 2\left(G_{b}-\frac{1}{4}F_{b}\right) & =2\left[\left(G^{a}-\frac{1}{4}F^{a}\right)\cdot L_{ab}\right]\circ\varphi_H \\ & -y^{c}\left\{\left[\left(g_{c}^{a}\cdot\rho_{a}^{i}\right)\circ h\circ\pi\right]\cdot L_{ib}\right\}\circ\varphi_H. \end{array} \tag{7.1.6}$$

Theorem 7.1.2 *Dual, if*

$$\Gamma\left((\rho,\eta)T\varphi_H,\varphi_H\right)\left(\overset{*}{S}\right)=S, \tag{7.1.4$'$}$$

then we obtain:

$$p_{b}\left(g^{ba}\circ h\circ \overset{*}{\pi}\right)\circ\varphi_L \quad =y^{b}\left(g_{b}^{a}\circ h\circ \pi\right) \tag{7.1.5$'$}$$

and

$$\begin{array}{ll} 2\left(G^{a}-\frac{1}{4}F^{a}\right) & =2\left[\left(G_{b}-\frac{1}{4}F_{b}\right)\cdot H^{ab}\right]\circ\varphi_L \\ & -p_{c}\left\{\left[\left(g^{ac}\cdot\rho_{a}^{i}\right)\circ h\circ\overset{*}{\pi}\right]\cdot H_{i}^{b}\right\}\circ\varphi_L. \end{array} \tag{7.1.6$'$}$$

7.2 The duality between Lie algebroids structures

The generalized tangent bundle

$$\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right)$$

can be endowed with a Lie algebroid structure

$$\left(\left[,\right]_{(\rho,\eta)TE},\left(\tilde{\rho},Id_E\right)\right).$$

The Lie bracket $[,]_{(\rho,\eta)TE}$ is defined by

$$(7.2.1)\qquad \begin{array}{l} \left[\left(\tilde{Z}_1^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y_1^a\frac{\partial}{\partial\tilde{y}^a}\right),\left(\tilde{Z}_2^\beta\frac{\partial}{\partial\tilde{z}^\beta}+Y_2^b\frac{\partial}{\partial\tilde{y}^b}\right)\right]_{(\rho,\eta)TE}= \\ =\left[\tilde{Z}_1^\alpha T_a,\tilde{Z}_2^\beta T_\beta\right]_{\pi^*(h^*F)}\oplus\left[\left(\rho_\alpha^i\circ h\circ\pi\right)\tilde{Z}_1^\alpha\frac{\partial}{\partial x^i}+Y_1^a\frac{\partial}{\partial\tilde{y}^a},\right. \\ \qquad\qquad\left.\left(\rho_\beta^i\circ h\circ\pi\right)\tilde{Z}_2^\beta\frac{\partial}{\partial x^i}+Y_2^b\frac{\partial}{\partial\tilde{y}^b}\right]_{TE}, \end{array}$$

for any sections $\left(\tilde{Z}_1^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y_1^a\frac{\partial}{\partial\tilde{y}^a}\right)$ and $\left(\tilde{Z}_2^\beta\frac{\partial}{\partial\tilde{z}^\beta}+Y_2^b\frac{\partial}{\partial\tilde{y}^b}\right)$.

The anchor map $(\tilde{\rho},Id_E)$ is a $\mathbf{B^v}$-morphism of $((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$ source and (TE,τ_E,E) target, where

$$(5.2.2)\qquad \begin{array}{c} (\rho,\eta)\,TE\ \xrightarrow{\tilde{\rho}}\ TE \\ \left(\tilde{Z}^\alpha\frac{\partial}{\partial\tilde{z}^\alpha}+Y^a\frac{\partial}{\partial\tilde{y}^a}\right)(u_x)\longmapsto\left(\left(\rho_\alpha^i\circ h\circ\pi\right)\tilde{Z}^\alpha\frac{\partial}{\partial x^i}+Y^a\frac{\partial}{\partial\tilde{y}^a}\right)(u_x) \end{array}$$

The generalized tangent bundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

can be endowed with a Lie algebroid structure

$$\left([,]_{(\rho,\eta)T\overset{*}{E}},\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)\right).$$

The Lie bracket $[,]_{(\rho,\eta)T\overset{*}{E}}$ is defined by

$$(7.2.1)'\qquad \begin{array}{l} \left[\left(\tilde{Z}_1^\alpha\frac{\overset{*}{\partial}}{\partial\tilde{z}^\alpha}+Y_{1a}\frac{\partial}{\partial\tilde{p}_a}\right),\left(\tilde{Z}_2^\beta\frac{\overset{*}{\partial}}{\partial\tilde{z}^\beta}+Y_{2b}\frac{\partial}{\partial\tilde{p}_b}\right)\right]_{(\rho,\eta)T\overset{*}{E}}= \\ =\left[\tilde{Z}_1^\alpha T_a,\tilde{Z}_2^\beta T_\beta\right]_{\overset{*}{\pi}^*(h^*F)}\oplus\left[\left(\rho_\alpha^i\circ h\circ\overset{*}{\pi}\right)\tilde{Z}_1^\alpha\frac{\partial}{\partial x^i}+Y_{1a}\frac{\partial}{\partial\tilde{p}_a},\right. \\ \qquad\qquad\left.\left(\rho_\beta^i\circ h\circ\overset{*}{\pi}\right)\tilde{Z}_2^\beta\frac{\partial}{\partial x^i}+Y_{2b}\frac{\partial}{\partial\tilde{p}_b}\right]_{T\overset{*}{E}}, \end{array}$$

for any sections $\left(\tilde{Z}_1^\alpha\frac{\overset{*}{\partial}}{\partial\tilde{z}^\alpha}+Y_{1a}\frac{\partial}{\partial\tilde{p}_a}\right)$ and $\left(\tilde{Z}_2^\beta\frac{\overset{*}{\partial}}{\partial\tilde{z}^\beta}+Y_{2b}\frac{\partial}{\partial\tilde{p}_b}\right)$.

The anchor map $\left(\overset{*}{\tilde{\rho}},Id_{\overset{*}{E}}\right)$ is a $\mathbf{B^v}$-morphism of $\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$ source and $\left(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right)$ target, where

$$(5.2.2)'\qquad \begin{array}{c} (\rho,\eta)\,T\overset{*}{E}\ \xrightarrow{\overset{*}{\tilde{\rho}}}\ T\overset{*}{E} \\ \left(\tilde{Z}^\alpha\frac{\overset{*}{\partial}}{\partial\tilde{z}^\alpha}+Y_a\frac{\partial}{\partial\tilde{p}_a}\right)(\overset{*}{u}_x)\longmapsto\left(\left(\rho_\alpha^i\circ h\circ\overset{*}{\pi}\right)\tilde{Z}^\alpha\frac{\overset{*}{\partial}}{\partial x^i}+Y_a\frac{\partial}{\partial\tilde{p}_a}\right)(\overset{*}{u}_x) \end{array}$$

Theorem 7.2.1 *If the* $\mathbf{B^v}$*-morphism* $((\rho,\eta)\,T\varphi_L,\varphi_L)$ *is morphism of Lie algebroids, then we obtain:*

$$(7.2.3)\qquad \left(L_{\alpha\beta}^\gamma\circ h\circ\pi\right)\circ\varphi_H=L_{\alpha\beta}^\gamma\circ h\circ\overset{*}{\pi},$$

$$\begin{aligned}&\left[\left(L_{\alpha\beta}^{\gamma}\rho_{\gamma}^{k}\right)\circ h\circ\pi\cdot L_{kb}\right]\circ\varphi_{H}&&=\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot\frac{\overset{*}{\partial}}{\partial x^{i}}\left[\left(\rho_{\beta}^{j}\circ h\circ\pi\cdot L_{jb}\right)\circ\varphi_{H}\right]\\ (7.2.4)\qquad&&&-\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\cdot\frac{\overset{*}{\partial}}{\partial x^{j}}\left[\left(\rho_{\alpha}^{i}\circ h\circ\pi\cdot L_{ib}\right)\circ\varphi_{H}\right]\\ &&&+\left(\rho_{\alpha}^{i}\circ h\circ\pi\cdot L_{ia}\right)\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{a}}\left[\left(\rho_{\beta}^{j}\circ h\circ\pi\cdot L_{jb}\right)\circ\varphi_{H}\right]\\ &&&-\left(\rho_{\beta}^{j}\circ h\circ\pi\cdot L_{ja}\right)\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{a}}\left[\left(\rho_{\alpha}^{i}\circ h\circ\pi\cdot L_{ib}\right)\circ\varphi_{H}\right],\end{aligned}$$

$$\begin{aligned}&0&&=\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot\frac{\overset{*}{\partial}}{\partial x^{i}}\left(L_{ba}\circ\varphi_{H}\right)\\ (7.2.5)\qquad&&&+\left(\rho_{\alpha}^{i}\circ h\circ\pi\cdot L_{bc}\right)\circ\varphi_{H}\frac{\partial}{\partial p_{c}}\left(L_{ba}\circ\varphi_{H}\right)\\ &&&-L_{bc}\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{c}}\left[\left(\rho_{\alpha}^{i}\circ h\circ\pi\cdot L_{ia}\right)\circ\varphi_{H}\right]\end{aligned}$$

and

$$\begin{aligned}&0&&=L_{ac}\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{c}}\left(L_{bd}\circ\varphi_{H}\right)\\ (7.2.6)\qquad&&&-L_{bc}\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{c}}\left(L_{ad}\circ\varphi_{H}\right).\end{aligned}$$

Proof. Developing the following equalities

$$\begin{aligned}&\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\left[\tilde{\partial}_{\alpha},\tilde{\partial}_{\beta}\right]_{(\rho,\eta)TE}\\ =&\left[\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\tilde{\partial}_{\alpha},\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\tilde{\partial}_{\beta}\right]_{(\rho,\eta)T\overset{*}{E}},\end{aligned}$$

$$\begin{aligned}&\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\left[\tilde{\partial}_{\alpha},\dot{\tilde{\partial}}_{b}\right]_{(\rho,\eta)TE}\\ =&\left[\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\tilde{\partial}_{\alpha},\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\dot{\tilde{\partial}}_{b}\right]_{(\rho,\eta)T\overset{*}{E}}\end{aligned}$$

and

$$\begin{aligned}&\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\left[\dot{\tilde{\partial}}_{a},\dot{\tilde{\partial}}_{b}\right]_{(\rho,\eta)TE}\\ =&\left[\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\dot{\tilde{\partial}}_{a},\Gamma\left(\left(\rho,\eta\right)T\varphi_{L},\varphi_{L}\right)\dot{\tilde{\partial}}_{b}\right]_{(\rho,\eta)T\overset{*}{E}}\end{aligned}$$

it results the conclusion of the theorem. *q.e.d.*

Corollary 7.2.1 *In particular, if* $(\rho,\eta,h)=(Id_{TM},Id_{M},Id_{M})$*, then we obtain:*

$$\begin{aligned}(7.2.4)'\qquad&0&&=\frac{\overset{*}{\partial}}{\partial x^{i}}\left(L_{jb}\circ\varphi_{H}\right)-\frac{\overset{*}{\partial}}{\partial x^{j}}\left(L_{ib}\circ\varphi_{H}\right)\\ &&&+L_{ia}\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{a}}\left(L_{jb}\circ\varphi_{H}\right)-L_{ja}\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{a}}\left(L_{ib}\circ\varphi_{H}\right)\end{aligned}$$

$$\begin{aligned}(5.2.5)'\qquad&0&&=\frac{\overset{*}{\partial}}{\partial x^{i}}\left(L_{ba}\circ\varphi_{H}\right)+L_{bc}\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{c}}\left(L_{ba}\circ\varphi_{H}\right)\\ &&&-L_{bc}\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{c}}\left(L_{ia}\circ\varphi_{H}\right)\end{aligned}$$

and

$$\begin{aligned}&0&&=L_{ac}\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{c}}\left(L_{bd}\circ\varphi_{H}\right)\\ (7.2.6)'\qquad&&&-L_{bc}\circ\varphi_{H}\cdot\frac{\partial}{\partial p_{c}}\left(L_{ad}\circ\varphi_{H}\right).\end{aligned}$$

Theorem 7.2.2 *Dual, if the the* $\mathbf{B}^{\mathbf{v}}$*-morphism* $((\rho,\eta)\,T\varphi_H,\varphi_H)$ *is morphism of Lie algebroids, then we obtain:*

$$\left(L_{\alpha\beta}^{\gamma}\circ h\circ \overset{*}{\pi}\right)\circ\varphi_L = L_{\alpha\beta}^{\gamma}\circ h\circ\pi, \tag{7.2.7}$$

$$\begin{array}{ll}\left[\left(L_{\alpha\beta}^{\gamma}\rho_{\gamma}^{k}\right)\circ h\circ\overset{*}{\pi}\cdot H_{k}^{b}\right]\circ\varphi_L & =\rho_{\alpha}^{i}\circ h\circ\pi\cdot\frac{\partial}{\partial x^{i}}\left[\left(\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\cdot H_{j}^{b}\right)\circ\varphi_L\right]\\ & -\rho_{\beta}^{j}\circ h\circ\pi\cdot\frac{\partial}{\partial x^{j}}\left[\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot H_{i}^{b}\right)\circ\varphi_L\right]\\ & +\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot H_{i}^{c}\right)\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left[\left(\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\cdot H_{j}^{b}\right)\circ\varphi_L\right]\\ & -\left(\rho_{\beta}^{j}\circ h\circ\overset{*}{\pi}\cdot H_{j}^{c}\right)\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left[\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot H_{i}^{b}\right)\circ\varphi_L\right],\end{array} \tag{7.2.8}$$

$$\begin{array}{rl}0 & =\rho_{\alpha}^{i}\circ h\circ\pi\cdot\frac{\partial}{\partial x^{i}}\left(H^{ba}\circ\varphi_L\right)\\ & +\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot H^{bc}\right)\circ\varphi_L\frac{\partial}{\partial y^{c}}\left(H^{ba}\circ\varphi_L\right)\\ & -H^{bc}\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left[\left(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi}\cdot H_{i}^{a}\right)\circ\varphi_L\right]\end{array} \tag{7.2.9}$$

and

$$\begin{array}{rl}0 & =H^{ac}\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left(H^{bd}\circ\varphi_L\right)\\ & -H^{bc}\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left(H^{ad}\circ\varphi_L\right).\end{array} \tag{7.2.10}$$

Proof. Developing the following equalities

$$\begin{array}{c}\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\left[\overset{*}{\tilde{\partial}}_{\alpha},\overset{*}{\tilde{\partial}}_{\beta}\right]_{(\rho,\eta)T\overset{*}{E}}\\ =\left[\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\overset{*}{\tilde{\partial}}_{\alpha},\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\overset{*}{\tilde{\partial}}_{\beta}\right]_{(\rho,\eta)TE},\end{array}$$

$$\begin{array}{c}\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\left[\overset{*}{\tilde{\partial}}_{\alpha},\dot{\tilde{\partial}}^{b}\right]_{(\rho,\eta)T\overset{*}{E}}\\ =\left[\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\overset{*}{\tilde{\partial}}_{\alpha},\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\dot{\tilde{\partial}}^{b}\right]_{(\rho,\eta)TE}\end{array}$$

and

$$\begin{array}{c}\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\left[\dot{\tilde{\partial}}^{a},\dot{\tilde{\partial}}^{b}\right]_{(\rho,\eta)T\overset{*}{E}}\\ =\left[\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\dot{\tilde{\partial}}^{a},\Gamma\left((\rho,\eta)\,T\varphi_H,\varphi_H\right)\dot{\tilde{\partial}}^{b}\right]_{(\rho,\eta)TE}\end{array}$$

it results the conclusion of the theorem. *q.e.d.*

Corollary 7.2.2 *In particular, if* $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$*, then we obtain:*

$$\begin{array}{rl}0 & =\frac{\partial}{\partial x^{i}}\left(H_{j}^{b}\circ\varphi_L\right)-\frac{\partial}{\partial x^{j}}\left(H_{i}^{b}\circ\varphi_L\right)\\ & +H_{i}^{c}\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left(H_{j}^{b}\circ\varphi_L\right)-H_{j}^{c}\circ\varphi_L\cdot\frac{\partial}{\partial y^{c}}\left(H_{i}^{b}\circ\varphi_L\right)\end{array} \tag{7.2.8'}$$

$$(7.2.9)' \qquad \begin{array}{rl} 0 & = \frac{\partial}{\partial x^i}\left(H^{ba}\circ\varphi_L\right) + H^{bc}\circ\varphi_L\cdot\frac{\partial}{\partial y^c}\left(H^{ba}\circ\varphi_L\right) \\ & -H^{bc}\circ\varphi_L\cdot\frac{\partial}{\partial y^c}\left(H^a_i\circ\varphi_L\right) \end{array}$$

and

$$(7.2.10)' \qquad \begin{array}{rl} 0 & = H^{ac}\circ\varphi_L\cdot\frac{\partial}{\partial y^c}\left(H^{bd}\circ\varphi_L\right) \\ & -H^{bc}\circ\varphi_L\cdot\frac{\partial}{\partial y^c}\left(H^{ad}\circ\varphi_L\right). \end{array}$$

Definition 7.2.1 If $((\rho,\eta)\,T\varphi_L,\varphi_L)$ and $((\rho,\eta)\,T\varphi_H,\varphi_H)$ are Lie algebroids morphisms, then we will say that (E,π,M) *and* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ *are Legendre* (ρ,η,h)*-equivalent.*

We will write
$$(E,\pi,M)\underset{(\rho,\eta,h)}{\overset{\mathcal{L}}{\widetilde{\quad}}}\left(\overset{*}{E},\overset{*}{\pi},M\right).$$

Theorem 7.2.3 *If*
$$(E,\pi,M)\underset{(\rho,\eta,h)}{\overset{\mathcal{L}}{\widetilde{\quad}}}\left(\overset{*}{E},\overset{*}{\pi},M\right),$$
then, using the equalities (7.2.3) *and* (7.2.7) *it results that for any vcetor local* $(m+r)$*-chart* (U,s_U) *of* (E,π,M) *and for any vector local* $(m+r)$*-chart* $\left(U,\overset{*}{s}_U\right)$ *of* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ *we obtain:*

$$(7.2.11) \qquad \varphi_H\circ\varphi_L = Id_{\pi^{-1}(U)}$$

and

$$(7.2.12) \qquad \varphi_L\circ\varphi_H = Id_{\overset{*}{\pi}^{-1}(U)}.$$

Therefore, locally, φ_L *is diffeomorphism and* $\varphi_L^{-1}=\varphi_H$.

7.3 The duality betwen adapted (ρ,η)-basis

If $(\rho,\eta)\,\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M), then the adapted (ρ,η)-base of sections is

$$(7.3.1) \qquad \left(\frac{\partial}{\partial\tilde{z}^\alpha}-(\rho,\eta)\,\Gamma^a_\alpha\frac{\partial}{\partial\tilde{y}^a},\frac{\partial}{\partial\tilde{y}^a}\right)\overset{put}{=}\left(\frac{\delta}{\delta\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{y}^a}\right)\overset{put}{=}\left(\tilde{\delta}_\alpha,\dot{\tilde{\partial}}_a\right).$$

If $(\rho,\eta)\,\overset{*}{\Gamma}$ is a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then the adapted (ρ,η)-base of sections is

$$(7.3.1)' \qquad \left(\frac{\overset{*}{\partial}}{\partial\tilde{z}^\alpha}+(\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\frac{\partial}{\partial\tilde{p}_b},\frac{\partial}{\partial\tilde{p}_a}\right)\overset{put}{=}\left(\frac{\overset{*}{\delta}}{\delta\tilde{z}^\alpha},\frac{\partial}{\partial\tilde{p}_a}\right)\overset{put}{=}\left(\overset{*}{\tilde{\delta}}_\alpha,\dot{\tilde{\partial}}^a\right).$$

Theorem 7.3.1 *If*
$$\Gamma\left((\rho,\eta)\,T\varphi_L,\varphi_L\right)\left(\tilde{\delta}_\alpha\right)=\overset{*}{\tilde{\delta}}_\alpha,$$
then

$$(7.3.2) \qquad (\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}=\left[\left(\rho^i_\alpha\circ h\circ\pi\right)\cdot L_{ib}-(\rho,\eta)\,\Gamma^a_\alpha\cdot L_{ab}\right]\circ\varphi_H.$$

Proof. After some calculations, it results the conclusion of the theorem. *q.e.d.*

Corollary 7.3.1 *In particular, if* $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$*, then the equality*

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\left(\tilde{\delta}_k\right)=\overset{*}{\tilde{\delta}}_k,$$

implies the equality

$$\overset{*}{\Gamma}_{bk}=\left[L_{kb}-\Gamma_k^a\cdot L_{ab}\right]\circ\varphi_H. \tag{7.3.2$'$}$$

Theorem 7.3.2 *Dual, if*

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\left(\overset{*}{\tilde{\delta}}_\alpha\right)=\tilde{\delta}_\alpha,$$

then

$$-\left(\rho,\eta\right)\Gamma_\alpha^a=\left[\left(\rho_\alpha^i\circ h\circ\overset{*}{\pi}\right)\cdot H_i^a+\left(\rho,\eta\right)\overset{*}{\Gamma}_{b\alpha}\cdot H^{ba}\right]\circ\varphi_L. \tag{7.3.3}$$

Proof. After some calculations, it results the conclusion of the theorem. *q.e.d.*

Corollary 7.3.2 *In particular, if* $(\rho,\eta,h)=(Id_{TM},Id_M,Id_M)$*, then the equality*

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\left(\overset{*}{\tilde{\delta}}_k\right)=\tilde{\delta}_k,$$

implies the equality

$$-\Gamma_k^a=\left[H_k^a+\overset{*}{\Gamma}_{bk}\cdot H^{ba}\right]\circ\varphi_L. \tag{7.3.3$'$}$$

Definition 7.3.2 If

$$\left(E,\pi,M\right)\overbrace{}^{\mathcal{L}}_{(\rho,\eta,h)}\left(\overset{*}{E},\overset{*}{\pi},M\right).$$

and

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_L,\varphi_L\right)\left(\tilde{\delta}_\alpha\right)=\overset{*}{\tilde{\delta}}_\alpha, \tag{7.3.4}$$

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\left(\overset{*}{\tilde{\delta}}_\alpha\right)=\tilde{\delta}_\alpha, \tag{7.3.4$'$}$$

then we will say that (E,π,M) *and* $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ *are horizontal Legendre* (ρ,η,h)*-equivalent.*

We will write

$$\left(E,\pi,M\right)\overbrace{}^{\mathcal{HL}}_{(\rho,\eta,h)}\left(\overset{*}{E},\overset{*}{\pi},M\right).$$

The dual natural (ρ,η)-base of the natural (ρ,η)-base $\left(\tilde{\partial}_\alpha, \dot{\tilde{\partial}}_a\right)$ is denoted $(d\tilde{z}^\alpha, d\tilde{y}^a)$ and the dual adapted (ρ,η)-base of the adapted (ρ,η)-base $\left(\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_a\right)$ is denoted

$$(7.3.5)\qquad (d\tilde{z}^\alpha, \delta\tilde{y}^a) \overset{put}{=} \left(d\tilde{z}^\alpha, d\tilde{y}^a + (\rho,\eta)\,\Gamma^a_\alpha \cdot d\tilde{z}^\alpha\right).$$

The dual natural (ρ,η)-base of the natural (ρ,η)-base $\left(\overset{*}{\tilde{\partial}}_\alpha, \dot{\tilde{\partial}}^a\right)$ is denoted $(d\tilde{z}^\alpha, d\tilde{p}_a)$ and the dual adapted (ρ,η)-base of the adapted (ρ,η)-base $\left(\overset{*}{\tilde{\delta}}_\alpha, \dot{\tilde{\partial}}_a\right)$ is denoted

$$(7.3.5)'\qquad (d\tilde{z}^\alpha, \delta\tilde{p}_a) \overset{put}{=} \left(d\tilde{z}^\alpha, d\tilde{p}_a - (\rho,\eta)\,\overset{*}{\Gamma}_{a\alpha} \cdot d\tilde{z}^\alpha\right).$$

Theorem 7.3.3 *The equality* (7.3.4) *is equivalent with the equality:*

$$(7.3.6)\qquad \Gamma\left((\rho,\eta)\,T\varphi_L, \varphi_L\right)^*(\delta\tilde{p}_a) = L_{ab}\cdot\delta\tilde{y}^b$$

and the equality (7.3.4)$'$ *is equivalent with the equality:*

$$(7.3.6)'\qquad \Gamma\left((\rho,\eta)\,T\varphi_H, \varphi_H\right)^*(\delta\tilde{y}^a) = H^{ab}\cdot\delta\tilde{p}_b$$

Theorem 7.3.4 *If*

$$(E,\pi,M)\,\underset{(\rho,\eta,h)}{\overset{\mathcal{HL}}{\widetilde{\qquad}}}\left(\overset{*}{E},\overset{*}{\pi},M\right),$$

then we obtain:

$$(7.3.7)\qquad (\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha} = \left[\left(\rho^i_\alpha\circ h\circ\pi\right)\cdot L_{ib} - (\rho,\eta)\,\Gamma^a_\alpha\cdot L_{ab}\right]\circ\varphi_H$$

and

$$(7.3.7)'\qquad -(\rho,\eta)\,\Gamma^a_\alpha = \left[\left(\rho^i_\alpha\circ h\circ\overset{*}{\pi}\right)\cdot H^a_i + (\rho,\eta)\,\overset{*}{\Gamma}_{b\alpha}\cdot H^{ba}\right]\circ\varphi_L.$$

If the Lagrangian L is regular, then we will define the real local functions $\tilde{L}^{ab}$ such that

$$\left\|\tilde{L}^{ab}(u_x)\right\| = \left\|L_{ab}(u_x)\right\|^{-1},\ \forall u_x\in\pi^{-1}(U).$$

If the Hamiltonian H is regular, then we will define the real local functions $\tilde{H}_{ab}$ such that

$$\left\|\tilde{H}_{ab}\left(\overset{*}{u}_x\right)\right\| = \left\|H^{ab}\left(\overset{*}{u}_x\right)\right\|^{-1},\ \forall\overset{*}{u}_x\in\overset{*}{\pi}{}^{-1}(U).$$

Remark 7.3.1 If the Lagrangian L is regular and

$$(E,\pi,M)\,\underset{(\rho,\eta,h)}{\overset{\mathcal{HL}}{\widetilde{\qquad}}}\left(\overset{*}{E},\overset{*}{\pi},M\right)$$

then, using the equalities (7.3.7) and (7.3.7)$'$, we obtain:

$$(7.3.8)\qquad \left(\rho^i_\alpha\circ h\circ\pi\right)\cdot L_{ib}\cdot\tilde{L}^{ab} = -\left[\left(\rho^i_\alpha\circ h\circ\overset{*}{\pi}\right)\cdot H^b_i\right]\circ\varphi_L$$

and

$$\tilde{L}^{ab} = H^{ab} \circ \varphi_L. \tag{7.3.9}$$

Therefore, the Hamiltonian H is regular and

$$\tilde{H}_{ab} = L_{ab} \circ \varphi_H. \tag{7.3.10}$$

It is known that the following equalities hold good

$$\left[\tilde{\delta}_\alpha, \tilde{\delta}_\beta\right]_{(\rho,\eta)TE} = \left(L^\gamma_{\alpha\beta} \circ h \circ \pi\right) \tilde{\delta}_\gamma + (\rho, \eta, h)\, \mathbb{R}^a{}_{\alpha\beta} \dot{\tilde{\partial}}_a, \tag{7.3.11}$$

and

$$\left[\overset{*}{\tilde{\delta}}_\alpha, \overset{*}{\tilde{\delta}}_\beta\right]_{(\rho,\eta)T\overset{*}{E}} = \left(L^\gamma_{\alpha\beta} \circ h \circ \overset{*}{\pi}\right) \overset{*}{\tilde{\delta}}_\gamma + (\rho, \eta, h)\, \mathbb{R}_{b\ \alpha\beta} \tilde{\partial}^{\cdot b}, \tag{7.3.11$'$}$$

Theorem 7.3.5 *If*

$$(E, \pi, M) \overset{\mathcal{HL}}{\widetilde{\,(\rho,\eta,h)\,}} \left(\overset{*}{E}, \overset{*}{\pi}, M\right),$$

then, we obtain:

$$(\rho, \eta, h)\, \mathbb{R}_{b\ \alpha\beta} = \left[(\rho, \eta, h)\, \mathbb{R}^a{}_{\alpha\beta} \cdot L_{ab}\right] \circ \varphi_H \tag{7.3.12}$$

and

$$(\rho, \eta, h)\, \mathbb{R}^a{}_{\alpha\beta} = \left[(\rho, \eta, h)\, \mathbb{R}_{b\ \alpha\beta} \cdot H^{ba}\right] \circ \varphi_L. \tag{7.3.12$'$}$$

Theorem 7.3.6 *If*

$$(E, \pi, M) \overset{\mathcal{HL}}{\widetilde{\,(\rho,\eta,h)\,}} \left(\overset{*}{E}, \overset{*}{\pi}, M\right),$$

then we obtain

$$\begin{array}{rl} \left(\frac{\partial(\rho,\eta)\Gamma^a_\alpha}{\partial y^b} \cdot L_{ac}\right) \circ \varphi_H & = L_{ba} \circ \varphi_H \cdot \frac{\partial(\rho,\eta)\Gamma_{c\alpha}}{\partial p_a} \\ & + \left(\rho^i_\alpha \circ h \circ \overset{*}{\pi}\right) \cdot \frac{\overset{*}{\partial}}{\partial x^i} \left(L_{bc} \circ \varphi_H\right) \\ & + (\rho, \eta)\, \Gamma_{a\alpha} \cdot \frac{\partial}{\partial p_a} \left(L_{bc} \circ \varphi_H\right) \end{array} \tag{7.3.13}$$

and

$$\begin{array}{rl} -\left(\frac{\partial(\rho,\eta)\Gamma^a_\alpha}{\partial y^b} \cdot L_{ac}\right) \circ \varphi_H & = L_{ba} \circ \varphi_H \cdot \frac{\partial(\rho,\eta)\Gamma_{c\alpha}}{\partial p_a} \\ & + \left(\rho^i_\alpha \circ h \circ \overset{*}{\pi}\right) \cdot \frac{\overset{*}{\partial}}{\partial x^i} \left(L_{bc} \circ \varphi_H\right) \\ & + (\rho, \eta)\, \Gamma_{a\alpha} \cdot \frac{\partial}{\partial p_a} \left(L_{bc} \circ \varphi_H\right) \end{array} \tag{7.3.13$'$}$$

Proof. Developing the following equalities

$$\begin{array}{c} \Gamma\left((\rho, \eta)\, T\varphi_L, \varphi_L\right) \left(\left[\tilde{\delta}_\alpha, \dot{\tilde{\partial}}_a\right]_{(\rho,\eta)TE}\right) \\ = \left[\Gamma\left((\rho, \eta)\, T\varphi_L, \varphi_L\right) \tilde{\delta}_\alpha, \Gamma\left((\rho, \eta)\, T\varphi_L, \varphi_L\right) \dot{\tilde{\partial}}_a\right]_{(\rho,\eta)T\overset{*}{E}} \end{array}$$

and

$$\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\left(\left[\overset{*}{\tilde{\delta}}_\alpha,\overset{\cdot}{\tilde{\partial}}^a\right]_{(\rho,\eta)T\overset{*}{E}}\right)$$
$$=\left[\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\overset{*}{\tilde{\delta}}_\alpha,\Gamma\left(\left(\rho,\eta\right)T\varphi_H,\varphi_H\right)\overset{\cdot}{\tilde{\partial}}_a\right]_{(\rho,\eta)TE}$$

it results the conclusion of the theorem. *q.e.d.*

7.4 The duality between distinguished linear (ρ,η)-connections

Let $(\rho,\eta)\,\Gamma$ be a (ρ,η)-connection for the vector bundle (E,π,M) and let

(7.4.1)
$$(X,T)\xrightarrow{(\rho,\eta)D}(\rho,\eta)\,D_XT$$

be a covariant (ρ,η)-derivative for the tensor algebra of generalized tangent bundle

$$((\rho,\eta)\,TE,(\rho,\eta)\,\tau_E,E)$$

which preserves the horizontal and vertical IDS by parallelism.

If (U,s_U) is a vector local $(m+r)$-chart for (E,π,M), then the real local functions

$$\left((\rho,\eta)\,H^\alpha_{\beta\gamma},(\rho,\eta)\,H^a_{b\gamma},(\rho,\eta)\,V^\alpha_{\beta c},(\rho,\eta)\,V^a_{bc}\right)$$

defined on $\pi^{-1}(U)$ and determined by the following equalities:

(7.4.2)
$$\begin{array}{ll}(\rho,\eta)\,D_{\tilde{\delta}_\gamma}\tilde{\delta}_\beta=(\rho,\eta)\,H^\alpha_{\beta\gamma}\tilde{\delta}_\alpha, & (\rho,\eta)\,D_{\tilde{\delta}_\gamma}\overset{\cdot}{\tilde{\partial}}_b=(\rho,\eta)\,H^a_{b\gamma}\overset{\cdot}{\tilde{\partial}}_a\\ (\rho,\eta)\,D_{\overset{\cdot}{\tilde{\partial}}_c}\tilde{\delta}_\beta=(\rho,\eta)\,V^\alpha_{\beta_c}\tilde{\delta}_\alpha, & (\rho,\eta)\,D_{\overset{\cdot}{\tilde{\partial}}_c}\overset{\cdot}{\tilde{\partial}}_b=(\rho,\eta)\,V^a_{bc}\overset{\cdot}{\tilde{\partial}}_a\end{array}$$

are the components of a distinguished linear (ρ,η)-connection $((\rho,\eta)\,H,(\rho,\eta)\,V)$.

Let $(\rho,\eta)\,\overset{*}{\Gamma}$ be a (ρ,η)-connection for the vector bundle $\left(\overset{*}{E},\overset{*}{\pi},M\right)$ and let

(7.4.1)′
$$(X,T)\xrightarrow{(\rho,\eta)\overset{*}{D}}(\rho,\eta)\,\overset{*}{D}_XT$$

be a covariant (ρ,η)-derivative for the tensor algebra of generalized tangent bundle

$$\left((\rho,\eta)\,T\overset{*}{E},(\rho,\eta)\,\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

which preserves the horizontal and vertical IDS by parallelism.

If $\left(U,\overset{*}{s}_U\right)$ is a vector local $(m+r)$-chart for $\left(\overset{*}{E},\overset{*}{\pi},M\right)$, then the real local functions

$$\left((\rho,\eta)\,\overset{*}{H}{}^\alpha_{\beta\gamma},(\rho,\eta)\,\overset{*}{H}{}^a_{b\gamma},(\rho,\eta)\,\overset{*}{V}{}^{\alpha c}_{\beta},(\rho,\eta)\,\overset{*}{V}{}^{ac}_{b}\right)$$

defined on $\overset{*}{\pi}{}^{-1}(U)$ and determined by the following equalities:

(7.4.2)′
$$\begin{array}{ll}(\rho,\eta)\,\overset{*}{D}_{\overset{*}{\tilde{\delta}}_\gamma}\overset{*}{\tilde{\delta}}_\beta=(\rho,\eta)\,\overset{*}{H}{}^\alpha_{\beta\gamma}\overset{*}{\tilde{\delta}}_\alpha, & (\rho,\eta)\,\overset{*}{D}_{\overset{*}{\tilde{\delta}}_\gamma}\overset{\cdot}{\tilde{\partial}}{}^a=(\rho,\eta)\,\overset{*}{H}{}^a_{b\gamma}\overset{\cdot}{\tilde{\partial}}{}^b\\ (\rho,\eta)\,\overset{*}{D}_{\overset{\cdot}{\tilde{\partial}}{}^c}\overset{*}{\tilde{\delta}}_\beta=(\rho,\eta)\,\overset{*}{V}{}^{\alpha c}_{\beta}\overset{*}{\tilde{\delta}}_\alpha, & (\rho,\eta)\,\overset{*}{D}_{\overset{\cdot}{\tilde{\partial}}{}^c}\overset{\cdot}{\tilde{\partial}}{}^b=(\rho,\eta)\,\overset{*}{V}{}^{bc}_{a}\overset{\cdot}{\tilde{\partial}}{}^a\end{array}$$

are the components of a distinguished linear (ρ,η)-connection

$$\left((\rho,\eta)\overset{*}{H},(\rho,\eta)\overset{*}{V}\right).$$

Theorem 7.4.1 *If*

$$(E,\pi,M)\underset{(\rho,\eta,h)}{\overset{\mathcal{HL}}{\frown}}\left(\overset{*}{E},\overset{*}{\pi},M\right)$$

and

$$\Gamma\left((\rho,\eta)T\varphi_L,\varphi_L\right)\left((\rho,\eta)D_XY\right)=(\rho,\eta)\overset{*}{D}_{\Gamma((\rho,\eta)T\varphi_L,\varphi_L)X}\Gamma\left((\rho,\eta)T\varphi_L,\varphi_L\right)Y,$$

for any $X,Y\in\Gamma\left((\rho,\eta)TE,(\rho,\eta)\tau_E,E\right)$, then we obtain:

$$(\rho,\eta)H^{\alpha}_{\beta\gamma}\circ\varphi_H=(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\gamma},\tag{7.4.3}$$

$$\begin{aligned}\left((\rho,\eta)H^{a}_{b\gamma}\cdot L_{ac}\right)\circ\varphi_H&=\left(\rho^k_\gamma\circ h\circ\overset{*}{\pi}\right)\cdot\tfrac{\overset{*}{\partial}}{\partial x^k}\left(L_{bc}\circ\varphi_H\right)\\&+(\rho,\eta)\overset{*}{\Gamma}_{b\gamma}\cdot\tfrac{\partial}{\partial p_b}\left(L_{bc}\circ\varphi_H\right)\\&-(\rho,\eta)\overset{*}{H}{}^{a}_{b\gamma}\cdot\left(L_{ac}\circ\varphi_H\right),\end{aligned}\tag{7.4.4}$$

$$(\rho,\eta)V^{\alpha}_{\beta d}\circ\varphi_H=(\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\beta}\cdot\left(L_{cd}\circ\varphi_H\right)\tag{7.4.5}$$

and

$$\begin{aligned}\left((\rho,\eta)V^{a}_{bc}\cdot L_{ad}\right)\circ\varphi_H&=\left(L_{ce}\circ\varphi_H\right)\cdot\tfrac{\partial}{\partial p_e}\left(L_{bd}\circ\varphi_H\right)\\&-\left(L_{ce}\circ\varphi_H\right)\cdot(\rho,\eta)\overset{*}{V}{}^{ef}_{d}\cdot\left(L_{bf}\circ\varphi_H\right).\end{aligned}\tag{7.4.6}$$

Theorem 7.4.2 *Dual, if*

$$(E,\pi,M)\underset{(\rho,\eta,h)}{\overset{\mathcal{HL}}{\frown}}\left(\overset{*}{E},\overset{*}{\pi},M\right)$$

and

$$\Gamma\left((\rho,\eta)T\varphi_H,\varphi_H\right)\left((\rho,\eta)\overset{*}{D}_XY\right)=(\rho,\eta)D_{\Gamma((\rho,\eta)T\varphi_H,\varphi_H)X}\Gamma\left((\rho,\eta)T\varphi_H,\varphi_H\right)Y,$$

for any $X,Y\in\Gamma\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$, then we obtain:

$$(\rho,\eta)\overset{*}{H}{}^{\alpha}_{\beta\gamma}\circ\varphi_L=(\rho,\eta)H^{\alpha}_{\beta\gamma},\tag{7.4.3$'$}$$

$$\begin{aligned}\left((\rho,\eta)\overset{*}{H}{}^{a}_{b\gamma}\cdot H^{bc}\right)\circ\varphi_L&=\left(\rho^k_\gamma\circ h\circ\pi\right)\cdot\tfrac{\partial}{\partial x^k}\left(H^{ac}\circ\varphi_L\right)\\&+(\rho,\eta)\Gamma^b_\gamma\cdot\tfrac{\partial}{\partial y^b}\left(H^{ac}\circ\varphi_L\right)\\&-(\rho,\eta)H^{a}_{b\gamma}\cdot\left(H^{bc}\circ\varphi_L\right),\end{aligned}\tag{7.4.4$'$}$$

$$(\rho,\eta)\overset{*}{V}{}^{\alpha c}_{\beta}\circ\varphi_L=(\rho,\eta)V^{\alpha}_{\beta c}\cdot\left(H^{cd}\circ\varphi_L\right)\tag{7.4.5$'$}$$

and

$$\begin{aligned}\left((\rho,\eta)\overset{*}{V}{}^{bc}_{a}\cdot H^{ad}\right)\circ\varphi_L&=\left(H^{ce}\circ\varphi_H\right)\cdot\tfrac{\partial}{\partial y^e}\left(H^{bd}\circ\varphi_L\right)\\&-\left(H^{ce}\circ\varphi_L\right)\cdot(\rho,\eta)V^{d}_{ef}\cdot\left(H^{bf}\circ\varphi_L\right).\end{aligned}\tag{7.4.6$'$}$$

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